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The mass transfer research in complex porous media and pipelines by spectral methods

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The method of solving problems of mathematical physics, in particular for pressure distribution finding in the water in the underground gas storage layers on the basis of the biorthogonal polynomials constructed by the authors is proposed in the paper. The way of the problem solving by the method of separation of variables on the basis of the biorthogonal polynomials is studied. The solution of the problem is found in the form of the series sum of the biorthogonal and quasi-spectral polynomials. The comparative analysis for the different values of parameters is performed. The impact of the methods parameters, in particular the partial sum order, the bit grid and the calculation error on the accuracy of the solution obtained is studied. The calculation results are presented in the form of the tables. The algorithm of the process of the gas motion in the pipelines using fractional derivatives is constructed.

Keywords: *spectral methods, mathematical model, pressure distribution, orthogonal polynomials, biorthogonal polynomials, quasi-orthogonal polynomials, partial differential equations, fractional derivatives equations.*

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1. Introduction

The setting of the parameters of the underground gas storage work for the gas taking off (pumping) process keeping in the presence of the water in the layers is an important problem for their effective exploitation. The water presence is found practically in all of the gas storages (gas deposits) created in the depleted deposits. Nowadays, with a large number of researches an exhaustive theory for describing of the processes occurring in the layers-collectors of the gas storages in the presence of water does not exist yet. Due to the increase in the cost of the energy carriers the requirements for the calculation of the underground gas storages work parameters increase. Since, there is the water in the underground gas storages layers the part of the gas can diffuse or dissolve in it. The proportion of the gas present in the water largely depends on the pressure distribution in it. It requires the formation of appropriate adaptive models and methods which using the measured parameters (pressure, flow, etc.), allow to determine the pressure distribution in the water, that will give an opportunity to estimate the amount of the gas in it. To solve this problem the spectral methods are used in this paper.

The spectral methods are used both in theoretical studies and for solving a wide class of problems of mathematics and mechanics. Their essence is that the functions included into the model are presented in the form of orthogonal series in accordance to the selected basis. The solution finding is reduced to the coefficients calculating of the orthogonal series of the desired solution. It is shown [1, 2] that the choice of the orthogonal basis should be coordinated with the definition domain of the desired solution. The positive sides are those that many orthogonal bases are studied well enough, they are easy to use, and the solving algorithms constructed on their basis are easy for automation. The negative side is that the summing of corresponding series is, as a rule, an incorrect problem. Further not all the criteria that are set to the solutions of the problems can be satisfied by means of using of one orthogonal basis. Due to this fact, to meet the broader criteria we modify the existing bases or construct new ones. One of the methods of these comments consideration is the use of the biorthogonal expansions. Nowadays, there are a few papers devoted to their research and practical application. It is mainly due to the fact that the formation of these biorthogonal bases is related to significant difficulties of calculation and they are not studied enough.

2. Formulation of the problem

An infinite layer of the thickness l, $0 < y < l$ is considered. The pressure values on the boundaries are equal to $\varphi_1(t)$ and $\varphi_2(t)$. The initial pressure is $f(x)$. The calculation of the water pressure distribution $p(y,t)$ in a flat infinite medium of the thickness l is determined as the solution of one-dimensional filtration equation [3, 4]

$$\frac{\partial}{\partial y}\left[\frac{kh}{\mu}\left(\frac{\partial p}{\partial y}+\rho g\right)\right]=2\alpha mh\frac{\partial p}{\partial t}, \tag{1}$$

where k is the layer permeability, μ is the dynamic viscosity of the water, α is the water saturation coefficient, m is the porosity of the layer. Since the layer thickness is insignificant and is considered to be constant, the equation (1) has the form

$$\frac{\partial}{\partial y}\left(\frac{k}{\mu}\frac{\partial p}{\partial y}\right)=2\alpha m\frac{\partial p}{\partial t}.$$

In this case, the problem is as follows:
find the solution of the equation

$$\varkappa\frac{\partial^2 p}{\partial y^2}=\frac{\partial p}{\partial t} \quad (0<y<l) \tag{2}$$

under the initial condition

$$p(y,0)=f(y), \tag{3}$$

and the boundary conditions

$$p(0,t)=\varphi_1(t), \quad p(l,t)=\varphi_2(t). \tag{4}$$

As the boundary conditions we will consider the following: on the bottom surface of the layer, the pressure can be considered to be constant, equal to hydrostatic pressure of the water p_n; on the upper surface, the pressure value is calculated on the basis of the hydraulic coupling GCP — obliteration zone — GWC [1–3] and is also considered to be constant p_v.

The initial distribution of the water pressure in the water layer

$$p(y,0)=\rho g\left(h_0+y\right), \quad 0<y<l.$$

The solution of the problem is a partial solution of a more general problem. Here $\varkappa = k/\left(2\mu\alpha m\right)$.

3. Solving the problem

Since the boundless layer of the thickness l, $0 < y < l$, is considered and functions which help us to solve this problem are considered on the interval $[-1, 1]$, then we make a replacement $y = 0.5l(x + 1)$. Then

$$p\left(y,t\right)=p\left(\frac{l\left(x+1\right)}{2},t\right)=P\left(x,t\right), \tag{5}$$

$$\frac{\partial^2 p(y,t)}{\partial y^2} = \frac{4}{l^2}\frac{\partial^2 P(x,t)}{\partial x^2}.$$

Considering the formula (5), we write the equation (2) and the conditions (3), (4) in the form

$$\varkappa\frac{4}{l^2}\frac{\partial^2 P(x,t)}{\partial x^2} = \frac{\partial P(x,t)}{\partial t}, \tag{6}$$

$$P(x,0) = \rho g\frac{l(x+1)}{2}, \tag{7}$$

$$P(-1,t) = \varphi_1 = \text{const}, \quad P(1,t) = \varphi_2 = \text{const}. \tag{8}$$

The solution of the problem (6)–(8) we find in the form [5,6]

$$P(x,t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x) G_i(t), \tag{9}$$

where $V(x)$ are the basis functions, for which the following relations are true

$$\frac{d^2 V_i^{n+\bar{i}}(x)}{dx^2} = -\frac{V_i^{n+\bar{i}}(x)}{\lambda^n_{i+(-1)^{\bar{i}+1}}} + \frac{\tau^n_{i+(-1)^{\bar{i}+1}}}{\lambda^n_{i+(-1)^{\bar{i}+1}}} T'_{n+1+\bar{i}}(x), \quad i=1,\ldots,n,$$

$$\frac{d^2 V_{n+1}^{n+1}(x)}{dx^2} = \frac{(n+2)^2\pi}{8}\sum_{k=1}^{\frac{n}{2}}\frac{\bar{c}_2^{2k}}{\lambda^n_{2k}}\frac{V_{2k-1}^{n+1}(x)}{N^n_{2k-1}} + \frac{1}{15}n(n+1)(n+3)(n+4)T'_{n+2}(x),$$

$$\frac{d^2 V_{n+2}^{n}(x)}{dx^2} = \frac{(n+1)^2\pi}{2}\sum_{k=1}^{\frac{n}{2}}\frac{\bar{c}_1^{2k-1}}{\lambda^n_{2k-1}}\frac{V_{2k}^{n}(x)}{N^n_{2k}} + \frac{1}{15}(n-1)n(n+2)(n+3)T'_{n+1}(x),$$

where $V_{n+1}^{n+1}(x) = T'_{n+2}(x)$, $V_{n+2}^{n}(x) = T'_{n+1}(x)$, $\bar{i} = 0$ for the even values of i, and $\bar{i} = 1$ for the odd values of i. $T_{n+1} = T_{n+1}(x)$ and $T_{n+2} = T_{n+2}(x)$ are the Chebyshev polynomials of degree $n+1$ and $n+2$, λ_i^n, $i=1,\ldots,n$ are the eigenvalues of the integral operator $\pi_1^\infty L = \pi_1^\infty \int_{-1}^{x}\int_{-1}^{x_1}$, $U_{2i}^{2s}(x) = \sum_{j=1}^{s} c_{2j}^{2i}\tilde{T}_{2j}(x)$, $U_{2i-1}^{2s-1}(x) = \sum_{j=1}^{s} c_{2j-1}^{2i-1}\tilde{T}_{2j-1}(x)$ are the eigenfunctions of this operator, $\tilde{T}_j(x)$ are modified Chebyshev polinomials, $\bar{U}_{2i}^{2s}(x) = \sum_{j=1}^{s}\bar{c}_{2j}^{2i}T_{2j}(x)$, $\bar{U}_{2i-1}^{2s-1}(x) = \sum_{j=1}^{s}\bar{c}_{2j-1}^{2i-1}T_{2j-1}(x)$ are the eigenfunctions of this turned operator, $\tau_{2i}^{2s}(x) = \frac{c_{2s}^{2i}}{4(2s+1)(2s+2)}$, $\tau_{2i-1}^{2s}(x) = \frac{c_{2s-1}^{2i-1}}{4(2s)(2s+1)}$, $\bar{\tau}_{2i}^{2s}(x) = \frac{\bar{c}_{2s}^{2i}}{4(2s)(2s+1)}$, $\bar{\tau}_{2i-1}^{2s}(x) = \frac{\bar{c}_{2s-1}^{2i-1}}{4(2s-1)(2s)}$, are the biorthogonal functions $V_{2i-\bar{i}}^{n+\bar{i}}(x) = \int_{-1}^{x} U_{2i-1+\bar{i}}^{n-\bar{i}-(-1)^{\bar{i}}}(x_1)dx_1$, $\bar{V}_{2i-\bar{i}}^{n+\bar{i}}(x) = -\sqrt{1-x^2}\int_{-1}^{x}\frac{\bar{U}_{2i-1+\bar{i}}^{n-\bar{i}-(-1)^{\bar{i}}}(x_1)}{\sqrt{1-x_1^2}}dx_1$, $i=1,\ldots,n/2$, $N_i^n = \int_{-1}^{1}\frac{V_i^{n+\bar{i}}(x)\bar{V}_i^{n+\bar{i}}(x)}{\sqrt{1-x^2}}dx$, $i=1,\ldots,n$ is the norm of the biorthogonal functions [5,6]. Let us substitute (9) into the equation (6) and obtain

$$\varkappa\frac{4}{l^2}\sum_{i=1}^{n+2}\frac{\partial^2 V_i^{n+\bar{i}}(x)}{\partial x^2}G_i(t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x)\frac{\partial G_i(t)}{\partial t}. \tag{10}$$

Let us multiply the equation (10) by $V_i^{n+\bar{i}}(x)$, $i=1,\ldots,n$ and integrate it with respect to x with the weight $r(x) = (1-x^2)^{-1/2}$ from $x=-1$ to $x=1$. We will obtain the system of n differential equations of the first order of the kind

$$\frac{4}{l^2}\left(-\frac{N_i}{\lambda^n_{i+(-1)^{\bar{i}+1}}}G_i(t) + \frac{(n+1+\bar{i})^2}{2^{2\bar{i}+1}}\pi\frac{\bar{c}_{1+\bar{i}}^{i+(-1)^{\bar{i}+1}}}{\lambda^n_{i+(-1)^{\bar{i}+1}}}G_{n+2-\bar{i}}(t)\right) = N_i\frac{\partial G_i(t)}{\partial t}, \quad i=1,\ldots,n. \tag{11}$$

From the boundary conditions (7), considering $V_i(-1) = V_i(1) = 0$, we have

$$\frac{\varphi_1 + \varphi_2}{2(n+1)^2} = G_{n+2}(t), \quad \frac{\varphi_2 - \varphi_1}{2(n+2)^2} = G_{n+1}(t). \tag{12}$$

Taking into account the formula (12), we write the system (11) in the form

$$\frac{\partial G_i(t)}{\partial t} + \frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} G_i(t) = \frac{\varkappa\pi}{2^{2\bar{i}} l^2 N_i} \frac{\bar{c}^{\,i+(-1)^{\bar{i}+1}}_{1+\bar{i}}}{\lambda^n_{i+(-1)^{\bar{i}+1}}} \left(\varphi_2 + (-1)^i \varphi_1\right), \quad i = 1, \ldots, n. \tag{13}$$

Therefore, the following functions will be the solutions of the equations of the system (13)

$$G_i(t) = \frac{\bar{c}^{\,i+(-1)^{\bar{i}+1}}_{1+\bar{i}} \pi}{2^{2\bar{i}+2} N_i} \left(\varphi_2 + (-1)^i \varphi_1\right) \left(1 - e^{-\frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} t}\right) + G_i(0) e^{-\frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} t}, \quad i = 1, \ldots, n, \tag{14}$$

where the coefficients $G_i(0)$ are found from the initial condition (7) by the formula

$$G_i(0) = \frac{1}{N_i} \int_{-1}^{1} \frac{P(x,0) \bar{V}_i^{n+\bar{i}}(x)}{\sqrt{1-x^2}} dx. \tag{15}$$

Thus, we have found the solution of the equation (6) on the interval $x \in [-1, 1]$, namely

$$P(x,t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x) \left(\frac{\bar{c}^{\,i+(-1)^{\bar{i}+1}}_{1+\bar{i}} \pi}{2^{2\bar{i}+2} N_i} \left(\varphi_2 + (-1)^i \varphi_1\right) \left(1 - e^{-\frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} t}\right) + G_i\left(0\right) e^{-\frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} t} \right).$$

Let us return to the variable y in obtained solution and obtain the solution of the equation (2)

$$p(y,t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}} \left(\frac{2y - l}{l}\right) \times \left(\frac{\bar{c}^{\,i+(-1)^{\bar{i}+1}}_{1+\bar{i}} \pi}{2^{2\bar{i}+2} N_i} \left(\varphi_2 + (-1)^i \varphi_1\right) \left(1 - e^{-\frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} t}\right) + G_i(0) e^{-\frac{4\varkappa}{l^2 \lambda^n_{i+(-1)^{\bar{i}+1}}} t} \right).$$

4. Computational experiment

The results of the solution of the equation (2) for $k = 4 \cdot 10^{-12}$, $\mu = 1.1 \cdot 10^{-6}\ (\mathrm{m^2/s})$, $\alpha = 0.8$, $m = 0.28$, $p_0 = 6.864655\,(\mathrm{MN/m^2})$, $p_1 = 5.3936575\,(\mathrm{MN/m^2})$, $\rho = 998\,(\mathrm{kg/m^3})$, $g = 9.8\,(\mathrm{m/s^2})$, $h_0 = 541\,(\mathrm{m})$ is indicated in the tables.

Table 1. The values of the water pressure in an infinite layer of the thickness $l = 10$ m for different values of the time t and the coordinate y(meters) at $n = 10$, the time value t is indicated in hours in the table.

t	y										
	0	1	2	3	4	5	6	7	8	9	10
0	70.0000	56.0267	52.6886	55.1106	54.3511	53.7224	55.4532	54.0919	54.9389	54.6221	55.0000
24	70.0000	60.4508	55.6144	54.4376	54.3707	54.4451	54.5601	54.6542	54.7525	54.8763	55.0000
120	70.0000	65.3781	61.3767	58.3828	56.4559	55.4035	54.9364	54.7972	54.8148	54.8966	55.0000
240	70.0000	66.7222	63.6759	61.0459	58.9397	57.3793	56.3153	55.6536	55.2837	55.0979	55.0000
360	70.0000	67.3318	64.7914	62.4891	60.5032	58.8729	57.5973	56.6414	55.9437	55.4259	55.0000
480	70.0000	67.6952	65.4725	63.4048	61.5476	59.9338	58.5711	57.4429	56.5106	55.7188	55.0000
720	70.0000	68.1024	66.2438	64.4597	62.7777	61.2156	59.7792	58.4621	57.2462	56.1038	55.0000

Table 2. The values of the water pressure in an infinite layer of the thickness $l = 10$ m for different values of the time t and the coordinate y(meters) at $n = 10$, the time value t is indicated in hours in the table.

t	y								
	1	2	3	4	5	6	7	8	9
168	66.0826	62.5512	59.6844	57.5939	56.2329	55.4537	55.0777	54.9471	54.9466
336	67.2362	64.6141	62.2544	60.2411	58.6131	57.3652	56.4553	55.8149	55.3603
504	67.7511	65.5780	63.5482	61.7136	60.1054	58.7315	57.5771	56.6069	55.7689
672	68.0429	66.1309	64.3045	62.5958	61.0249	59.5984	58.3087	57.1350	56.0455
840	68.2190	66.4656	64.7646	63.1356	61.5913	60.1360	58.7652	57.4662	56.2194
1008	68.3270	66.6710	65.0471	63.4676	61.9403	60.4677	59.0472	57.6710	56.3271
1176	68.3934	66.7973	65.2210	63.6721	62.1552	60.6721	59.2211	57.7973	56.3935
1344	68.4344	66.8752	65.3282	63.7980	62.2876	60.7980	59.3282	57.8752	56.4344
1512	68.4596	66.9231	65.3942	63.8756	62.3692	60.8756	59.3942	57.9231	56.4596
1680	68.4751	66.9526	65.4348	63.9233	62.4194	60.9233	59.4348	57.9526	56.4751
1848	68.4847	66.9708	65.4598	63.9528	62.4504	60.9528	59.4598	57.9708	56.4847
2016	68.4905	66.9820	65.4753	63.9709	62.4694	60.9709	59.4753	57.9820	56.4905

Table 3. The values of the water pressure in an infinite layer of the thickness $l = 10$ m for different values of the time t and the coordinate y(meters) at $n = 10$, the time value t is indicated in hours in the table.

t	y								
	1	2	3	4	5	6	7	8	9
720	68.1024	66.2438	64.4597	62.7777	61.2156	59.7792	58.4621	57.2462	56.1038
1440	68.4502	66.9053	65.3697	63.8469	62.3390	60.8469	59.3697	57.9053	56.4502
2160	68.4938	66.9881	65.4837	63.9808	62.4798	60.9808	59.4837	57.9881	56.4938
2880	68.4992	66.9985	65.4980	63.9976	62.4975	60.9976	59.4980	57.9985	56.4992

5. Solving of differential equations systems in the presence of fractional derivatives using the orthogonal method

During the modeling many physical processes, in particular the mass transfer, it is important to consider the process history. Mathematical modeling of physical processes is usually reduced to the differential equations construction (or their systems) in partial derivatives and the formulation of the appropriate problems of mathematical physics. The models of such kind do not take into account the history of the process. Therefore, fractional (differential and integral) calculations are increasingly used for the study of such processes [7–12]. The analytical methods for solving of the problems that arise are usually constructed on the basis of Laplace operational transform [13]. The available tables of correspondence between the originals and the images or the use of contour integration don't always lead to the desired result. The use of the approximate methods of inversion can't guarantee the necessary accuracy of the original restoring. One of the effective approaches to avoid this problem is the use of the spectral methods in different bases in particular in the Laguerre polynomials basis to apply the problems of the fractional calculus.

The proposed paper has a purpose to research the method of solving of differential fractional derivatives using the spectral method in the Chebyshev-Laguerre polynomials basis.

5.1. Fractional derivatives definition

Several types of the fractional derivatives and integrals are introduced in the literature. The fractional derivatives in Caputo and Riemann-Liouville terms are the most used. The fractional derivative operator in Caputo terms is defined as follows [9–12]:

$$ {}^{c}D_{\tau}^{\alpha} = \frac{{}^{c}\partial^{\alpha}}{\partial\tau^{\alpha}}\varphi(\tau) := \frac{1}{\Gamma(m+1-\alpha)} \int_0^{\tau} \left(\frac{\partial^{m+1}}{\partial\xi^{m+1}}\varphi(\xi) \right) \Big/ (\tau-\xi)^{\alpha-m} d\xi, \tag{16} $$

where $m = [\alpha]$, $[\cdot]$ is the integer part of real number, and in Riemann-Liouville terms

$$D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}\varphi(t) := \frac{1}{\Gamma(m+1-\alpha)}\frac{\partial^{m+1}}{\partial \xi^{m+1}}\int_0^t \frac{\varphi(\xi)}{(t-\xi)^{\alpha-m}}d\xi. \tag{17}$$

Taking into account the given definitions we can conclude that the fractional derivatives are nothing but an integral convolution of the desired solution and another function, e.g. the power function. It is known [2, 14, 15] that in such cases it is expedient to apply the Chebyshev-Laguerre polynomials because they are orthogonal on the semi-axis and during the expansion of the desired solution and the kernel by the given polynomials the integral convolution passes exactly to the convolution of the accordant series. Herewith, the sampling procedure which contributes a significant error into the end result in such operations is excluded.

5.2. Formulation of the problem

To solve many applied problems related to the nonstationary gas motion process in horizontal pipelines the linearized system of the partial differential equations that looks like the formula (18) [15] is used

$$\begin{cases} \dfrac{\partial \omega(x,t)}{\partial t} + \dfrac{\partial p(x,t)}{\partial x} + a\omega(x,t) - bp(x,t) = 0, \\ \dfrac{\partial \omega(x,t)}{\partial x} + \dfrac{1}{c^2}\dfrac{\partial p(x,t)}{\partial t} = 0, \end{cases} \tag{18}$$

where p, ω are the pressure and the mass velocity of gas motion accordingly; t is the time; x is the movable coordinate, $x \in [0, L]$; L is the length of pipeline, $a = \upsilon_1 + \upsilon_2$, $b = -\frac{1}{4}\left(\upsilon_1^2 + \upsilon_2^2\right)$, and υ_1 and υ_2 are the limits of change of gas motion velocity, c is the sound speed in gas.

It is evident that to formulate the accordant problem of mathematical physics, it is necessary to set the initial and limiting (boundary) conditions for the gas pressure and or the volumetric mass consumption which are the desired functions. The boundary conditions for the desired functions are set depending on known input data.

In Riemann-Liouville fractional derivatives terms (17), the system (18) will be written in the form

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{\partial}{\partial t}\displaystyle\int_0^t \frac{\omega(x,\zeta)}{(t-\zeta)^\alpha}d\zeta + \dfrac{\partial p}{\partial x} + a\omega - bp = 0, \\ \dfrac{\partial \omega}{\partial x} + \dfrac{1}{c^2}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{\partial}{\partial t}\displaystyle\int_0^t \frac{p(x,\zeta)}{(t-\zeta)^\alpha}d\zeta = 0. \end{cases} \tag{19}$$

The problem is to find the pressure distribution and the volumetric flow rate over the given domain under the given initial and boundary conditions on the desired functions (the pressure and volumetric flow rate).

5.3. Solving of the problem

Let us present the unknown functions $p(x,t)$ and $\omega(x,t)$ in the form of the series by the orthogonal Chrbyshev-Laguerre polynomials $L_n^\lambda(t)$, where $\lambda > -1$ is the arbitrary parameter [2]

$$p(x,t) = t^\lambda \sum_{m=0}^{\infty} \frac{p_m(x)}{r_m} L_m^\lambda(t), \quad \omega(x,t) = t^\lambda \sum_{m=0}^{\infty} \frac{\omega_m(x)}{r_m} L_m^\lambda(t), \tag{20}$$

where the coefficients $p_m(x)$, $\omega_m(x)$ are determined by the integral relations

$$p_m(x) = \int_0^\infty e^{-t} p(x,t) L_m^\lambda(t)dt \quad \text{and} \quad \omega_m(x) = \int_0^\infty e^{-t}\omega(x,t)L_m^\lambda(t)dt. \tag{21}$$

The integral equations kernel is also presented in a similar way (19)

$$k(t)=t^{\lambda}\sum_{m=0}^{\infty}\frac{k_m}{r_m}L_m^{\lambda}(t). \tag{22}$$

In such presentation the series coefficients will have the form

$$k_m=\int_0^{\infty}e^{-t}L_m^{\lambda}(t)k(t)dt, \tag{23}$$

and the normalizing multiplier r_m is calculated by the formula

$$r_m=\int_0^{\infty}e^{-x}L_m^{\lambda}(x)L_m^{\lambda}(x)dx=\frac{(1+\lambda)_m(\lambda)_m}{m!m!}\,{}_3F_2(-m,1,1-\lambda;\lambda+1,1-\lambda-m;1).$$

In our case generalized Fourier-Laguerre spectra for the function $k(t)=t^{-\alpha}$ are defined as follows

$$k_m=\frac{\Gamma(m+\lambda+\alpha)\Gamma(1-\alpha)}{\Gamma(m+1)\Gamma(\lambda+\alpha)}. \tag{24}$$

For the large values m

$$k_m\approx\frac{\Gamma(1-\beta)}{\Gamma(\lambda+\beta)}m^{\lambda+\alpha-1}.$$

The latter formula gives the opportunity to evaluate the impact of the free parameter λ on the convergence velocity of the accordant series. However, the function representation by the series of type (20) and (22) is advantages even more so that the agreement of the choice of parameter λ with the behavior of the function $k(t)$ accelerates the rate of the series convergence. Let us represent the functions $k(t)$ and $p(x,t)$ in the form of Fourier series by the polynomials $L_n^{\lambda_k}(t)$, $\lambda_k>-1$, and $L_n^{\lambda_p}(t)$, $\lambda_p>-1$, accordingly. Since [2]

$$\int_0^t(t-\tau)^{\lambda_k}L_m^{\lambda_k}(t-\tau)\tau^{\lambda_f}L_n^{\lambda_f}(\tau)d\tau=\frac{(n+m)!}{n!\,m!}B(\lambda_k+m+1,n+\lambda_f+1)t^{\lambda_k+\lambda_f+1}L_{n+m}^{\lambda_k+\lambda_f+1}(t),$$

then the equation

$$\frac{\partial\omega}{\partial x}+\frac{1}{c^2}\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{p(x,\zeta)}{(t-\zeta)^{\alpha}}d\zeta=0$$

has the form

$$\frac{\partial\omega}{\partial x}+\frac{1}{c^2}\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\sum_{m=0}^{\infty}\frac{m!\,k_m}{\Gamma(m+\lambda_k+1)}\sum_{n=0}^{\infty}\frac{n!\,p_n(x)}{\Gamma(n+\lambda_p+1)}$$
$$\times\frac{(n+m)!}{n!\,m!}B\left(\lambda_k+m+1,n+\lambda_p+1\right)t^{\lambda_k+\lambda_p+1}L_{n+m}^{\lambda_k+\lambda_p+1}(t)=0,$$

or

$$\frac{\partial\omega}{\partial x}+\frac{1}{c^2}\frac{1}{\Gamma(1-\alpha)}\sum_{m=0}^{\infty}\frac{m!\,k_m}{\Gamma(m+\lambda_k+1)}\sum_{n=0}^{\infty}\frac{n!\,p_n(x)}{\Gamma(n+\lambda_f+1)}$$
$$\times\frac{(n+m)!}{n!\,m!}B\left(\lambda_k+m+1,n+\lambda_p+1\right)\left(n+m+\lambda_k+\lambda_p+1\right)_1t^{\lambda_k+\lambda_p}L_{n+m}^{\lambda_k+\lambda_p}(t)=0.$$

If we regroup the summands in the double sum in the right side of the latter formula, we will obtain the equation

$$\frac{\partial \omega}{\partial x}+\frac{1}{c^2}\frac{1}{\Gamma(1-\alpha)}t^{\lambda_k+\lambda_p}\sum_{n=0}^{\infty}d_n(x)L_{n+m}^{\lambda_k+\lambda_p}(t)=0.$$

In the latter formula

$$d_n(x)=\sum_{m=0}^{n}k_m p_{n-m}(x)=\sum_{m=0}^{n}k_{n-m}p_m(x).$$

If we represent the mass consumption $\omega(x,t)$ as the series

$$\omega(x,t)=t^{\lambda_k+\lambda_p}\sum_{n=0}^{\infty}\frac{n!\,\omega_n(x)}{\Gamma(n+\lambda_p+1)}L_n^{\lambda_k+\lambda_p}(t),$$

we will obtain the following recurrent system of ordinary differential equations of the unknown coefficients $\omega_n(x)$ and $p_n(x)$

$$\frac{n!}{\Gamma(n+\lambda_p+1)}\frac{d\omega_n(x)}{dx}-\frac{1}{c^2}\frac{1}{\Gamma(1-\alpha)}d_n(x)=0. \tag{25}$$

A similar system is obtained from the first equation of the system (18)

$$\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{\omega(x,\zeta)}{(t-\zeta)^{\alpha}}d\zeta+\frac{\partial p}{\partial x}+a\omega-bp=0.$$

Applying similar expansions in the latter equation, we obtain

$$\frac{n!}{\Gamma(n+\lambda_p+1)}\frac{dp_n(x)}{dx}+\frac{1}{\Gamma(1-\alpha)}c_n(x)+a\omega_n(x)-bp_n(x)=0, \tag{26}$$

$$c_n(x)=\sum_{m=0}^{n}k_m\omega_{n-m}(x)=\sum_{m=0}^{n}k_{n-m}\omega_m(x).$$

The systems (23) and (24) are recurrent relatively the unknown generalized spectra. Solving them we find $\omega_n(x)$ and $p_n(x)$ for the arbitrary values n by the spectral method described above.

If we substitute the expansions (20) and (22) in the system (19) we will obtain the recurrent system of ordinary differential equations for determining of the unknown coefficients $\omega_n(x)$ and $p_n(x)$.

$$\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}c_n(x)+p_n'(x)+a\omega_n(x)-bp_n(x)=0,\\ \omega_n'(x)+\dfrac{1}{c^2}\dfrac{1}{\Gamma(1-\alpha)}d_n(x)=0.\end{cases} \tag{27}$$

The solution of the latter system we can find in the bases of the orthogonal or biorthogonal polynomials [2, 5, 6, 14].

6. Discussion and conclusions

The numerical values of the pressure presented in the tables 1–3 confirm authenticity of the obtained theoretical results and the efficiency of applying of the constructed biorthogonal polynomials for solving problems of the mathematical physics. The proposed algorithm is easy to automate and can be effectively used to solve other practical problems.

The proposed approach gives us an opportunity to construct the effective algorithm for solving of the differential equations or the systems of the differential equations in the presence of the fractional time derivative. More, if the input data is set in a discrete form the similar to the paper [2] the algorithm can be submitted in matrix form.

From the results obtained it follows that if $\lambda = -\alpha$ then the coefficients of function $k(t) = t^{-\alpha}$ are equal to zero for orders greater than one. So, for such a choice of the parameter λ we will have the following formula

$$d_n(x) = k_0 p_n(x), \quad c_n(x) = k_0 \omega_n(x).$$

However, as can be seen from the formula (24) such an approach to choosing of the parameter λ allows one to accelerate the convergence of accordant Fourier-Laguerre series.

It is necessary to note that the summing of Fourier-Laguerre series is sensitive to the parameter λ. Therefore, there is a need of additional researches in summing operations of these series because Chebyshev-Laguerre polynomials have significant disadvantage that for the large n their behavior is following

$$L_n^{\lambda}(t) = \mathrm{O}\left(e^{t/2} t^{-(2\lambda+1)/4} n^{(2\lambda-1)/4}\right).$$

This property of the polynomials considerably narrows the problems class when the Chebyshev-Laguerre polynomials are used because there are the computational difficulties during the series summing for the large values t. In practice this problem is solved by the introduction of the scaling multiplier. However, the change of the scaling multiplier requires the redefining of the problem and leads to instability in the desired function calculation. Therefore, Chebyshev-Laguerre transform is generalized as follows.

Introduce the integral transform

$$f_n = \int_0^{\infty} t^{\nu\lambda+\nu-1} e^{-\mu t^{\nu}} L_n^{\lambda}(\mu t^{\nu}) f(t) dt,$$

where $n = 0, 1, 2, \ldots$, $\mu > 0$, $|\nu| < \infty$, $\nu \neq 0$. Then the reverse formula has the form

$$f(t) = \sum_{n=0}^{\infty} \frac{n! f_n}{\Gamma(n+\lambda+1)} L_n^{\lambda}(\mu t^{\nu}).$$

Choosing the free parameters μ and ν allows us to construct the regularizing algorithm for calculating Fourier-Laguerre coefficients f_n and summing the corresponding orthogonal series.

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Дослідження масоперенесення в складних пористих середовищах та трубопроводах за допомогою спектральних методів

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На основі побудованих авторами біортогональних поліномів запропоновано метод розв'язування задач математичної фізики, зокрема для знаходження розподілу тиску у воді в пластах підземних сховищ газу. Досліджено спосіб розв'язування задачі методом розділення змінних у базисі біортогональних поліномів. Розв'язок задачі знайдено у вигляді суми ряду біортогональних та квазіспектральних поліномів. Проведено порівняльний аналіз для різних значень параметрів. Вивчено вплив параметрів методів, зокрема порядку часткової суми, розрядної сітки та похибки обчислення на точність отриманого розв'язку. Результати обчислень подано у вигляді таблиць. Побудовано алгоритм дослідження процесу руху газу в трубопроводах з використанням похідних дробових порядків.

Ключові слова: *спектральні методи, математична модель, розподіл тиску, ортогональні, біортогональні та квазіортогональні поліноми, диференціальні рівняння в частинних похідних та похідних дробових порядків.*

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