GEODESY

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MODELING OF THE REGIONAL GRAVITATIONAL FIELD USING FIRST AND SECOND DERIVATIVE OF SPHERICAL FUNCTIONS

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Aim. The main task of the geodesy is to determine the shape and size of the Earth and its changes over time. An integral part of determining the shape and size of the Earth is the construction of a geoid surface model, both global and regional. Today larger and more precise arrays of input data (geopotential functionals) appear to construct such a model, methods for their processing are rapidly becoming generated. One of such methods is an extention in a series of spherical functions on a spherical trapezium. Since the functionals of the gravitational field are mostly obtained using the differentiation operator, for the use of this method it is necessary to investigate the first and second derivatives of spherical functions on a spherical trapezium, which is the purpose of this work. **Method**. Spherical functions on a spherical trapezium can not be presented in explicit form, nor do they have simple recurrence relations. To calculate them, the extention in a hypergeometric series should be used. Therefore, to calculate the first and second derivative from the above-mentioned functions, the derivative from the hypergeometric series should be used. **Results**. As a result of certain mathematical transformations, we obtain an expression for finding the first and second derivatives of spherical functions on a spherical trapezium, as well as a comparatively obtained result with spherical Legendre functions of the first kind. **The scientific novelty and practical significance.** The expression for the first and second derivative of spherical functions on a spherical trapezium was first determined. The result obtained with the spherical Legendre functions of the first kind is comparative. This enables the use of such functions as a basic system of functions on a spherical trapezium for modelling of regional gravitational or magnetic fields.

Key words: spherical functions, spherical trapezium, first and second derivative.

Introduction

There are many methods for modeling the Earth's gravitational field [Dzhuman, 2013; Sneeuw, 1994]. For modeling the global field we usually use the Legendre spherical functions of an integer degree / order [Pavlis et al., 2012; Hobson, 1931], in turn for modeling the regional field we use the Legendre spherical functions of an integer degree, using the real order. We can consider a spherical cap harmonic analysis (SCHA) as a classical method of modeling a regional gravitational (or magnetic) field [Haines, 1985; Haines, 1988; De Santis & Torta, 1997; Yankiv-Vitkovska & Dzhuman, 2018], the theoretical foundations of which were proposed by Thompson and Tait [Kelvin & Tait, 1896], and practical realization was in the making for more than a century!, and was first implemented by Haines for modeling the magnetic field in 1985 [Haines, 1985]. Based on this method a number of methods were developed that have been applied to solving the problem of computing a model of regional potential field. To such methods above all belong ASHA [De Santis, 1992; Dzhuman, 2014], TOSCA (named after the Puccini opera) [De Santis, 1991], RSHA [Thebault et. al., 2006] and other. Despite the large selection of these methods the author believes that their use on a part of a sphere other than a segment with a center on the north or south poles looks rather artificial. This is primarily due to the need for the transformation of coordinates, which causes loss of the physical content of the process. In turn the use of spherical functions on a spherical trapezium [Dzhuman, 2018] does not require any additional transformations. Such functions are orthogonal on an arbitrary spherical trapezium, so the solution is stable.

Aim

The purpose of this work is to find the expression of the first and second derivatives for spherical functions on a spherical trapezium. This will allow the use of these functions to model regional gravity or magnetic fields.

Method

The Legendre functions on a spherical meridian $[q_{\min}, q_{\max}]$ have the form [Dzhuman, 2018]

$$\left.\begin{array}{l} P_{km}(q)=\sin^m(q_0-|q-q_{mean}|)\cdot F\left(m-n_k,n_k+m+1,1+m,\dfrac{1-\cos(q_0-|q-q_{mean}|)}{2}\right),\ if\ q_{\min}\le q\le q_{mean} \\ P_{km}(q)=(-1)^{k+m}\sin^m(q_0-|q-q_{mean}|)\cdot F\left(m-n_k,n_k+m+1,1+m,\dfrac{1-\cos(q_0-|q-q_{mean}|)}{2}\right),\ if\ q_{mean}\le q\le q_{\max} \end{array}\right\} \quad (1)$$

where k and m are whole numbers, q_{mean} is average value, namely $q_{mean}=(q_{\min}+q_{\max})/2$. In turn the value n_k will depend on k, m and $q_0=(q_{\max}-q_{\min})/2$. They can be found using the equation [Macdonald, 1900; Hwang & Chen, 1997; Dzhuman, 2018]

$$\tilde{F}(n_k,m,\cos q_0)=0, \quad (2)$$

if $k-m$ is an odd number, or using the equation

$$n_k\cos q_{mean}\tilde{F}(n_k,m,\cos q_0)-$$
$$-(n_k-m)\tilde{F}(n_k-1,m,\cos q_0)=0, \quad (3)$$

if $k-m$ is an even number, where $\tilde{F}$ is the symbol of the hypergeometric series, namely [Hwang & Chen, 1997]

$$\tilde{F}(n,m,m)=F(m-n,m+n+1,m+1,\frac{1-m}{2}). \quad (4)$$

For example, the studied Legendre polynomials are shown in Fig. 1 with an interval of 5 ranging from $[20°;70°]$.

In turn we recall that the classical Legendre polynomials which are shown in Fig. 2 are at least visually very similar to the studied Legendre polynomials.

To find the derivative of functions (1) we must take derivative of the hypergeometric series, which in general is the following [Smirnov, 1954]:

$$\frac{d}{dz}F(a,b;c;z)=\frac{ab}{c}F(a+1,b+1;c+1;z). \quad (5)$$

Results

After minor mathematical transformations we get

$$\left.\begin{array}{l} \dfrac{dP_{km}(q)}{dq}=m\cdot ctg(q-q_{\min})P_{km}(q)+\dfrac{1}{2}\dfrac{(m-n_k)(n_k+m+1)}{1+m}\sin^{m+1}(q-q_{\min})\cdot \\ \qquad\cdot F\left(m-n_k+1,n_k+m+2,2+m,\dfrac{1-\cos(q-q_{\min})}{2}\right),\ if\ q_{\min}\le q\le q_{mean} \\ \dfrac{dP_{km}(q)}{dq}=-m\cdot ctg(q_{\max}-q)P_{km}(q)+(-1)^{k+m+1}\cdot\dfrac{1}{2}\dfrac{(m-n_k)(n_k+m+1)}{1+m}\sin^{m+1}(q_{\max}-q)\cdot \\ \qquad\cdot F\left(m-n_k+1,n_k+m+2,2+m,\dfrac{1-\cos(q_{\max}-q)}{2}\right),\ if\ q_{mean}\le q\le q_{\max} \end{array}\right\} \quad (6)$$

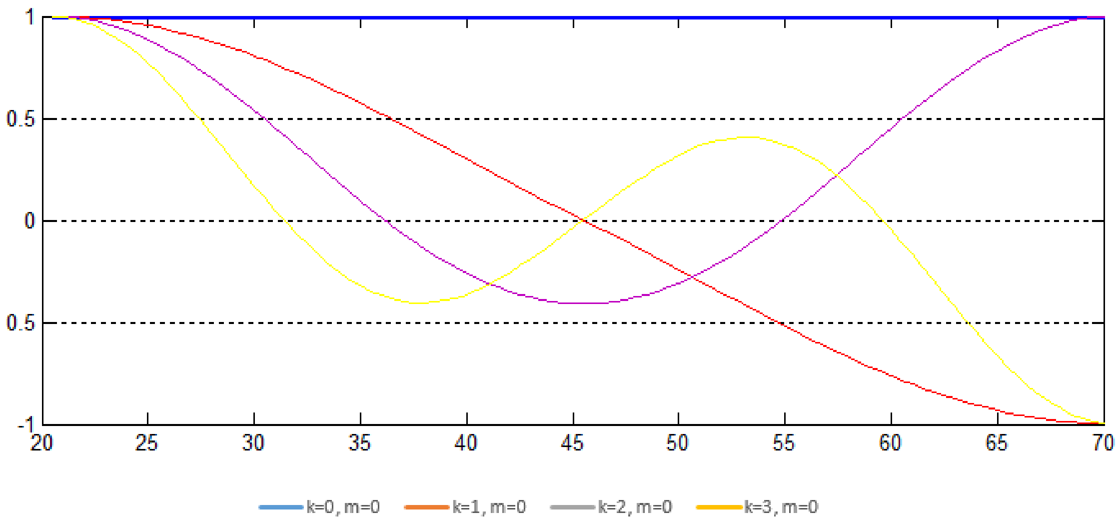


Fig. 1. Studied Legendre polynomials on the range $[20°;70°]$ *[Dzhuman, 2018]*

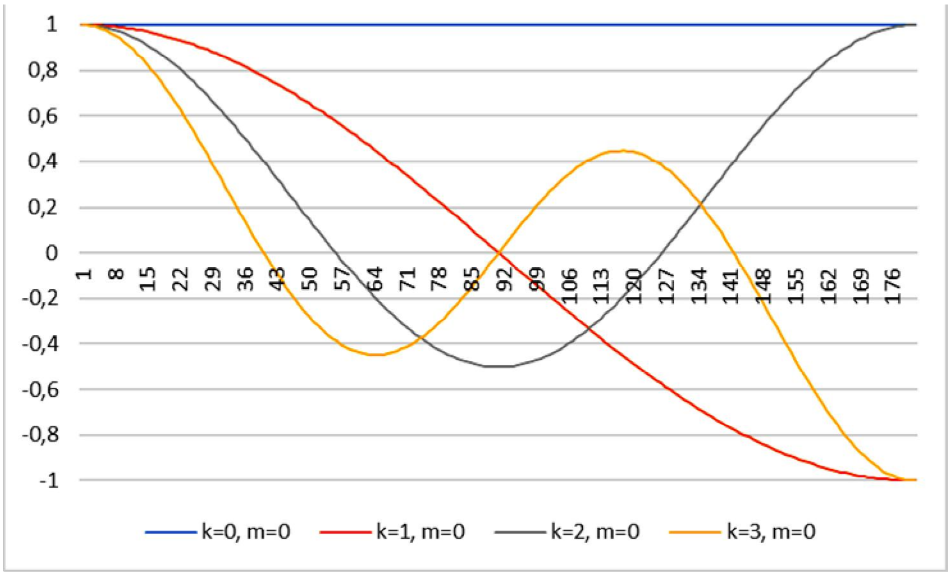
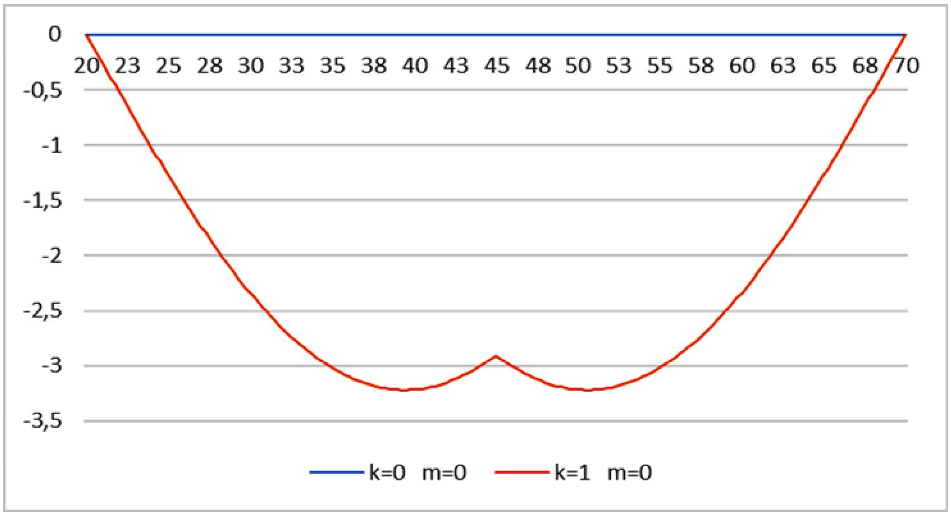


Fig. 2. The first four Legendre polynomials



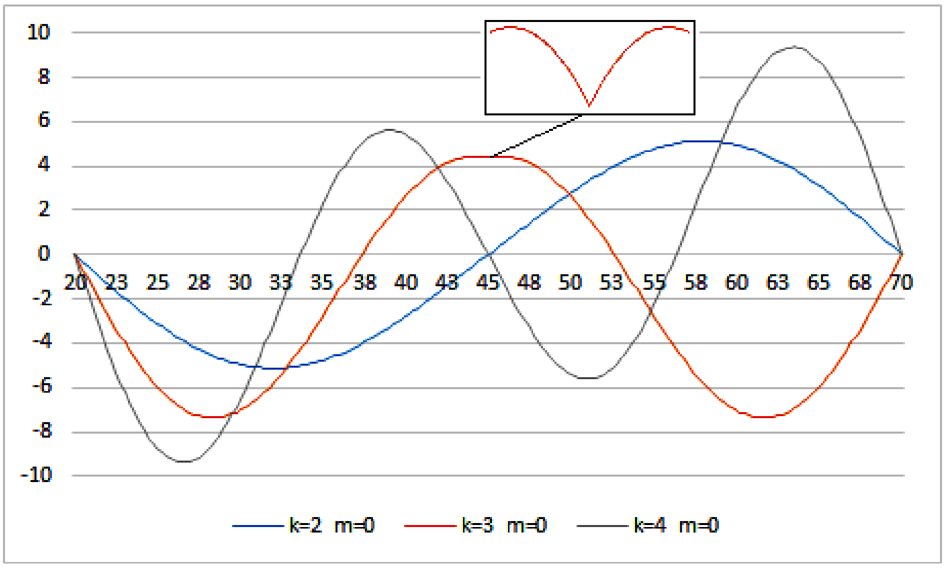


Fig. 3. Graph of functions $\frac{dP_{km}(q)}{dq}$ *for* $k = 0,4$ *and* $m = 0$

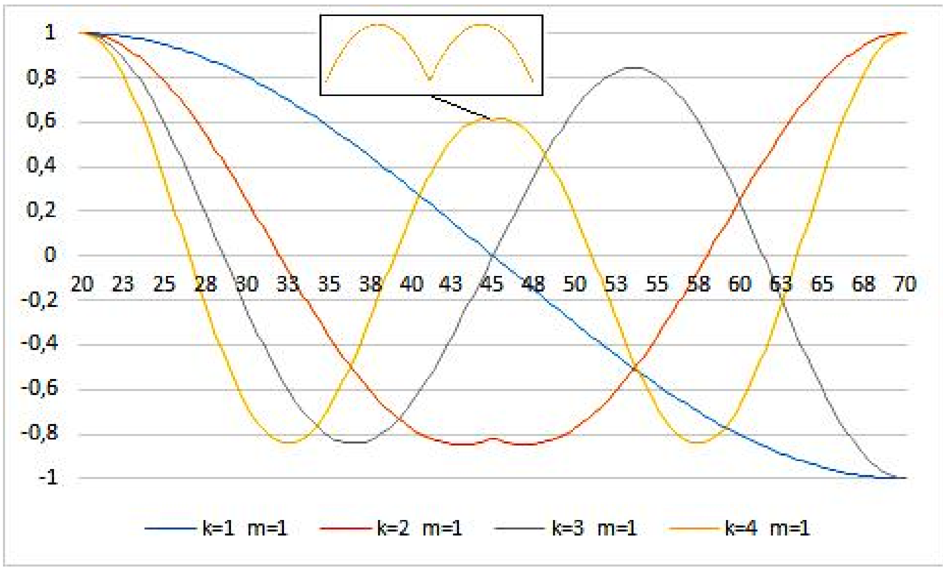


Fig. 4. Graph of functions $\frac{dP_{km}(q)}{dq}$ *for* $k=1,4$ *and* $m=1$

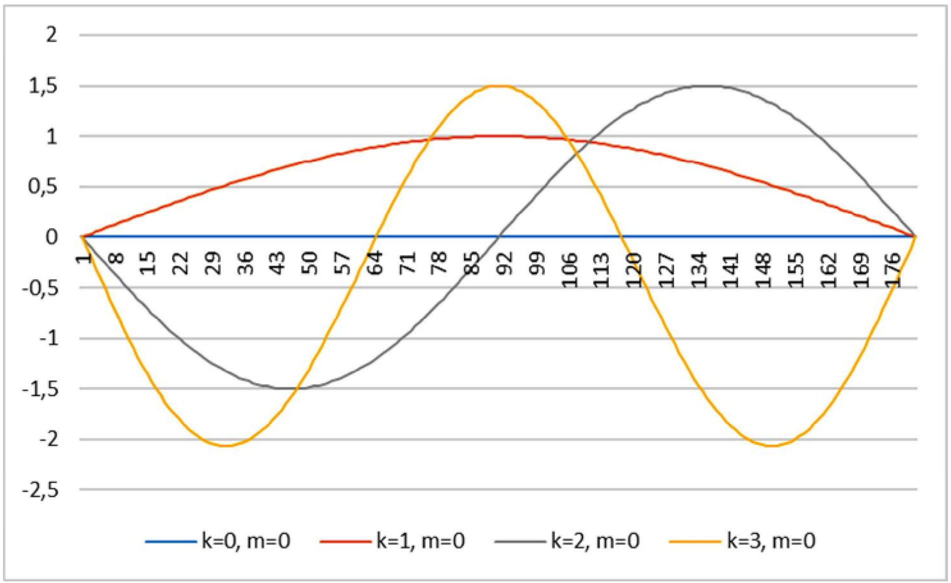


Fig. 5. Derivatives from Legendre polynomials

We can see from (6) that when $k+m=\text{even number}$ the function $\frac{dP_{km}(q)}{dq}$ will be odd, but for $k+m=\text{odd number}$ the function $\frac{dP_{km}(q)}{dq}$ will be even.

Graph of functions $\frac{dP_{km}(q)}{dq}$ for $k=0,4$ and $m=0,1$ is shown in figures 3 and 4.

From (6) and figures 3, 4 we can see, that the first derivatives $\frac{dP_{km}(q)}{dq}$ have no breaks.

Let us compare the obtained result with the derivatives of the first four Legendre polynomials (Fig. 5).

From figures 3–5 we can conclude that derivatives of the studied Legendre polynomials and ordinary Legendre polynomials are similar only when $k=\text{even number}$, or in general $k+m=\text{even number}$. Moreover the second derivatives of spherical functions on a spherical trapezium when $k+m=\text{odd number}$ will have a break of first kind.

Let us find the second derivatives of functions (1). They will look like

$$
\begin{cases}
\frac{d^2P_{km}(q)}{dq^2} = -\frac{m}{\sin^2(q-q_{\min})}P_{km} + m\cdot ctg(q-q_{\min})P'_{km}(q) + \frac{1}{2}(m-n_k)(n_k+m+1)\sin^m(q-q_{\min})\cdot \\
\cdot\cos(q-q_{\min})F\left(m-n_k+1, n_k+m+2, 2+m, \frac{1-\cos(q-q_{\min})}{2}\right) + \frac{1}{4}\sin^{m+2}(q-q_{\min})\cdot \\
\cdot\frac{(m-n_k)(m-n_k+1)(n_k+m+1)(n_k+m+2)}{(1+m)(2+m)}F\left(m-n_k+2, n_k+m+3, 3+m, \frac{1-\cos(q-q_{\min})}{2}\right), \\
if\ q_{\min} \le q \le q_{mean}; \\
\frac{d^2P_{km}(q)}{dq^2} = -\frac{m}{\sin^2(q_{\max}-q)}P_{km} - m\cdot ctg(q_{\max}-q)P'_{km}(q) + (-1)^{k+m}\frac{1}{2}(m-n_k)(n_k+m+1)\sin^m(q_{\max}-q)\cdot \\
\cdot\cos(q_{\max}-q)F\left(m-n_k+1, n_k+m+2, 2+m, \frac{1-\cos(q_{\max}-q)}{2}\right) + (-1)^{k+m}\frac{1}{4}\sin^{m+2}(q_{\max}-q)\cdot \\
\cdot\frac{(m-n_k)(m-n_k+1)(n_k+m+1)(n_k+m+2)}{(1+m)(2+m)}F\left(m-n_k+2, n_k+m+3, 3+m, \frac{1-\cos(q_{\max}-q)}{2}\right), \\
if\ q_{mean} \le q \le q_{\max}
\end{cases}
\quad (7)
$$

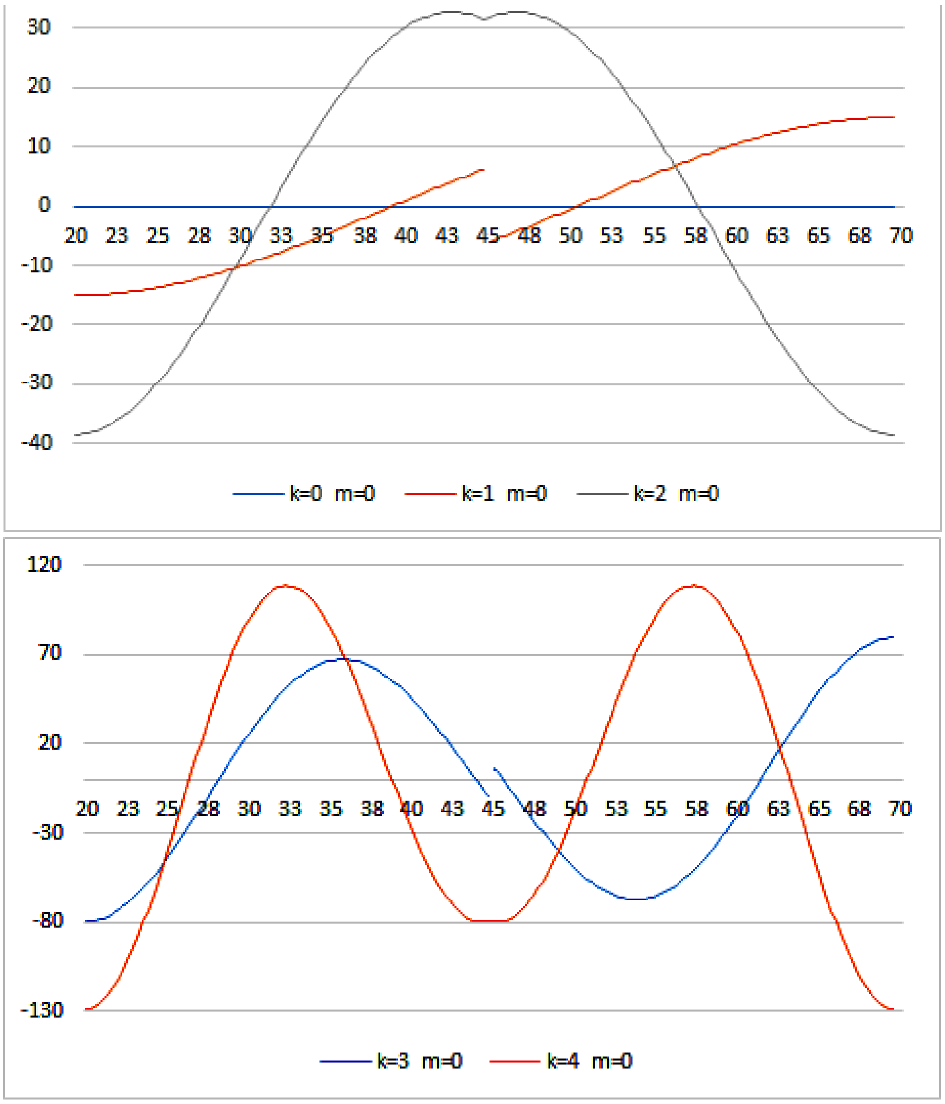


Fig. 6. Graph of functions $\frac{d^2P_{km}(q)}{dq^2}$ *for* $k=0,4$ *and* $m=0$

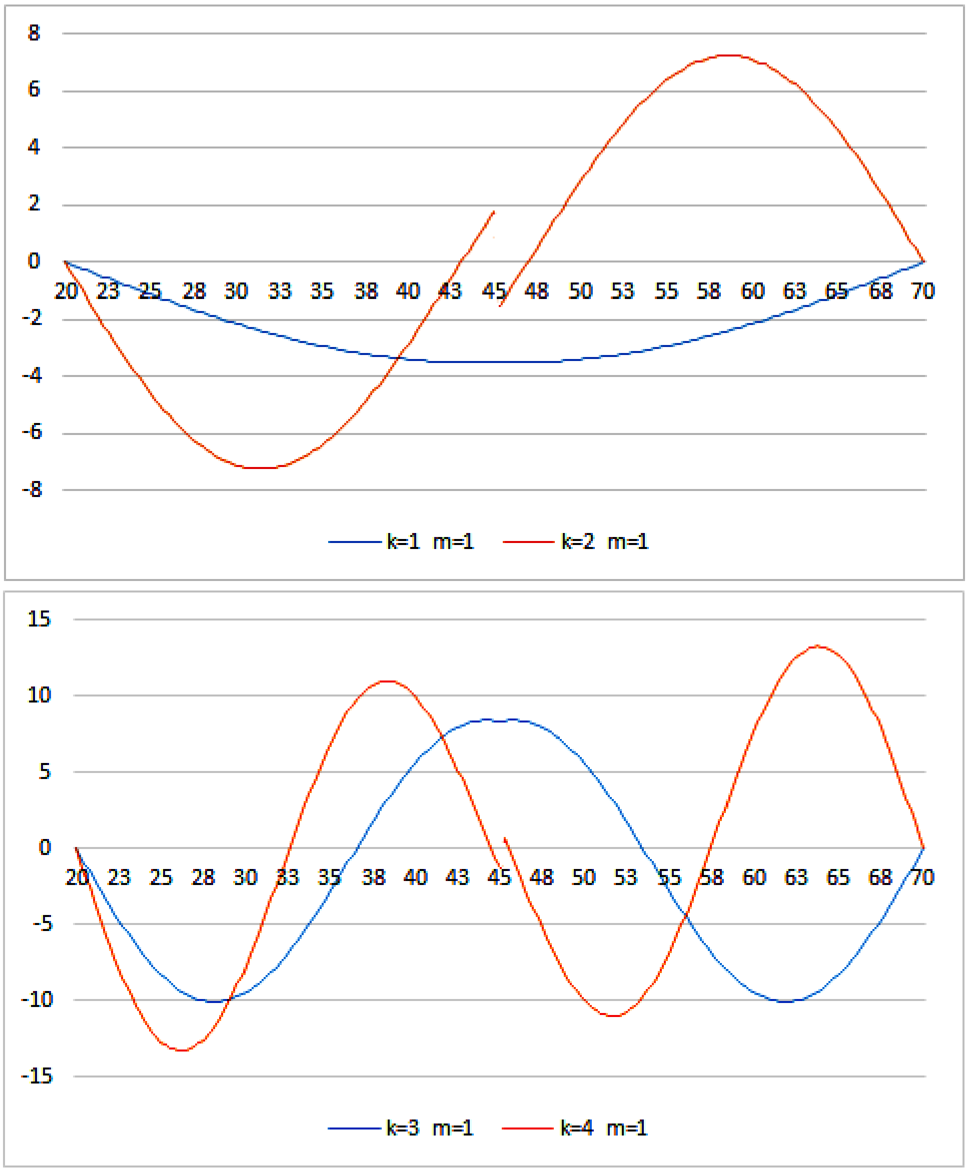


Fig. 7. Graph of functions $\frac{d^2P_{km}(q)}{dq^2}$ *for* $k=1,4$ *and* $m=1$

From (7) it is clear that with $k+m=\text{even number}$ functions $\frac{d^2P_{km}(q)}{dq^2}$ will be even, but for $k+m=\text{odd number}$ functions $\frac{d^2P_{km}(q)}{dq^2}$ will be odd. Also for every $k+m=\text{odd number}$ this function has a break of the first kind at the point q_{mean}. The size of the break is equal

$$\frac{d^2P_{km}(q_{mean}-0)}{dq^2}-\frac{d^2P_{km}(q_{mean}+0)}{dq^2}=$$
$$=2\cdot m\cdot ctg(q_{mean}-q_{\min})P'_{km}(q_{mean})+$$
$$+(m-n_k)(n_k+m+1)\sin^m(q_{mean}-q_{\min})\cdot$$
$$\cdot\cos(q_{mean}-q_{\min})F\left(m-n_k+1,n_k+m+2,2+m,\frac{1-\cos(q_{mean}-q_{\min})}{2}\right)+$$
$$+\frac{1}{2}\sin^{m+2}(q_{mean}-q_{\min})\cdot \tag{8}$$
$$\cdot\frac{(m-n_k)(m-n_k+1)(n_k+m+1)(n_k+m+2)}{(1+m)(2+m)}\cdot$$
$$\cdot F\left(m-n_k+2,n_k+m+3,3+m,\frac{1-\cos(q_{mean}-q_{\min})}{2}\right)$$

Graph of functions $\frac{d^2P_{km}(q)}{dq^2}$ for $k=0,4$ and $m=0,1$ is shown in figures 6–7.

It should also be noted that when $k+m=$ even number functions $\frac{d^3 P_{km}(q)}{dq^3}$ have a break of the first kind.

Scientific novelty and practical significance

In this paper we consider spherical functions on a spherical trapezium. Their use for modeling the regional gravitational field of the Earth requires knowledge of the expressions of the first and the second derivatives of these functions, since measurements in this case are functionals of the gravitational field. In this paper we find expressions for the first and second derivatives of these functions, and also compare them to the expressions of the first derivatives of ordinary spherical functions.

Conclusions

In this paper we can make the following conclusions:

– spherical functions on a spherical trapezium (1) represent a powerful mathematical apparatus for modeling the regional gravitational field of the Earth;

– for use of the above functions it is necessary to have the expression for the first and second derivatives, since the geopotential functionals are the first and the second derivatives of the geopotential function [Marchenko & Dzhuman, 2015]. In this paper an expression of the first and second derivatives of spherical functions on a spherical trapezium (6) is found as a function of a hypergeometric series;

– we compared the first and second derivatives of spherical functions on a spherical trapezium and the first and second derivatives of ordinary spherical functions.

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МОДЕЛЮВАННЯ РЕГІОНАЛЬНОГО ГРАВІТАЦІЙНОГО ПОЛЯ З ВИКОРИСТАННЯМ ПЕРШОЇ ТА ДРУГОЇ ПОХІДНИХ СФЕРИЧНИХ ФУНКЦІЙ

Мета. Основною задачею геодезії є визначення форми та розмірів Землі та їхні зміни з часом. Невід'ємною частиною визначення форми та розмірів Землі є побудова і глобальної, і регіональної моделі поверхні геоїда. Оскільки сьогодні з'являються все більші і точніші масиви вхідних даних (трансформант геопотенціалу) для побудови такої моделі, стрімкого розвитку набувають і методи для їх опрацювання. Одним із таких методів є розклад в ряд за сферичними функціями на сферичній трапеції. Оскільки трансформанти гравітаційного поля в більшості своїй отримуються з використанням оператора диференціювання, для використання цього методу необхідно дослідити перші та другі похідні від сферичних функцій на сферичній трапеції, що і є метою цієї роботи. **Методика.** Сферичні функції на сферичній трапеції неможливо представити у явній формі, а також вони не мають простих рекурентних співвідношень. Для їх обчислення треба використати розклад у гіпергеометричний ряд. Тому для обчислення першої та другої похідної від вищезгаданих функцій треба використати вираз похідної від гіпергеометричного ряду. Результати. В результаті певних математичних перетворень отримано вираз для знаходження перших та других похідних для сферичних функцій на сферичній трапеції, а також порівняно отриманий результат із сферичними функціями Лежандра першого роду. **Наукова новизна і практична значущість.** Вперше знайдено вираз для першої та другої похідної сферичних функцій на сферичній трапеції. Порівняно отриманий результат із сферичними функціями Лежандра першого роду. Це дасть змогу використовувати такі функції як базову систему функцій на сферичній трапеції для задач моделювання регіонального гравітаційного чи магнітного полів.

Ключові слова: сферичні функції, сферична трапеція, перша та друга похідна.

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