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Interpolation integral continued fraction with twofold node

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For a functional given on a continual set of nodes on the basis of the previously constructed interpolation integral continued fraction of the Newton type, an interpolant with a k-th twofold node has been constructed and investigated. It is proved that the constructed integral continued fraction is an interpolant of the Hermitian type.

Keywords: *Newton type interpolation continued fraction, twofold node of continued fraction, Hermitian type interpolant.*

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1. Introduction

For the first time, an interpolation integral continued fraction (interpolation ICF) was introduced in [1]. Moreover, in this article the fact has been proved that the introduced there definition of a kernel is a necessary condition that the integral continued fractions to be interpolative for the functionals $F\colon L_1(0,1) \to \mathbb{R}$ on the continual set of nodes

$$x^n(\cdot,\xi^n) = x_0(\cdot) + \sum_{i=1}^{n} H(\cdot - \xi_i)\big(x_i(\cdot) - x_{i-1}(\cdot)\big), \tag{1}$$

where $\Omega_{z^n} = \{z^n\colon 0 \leqslant z_1 \leqslant \ldots \leqslant z_n \leqslant 1\}$, $\xi^n = (\xi_1, \xi_2, \ldots, \xi_n) \in \Omega_{z^n}$. Here $Q[0,1]$ is the space of piecewise continuous functions on $[0,1]$ with the finite number of break points of the first type, $x_i(z) \in Q[0,1]$, $i = 0,1,\ldots$, are arbitrary fixed elements from the space $Q[0,1]$, and $H(t)$ is the Heaviside function.

Sufficient conditions of integral fraction interpolativity (see [1]) were discovered in the work [2]. These conditions state that the substitution rule takes place. In the article [3], the sufficient conditions for functional $F(x(\cdot))$ were given to fulfill the substitution rule.

For the shortened notation of a finite ICF, we will use following:

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n}}}} = \mathop{\mathrm{D}}_{i=1}^{n} \frac{a_i}{b_i} = \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots + \frac{a_n|}{|b_n}.$$

The ICF investigated in [1, 2] has the following problem. In (1) we put $x_i(z) \equiv x_i = \text{const}$, $i = 0,\ldots,n$, $x(z) \equiv x = \text{const}$. Then the interpolation ICF will not transform in a interpolation continued fraction for a function of a single variable. To solve this problem, in [4] the new class of the interpolation ICF of the following type has been introduced

$$Q_n(x(\cdot)) = K_0^I + \mathop{\mathrm{D}}_{i=1}^{n} \frac{q_i(x(\cdot))}{1}, \tag{2}$$

where $q_m(x(\cdot)) = \int_0^1 \int_{z_1}^1 \ldots \int_{z_{m-1}}^1 K_m^I(z^m) \prod\limits_{l=1}^{m} \left(x(z_l) - x_{l-1}(z_l)\right) dz_l$.

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In [4] it is proved that the necessary condition for interpolativity of ICF (2) for a smooth functional $F(x(\cdot)) : Q[0,1] \to \mathbb{P}^1$ on continual nodes set (1) is following: its kernel must be defined by formulas

$$\begin{aligned}
&K_p^I(\xi^p) = (-1)^p \prod_{i=1}^{p} (x_i(\xi_i) - x_{i-1}(\xi_i))^{-1} \frac{\partial^p}{\partial \xi_1 \dots \partial \xi_p} \mathop{\mathrm{D}}_{i=1}^{p} \frac{q_{p-i}(x^p(\cdot,\xi^p))}{-1},\\
&q_m(x^p(\cdot,\xi^p)) = \int_0^1 \int_{z_1}^1 \dots \int_{z_{m-1}}^1 K_m^I(z^m) \prod_{l=1}^{m} \big(x(z_l,\xi^p) - x_{l-1}(z_l)\big) dz_l,\\
&q_0\big(x^p(\cdot,\xi^p)\big) = F(x_0(\cdot)) - F\big(x^p(\cdot,\xi^p)\big) + 1, \quad K_0^I = F(x_0(\cdot)),\\
&K_1^I(\xi^1) = -(x_1(\xi_1) - x_0(\xi_1))^{-1} \frac{\partial}{\partial \xi_1} F(x_0(\cdot) + H(\cdot - \xi_1)(x_1(\cdot) - x_0(\cdot))),\\
&m = 1,2,\dots, \quad p = 2,3,\dots,n,
\end{aligned} \tag{3}$$

and a sufficient condition is holding of the following substitution rule

$$\begin{aligned}
&\frac{\partial^p}{\partial z_1 \partial z_2 \dots \partial z_p} \Big(F\big(x^{p+1}(\cdot, z^{p+1})\big)\big|_{z_{p+1}=z_p}\Big)\\
&\quad = \frac{\partial^p}{\partial z_1 \partial z_2 \dots \partial z_p} \big(F\big(x^{p+1}(\cdot, z^{p+1})\big)\big)\Big|_{z_{p+1}=z_p} \frac{x_{p+1}(z_p) - x_{p-1}(z_p)}{x_p(z_p) - x_{p-1}(z_p)}, \quad p = 1,\dots,n.
\end{aligned} \tag{4}$$

According to [5], for the ICF (2) the following theorem takes place.

Theorem 1. *If for a smooth functional $F(x(\cdot))$ substitution rule (4) holds then the representation*

$$F(x(\cdot)) = F(x_0(\cdot)) + \mathop{\mathrm{D}}_{i=1}^{n} \frac{q_i(x(\cdot))}{1} \tag{5}$$

takes place, where kernels $q_p(x(\cdot))$, $p = 0,\dots,n$, defined by forlulas (3) and the kernel $q_{n+1}(x(\cdot)) = q_{n+1}^R(x(\cdot))$ satisfy (3) for $p = n+1$, $x_{n+1}(z) \equiv x(z)$.

The kernel $q_{n+1}^R(x(\cdot))$ can be considered as residual.

The aim of this work is to solve the following problem.

Problem formulation. Let us consider the functional $F\colon L_1(0,1) \to \mathbb{R}^1$ on the continual nodes set (1) for $n = m+1$ and suppose that for this functional the substitution rule (4) holds. Then our task is to construct the interpolant $Q_{m+1}^E(x(\cdot))$ with k-th twofold nodes using interpolative ICF of Newton type (2), (3) and prove that the constructed ICF is an interpolant of Hermitian type, i.e.

$$Q_{m+1}^E\Bigg(x_0(\cdot) + \sum_{\substack{l=1\\ l\neq k+1}}^{m+1} (x_l(\cdot) - x_{l-1}(\cdot))H(\cdot - \xi_l)\Bigg) = F\Bigg(x_0(\cdot) + \sum_{\substack{l=1\\ l\neq k+1}}^{m+1} (x_l(\cdot) - x_{l-1}(\cdot))H(\cdot - \xi_l)\Bigg), \tag{6}$$

$$\begin{aligned}
&{Q_{m+1}^E}'\Bigg(x_0(\cdot) + \sum_{l=1}^{m+1} (x_l(\cdot) - x_{l-1}(\cdot))H(\cdot - \xi_l)\Bigg) v_k(\cdot) H(\cdot - \xi_{k+1})\\
&\qquad = F'\Bigg(x_0(\cdot) + \sum_{l=1}^{m+1} (x_l(\cdot) - x_{l-1}(\cdot))H(\cdot - \xi_l)\Bigg) v_k(\cdot) H(\cdot - \xi_{k+1}).
\end{aligned} \tag{7}$$

2. Solution of the problem

Let us consider the functional $F\colon Q[0,1] \to \mathbb{P}^1$. At first, we consider the case of two nodes, i.e. $m=1$. Let us consider possible cases according to [6], where the functional has the form $F(x(\cdot)) = \big(\int_0^1 x(t)dt\big)^3$, $x(t) \in Q[0,1]$.

Case 1. We need to construct an interpolation ICF with the single node $x_0(\cdot)$ and the double node $x^1(\cdot,\xi_1)$ that satisfies the continual interpolation conditions

$$
\begin{aligned}
&Q_2^E\big(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)\big) = F\big(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)\big),\\
&Q_2^{E'}\big(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)\big)v_1(\cdot)H(\cdot-\xi_2)\\
&\qquad = F'\big(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)\big)v_1(\cdot)H(\cdot-\xi_2).
\end{aligned}
$$

Let us consider the interpolants (2), (3) of degree $n=2$. We consider a sequence of functions $x_0(t)$, $x_1(t)$, $x_2(t)$ of the form $x_0(t)$, $x_1(t)$, $x_2(t) = x_0(t)+\alpha_1 v_1(t)$. Since we have three functions then we will search an interpolation ICF in the following form

$$
Q_2^E(x(\cdot)) = F(x_0(\cdot)) + \frac{q_1^E(x(\cdot))}{1+q_2^E(x(\cdot))}.
$$

We have to find expressions for $q_1^E(x(\cdot))$ and $q_2^E(x(\cdot))$. Since $q_1^E(x(\cdot))$ does not depend on $x_2(z)$, then

$$
q_1^E(x(\cdot)) = q_1(x(\cdot)) = -\int_0^1 \frac{x(z_1)-x_0(z_1)}{x_1(z_1)-x_0(z_1)} \frac{\partial}{\partial z_1} F\big(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H\big(\cdot - z_1\big)\big)dz_1.
$$

From (3), we have

$$
\begin{aligned}
q_2(x(\cdot)) = \int_0^1\int_{z_1}^1 &\frac{x(z_1)-x_0(z_1)}{x_1(z_1)-x_0(z_1)} \frac{x(z_2)-x_1(z_2)}{v_1(z_2)}\\
&\times \frac{\partial^2}{\partial z_1 \partial z_2}\left(\frac{q_1\big(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1)+\alpha_1 v_1(\cdot)H(\cdot-z_2)\big)}{\alpha_1\big(F(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1)+\alpha_1 v_1(\cdot)H(\cdot-z_2))-F(x(\cdot))\big)}\right).
\end{aligned}
$$

Let us consider a limit for $\alpha_1 \to 0$. We obtain

$$
\begin{aligned}
q_2^E(x(\cdot)) &= \int_0^1\int_{z_1}^1 \frac{x(z_1)-x_0(z_1)}{x_1(z_1)-x_0(z_1)} \frac{x(z_2)-x_1(z_2)}{v_1(z_2)} \frac{\partial}{\partial z_1}\big(q_1^E(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1)\\
&\quad\times \lim_{\alpha_1\to 0}\frac{1}{\alpha_1}\frac{\partial}{\partial z_2}(F(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1)+\alpha_1 v_1(\cdot)H(\cdot-z_2))-F(x(\cdot)))^{-1}\Big)\,dz_2\,dz_1\\
&= -\int_0^1\int_{z_1}^1 \frac{x(z_1)-x_0(z_1)}{x_1(z_1)-x_0(z_1)} \frac{x(z_2)-x_1(z_2)}{v_1(z_2)} \left(\frac{q_1^E(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1))}{\frac{(F(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot)))^2}{F'(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-z_1))v_1(\cdot)H(\cdot-z_2)}}\right)dz_2\,dz_1.
\end{aligned}
$$

Let us check the interpolation conditions. Interpolativity in the node $x^1(\cdot,\xi_1) = x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)$ follows from [4] because $q_2^E(x^1(\cdot,\xi_1)) = 0$. Now we will check interpolativity of the derivatives. By the definition of derivative by Gato, we have:

$$
\begin{aligned}
&Q_2^{E'}(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1))v_1(\cdot)H(\cdot-\xi_2)\\
&\quad= \lim_{\alpha_1\to 1}\frac{Q_2(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)+\alpha_1 v_1(\cdot)H(\cdot-\xi_2))}{\alpha_1}\\
&-Q_2(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1))) = \lim_{\alpha_1\to 1}\frac{F(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1)+\alpha_1 v_1(\cdot)H(\cdot-\xi_2))}{\alpha_1},
\end{aligned}
$$

$$-F(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1))) = F'(x_0(\cdot)+(x_1(\cdot)-x_0(\cdot))H(\cdot-\xi_1))v_1(\cdot)H(\cdot-\xi_2).$$

Example 1. Let us consider the functional $F(x(\cdot)) = \big(\int_0^1 x(t)dt\big)^3$, $x(t)\in Q[0,1]$. In the case of the single node $x_0(\cdot)$ and the double node $x^1(\cdot,\xi_1)$, we have

$$Q_2^E(x(\cdot)) = \left(\int_0^1 x(t)\,dt\right)^3 + \frac{13\displaystyle\int_0^1\left(\int_0^{z_1}x_0(t)\,dt+\int_{z_1}^1 x_1(t)\,dt\right)^2(x(z_1)-x_0(z_1))\,dz_1}{1-9\displaystyle\int_0^1\int_{z_1}^1\prod_{i=1}^2(x(z_i)-x_{i-1}(z_i))\frac{\int_0^{z_1}x_0(s)ds-\int_{z_1}^1 x_1(s)ds}{\left(\left(\int_0^{z_1}x_0(s)ds+\int_{z_1}^1 x_1(s)ds\right)^3-\left(\int_0^1 x_0(s)ds\right)^3\right)^4}A\,dz^2},$$

where

$$\begin{aligned} A = &\left(\int_0^{z_1}x_0(t)dt+\int_{z_1}^1 x_1(t)dt\right)^3\left(\left(\int_0^{z_1}x_0(s)ds+\int_{z_1}^1 x_1(s)ds\right)^3-\left(\int_0^1 x_0(s)ds\right)^3\right)\\ &-2\left(\int_{z_1}^1\left(\int_0^s x_0(t)dt+\int_s^1 x_1(t)dt\right)^2(x_1(s)-x_0(s))ds\right)\\ &\times\left(2\left(\int_0^{z_1}x_0(s)ds+\int_{z_1}^1 x_1(s)ds\right)^3+\left(\int_0^1 x_0(s)ds\right)^3\right). \end{aligned}$$

Case 2. We need to construct an interpolation ICF with the double node $x_0(t)$ and the single node $x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2)$, which will satisfy the continual interpolation conditions

$$\begin{aligned} &Q_2^E\big(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2)\big) = F\big(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2)\big),\\ &Q_2^{E\prime}(x_0(\cdot))v_0(\cdot)H(\cdot-\xi_1) = F'(x_0(\cdot))v_0(\cdot)H(\cdot-\xi_1). \end{aligned}$$

Let us represent the sequence of functions $x_0(t)$, $x_1(t)$, $x_2(t)$ in the form $x_0(t)$, $x_1(t)=x_0(t)+\alpha_0 v_0(t)$, $x_2(t)$. We will construct an interpolation ICF in the form

$$Q_2^E(x(\cdot)) = F(x_0(\cdot)) + \frac{q_1^E(x(\cdot))}{1+q_2^E(x(\cdot))}.$$

At first we build $q_1^E(x(\cdot))$ and $q_2^E(x(\cdot))$. Using (3), we have

$$\begin{aligned} q_1(x(\cdot)) &= -\int_0^1 \frac{x(z_1)-x_0(z_1)}{\alpha_0 v_0(z_1)}\frac{\partial}{\partial z_1}F\big(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)\big)dz_1,\\ q_2(x(\cdot)) &= \int_0^1\int_{z_1}^1 \frac{x(z_1)-x_0(z_1)}{v_0(z_1)}\frac{x(z_2)-x_0(z_2)-\alpha_0 v_0(z_2)}{x_2(z_2)-x_0(z_2)-\alpha_0 v_0(z_2)}\\ &\times\frac{\partial^2}{\partial z_1\partial z_2}\left(\frac{q_1(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))}{\alpha_1(F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))-F(x_0(\cdot)))}\right)dz_2\,dz_1. \end{aligned}$$

Let us consider a limit for $\alpha_1\to 0$. We obtain

$$q_1^E(x(\cdot)) = -\int_0^1\frac{x(z_1)-x_0(z_1)}{v_0(z_1)}\frac{\partial}{\partial z_1}(F'(x_0(\cdot))v_0(\cdot)H(\cdot-z_1))dz_1,$$

$$q_2^E(x(\cdot)) = \int_0^1 \int_{z_1}^1 \frac{x(z_1)-x_0(z_1)}{v_0(z_1)} \frac{x(z_2)-x_0(z_2)}{x_2(z_2)-x_0(z_2)} \frac{\partial}{\partial z_2} \big(q_1^E(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot - z_2)\big)$$
$$\times \lim_{\alpha_1\to 0} \frac{1}{\alpha_1}\frac{\partial}{\partial z_1} F\big(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2)\big) - F(x_0(\cdot)))^{-1} dz_2\, dz_1$$
$$= -\int_0^1 \int_{z_1}^1 \frac{x(z_1)-x_0(z_1)}{v_0(z_1)} \frac{x(z_2)-x_0(z_2)}{x_2(z_2)-x_0(z_2)} \frac{\partial^2}{\partial z_1 \partial z_2} \left(\frac{q_1^E(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_2))}{\frac{(F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_2))-F(x_0(\cdot)))^2}{F'(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_2))v_0(\cdot)H(\cdot-z_1)}} \right) dz_2\, dz_1.$$

Let us check the interpolation conditions in the node $\tilde{x}(\cdot,\xi_2) = x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2)$. To do this, we will need following lemma.

Lemma 2. *Let the functional $F(x(\cdot))$ is differentiable by Gato and a linear functional $F'(x_0(\cdot))u(\cdot)$ has the representation*

$$F'(x_0(\cdot))u(\cdot) = \int_0^1 M(t)u(t)\,dt, \tag{8}$$

where $M(t) \in C[0,1]$. Then the following equality

$$\lim_{\xi_1\to 1}\left(\frac{x_1(\xi_1)-x_0(\xi_1)}{x_2(\xi_1)-x_0(\xi_1)} \frac{d}{d\xi_1} F(x^2(\cdot,(\xi_1,\xi_1)^T)) - \frac{d}{d\xi_1}F(x^1(\cdot,\xi_1))\right) = 0 \tag{9}$$

takes place.

Proof. Since the functional $F(x(\cdot))$ is differentiable by Gato then the following equalities hold:

$$\begin{aligned} &\frac{d}{d\xi_1}F(x^1(\cdot,\xi_1)) = F'(x^1(\cdot,\xi_1))\frac{d}{d\xi_1}H(\cdot-\xi_1)(x_1(\cdot)-x_0(\cdot)),\\ &\frac{d}{d\xi_1}F\left(x^2\left(\cdot,(\,\xi_1,\xi_1\,)^T\right)\right) = F'\left(x^2\left(\cdot,(\,\xi_1,\xi_1\,)^T\right)\right)\frac{d}{d\xi_1}H(\cdot-\xi_1)(x_2(\cdot)-x_0(\cdot)). \end{aligned} \tag{10}$$

If we substitute the equalities (10) into the left part of the equation (9) and use the representation (8) then we have the following equalities:

$$\begin{aligned} &\lim_{\xi_1\to 1}\left(\frac{x_1(\xi_1)-x_0(\xi_1)}{x_2(\xi_1)-x_0(\xi_1)} \frac{d}{d\xi_1} F(x^2(\cdot,(\xi_1,\xi_1)^T)) - \frac{d}{d\xi_1}F(x^1(\cdot,\xi_1))\right)\\ &= \lim_{\xi_1\to 1}\left(\frac{x_1(\xi_1)-x_0(\xi_1)}{x_2(\xi_1)-x_0(\xi_1)} F'(x^2(\cdot,(\xi_1,\xi_1)^T))\frac{d}{d\xi_1}H(\cdot-\xi_1)(x_2(\cdot)-x_0(\cdot)) \right.\\ &\qquad \left. -F'(x^1(\cdot,\xi_1))\frac{d}{d\xi_1}H(\cdot-\xi_1)(x_1(\cdot)-x_0(\cdot))\right)\\ &= \left(\frac{x_1(\xi_1)-x_0(\xi_1)}{x_2(\xi_1)-x_0(\xi_1)} F'(x_0(\cdot)) \lim_{\xi_1\to 1}\frac{d}{d\xi_1}H(\cdot-\xi_1)(x_2(\cdot)-x_0(\cdot)) \right.\\ &\qquad \left. -F'(x_0(\cdot))\lim_{\xi_1\to 1}\frac{d}{d\xi_1}H(\cdot-\xi_1)(x_1(\cdot)-x_0(\cdot))\right)\\ &= \lim_{\xi_1\to 1}\left(\frac{x_1(\xi_1)-x_0(\xi_1)}{x_2(\xi_1)-x_0(\xi_1)} \int_0^1 M(t)\frac{d}{d\xi_1}H(t-\xi_1)(x_2(t)-x_0(t))dt \right.\\ &\qquad \left. -\int_0^1 M(t)\frac{d}{d\xi_1}H(t-\xi_1)(x_1(t)-x_0(t))dt\right) = 0. \end{aligned}$$

Lemma is proved. ■

Let us consider in the notations of Lemma 2 the substitution $x_1(t) = x_0(t) + \alpha_0 v_0(t)$. We obtain

$$\lim_{\xi_1\to 1}\left(\frac{\alpha_0 v_0(\cdot)}{x_2(\xi_1)-x_0(\xi_1)}\frac{d}{d\xi_1}F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_1)) - \frac{d}{d\xi_1}F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-\xi_1))\right) = 0.$$

Let us denote that the substitution rule (4) has the form

$$\begin{aligned}&\frac{d}{dz_1}\left(F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))\right)\\ &\quad= \frac{d}{dz_1}F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)\alpha_0 v_0(\cdot))H(\cdot-z_2))\Big|_{z_2=z_1}\frac{x_2(z_1)-x_0(z_1)}{\alpha_0 v_0(z_1)}.\end{aligned}$$

Let us make necessary calculations.

1. $q_1^E(\tilde{x}(\cdot,\xi_2)) = -\displaystyle\int_{\xi_2}^1 \frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\frac{\partial}{\partial z_1}\left(F'(x_0(\cdot)v_0(\cdot)H(\cdot-z_1))\right)dz_1.$

2. $q_2^E(\tilde{x}(\cdot,\xi_2)) = \displaystyle\int_{\xi_2}^1 \frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\int_{z_1}^1\frac{\partial^2}{\partial z_1\partial z_2}$

$$\begin{aligned}
&\left(\frac{q_1(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))}{F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))-F(x_0(\cdot))}\right)dz_2\,dz_1\\
&= \int_{\xi_2}^1 \frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\frac{\partial}{\partial z_1}\\
&\left(\frac{q_1(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))}{F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))-F(x_0(\cdot))}\right)\Bigg|_{z_2=z_1}^1 dz_1\\
&= \int_{\xi_2}^1 \frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\frac{\partial}{\partial z_1}\frac{q_1(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1))}{F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1))-F(x_0(\cdot))}dz_1\\
&\quad- \int_{\xi_2}^1 \frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\frac{\partial}{\partial z_1}\\
&\left(\frac{q_1(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))}{F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2))-F(x_0(\cdot))}\right)\Bigg|_{z_2=z_1}^1 dz_1\\
&= -\int_{\xi_2}^1 \frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\Bigg(\frac{\frac{\partial}{\partial z_1}(F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)))}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot))}\,dz_1\\
&\quad+ q_1^E(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))\\
&\qquad\times\frac{\frac{\partial}{\partial z_1}(F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1)+(x_2(\cdot)-x_0(\cdot)-\alpha_0 v_0(\cdot))H(\cdot-z_2)))\Big|_{z_2=z_1}}{(F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot)))^2}\Bigg)dz_1\\
&= -\int_{\xi_2}^1 \frac{\frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\frac{\partial}{\partial z_1}(F'(x_0(\cdot)v_0(\cdot)H(\cdot-z_1)))}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot))}dz_1\\
&\quad+ \lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\int_{\xi_2}^1\frac{q_1^E(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))\frac{\partial}{\partial z_1}F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))}{(F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot)))^2}dz_1\\
&= -\int_{\xi_2}^1 \frac{\frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\frac{\partial}{\partial z_1}(F'(x_0(\cdot)v_0(\cdot)H(\cdot-z_1)))}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot))}dz_1
\end{aligned}$$

$$+\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\int_{\xi_2}^{1}\int_{z_1}^{1}\frac{x_2(s_1)-x_0(s_1)}{v_0(s_1)}\frac{d}{ds_1}F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1))ds_1$$
$$\times\frac{\partial}{\partial z_1}(F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1)))^{-1}dz_1.$$

In the last equality, we use the substitution rule. Now we calculate the last integral. We obtain

$$\begin{aligned}
q_2^E(\tilde{x}(\cdot,\xi_2)) &= -\int_{\xi_2}^{1}\frac{\frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\frac{\partial}{\partial z_1}(F'(x_0(\cdot)v_0(\cdot)H(\cdot-z_1)))}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot))}dz_1\\
&\quad+\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\int_{z_1}^{1}\frac{x_2(s_1)-x_0(s_1)}{v_0(s_1)}\frac{d}{ds_1}F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1))\,ds_1\\
&\quad\times(F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1)))^{-1}\Big|_{z_1=\xi_2}^{z_1=1}\\
&= -\int_{\xi_2}^{1}\frac{\frac{x_2(z_1)-x_0(z_1)}{v_0(z_1)}\frac{\partial}{\partial z_1}(F'(x_0(\cdot)v_0(\cdot)H(\cdot-z_1)))}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot))}dz_1\\
&\quad+\lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\left(\lim_{z_1\to 1}\frac{\int_{z_1}^{1}\frac{x_2(s_1)-x_0(s_1)}{v_0(s_1)}\frac{d}{ds_1}F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1))ds_1}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-z_1))-F(x_0(\cdot))}\right.\\
&\quad\left.-\frac{\int_{\xi_2}^{1}\frac{x_2(s_1)-x_0(s_1)}{v_0(s_1)}\frac{d}{ds_1}F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-z_1))ds_1}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2))-F(x_0(\cdot))}\right)\\
&= -1-\frac{\int_{\xi_2}^{1}\frac{x_2(s_1)-x_0(s_1)}{v_0(s_1)}\frac{d}{ds_1}F'(x_0(\cdot)v_0(\cdot)H(\cdot-z_1))ds_1}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2))-F(x_0(\cdot))}\\
&= -1+\frac{q_1^E(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2))}{F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2))-F(x_0(\cdot))}.
\end{aligned}$$

Here we calculate the limit using L'Hopital's rule, Lemma 2 and $Q_2^E(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2)) = F(x_0(\cdot)+(x_2(\cdot)-x_0(\cdot))H(\cdot-\xi_2))$. Therefore, the first interpolation condition takes place. Now we check interpolativity of derivatives. Using the definition by Gato, we have

$$\begin{aligned}
Q_2^{E\prime}(x_0(\cdot))v_0(\cdot)H(\cdot-\xi_1) &= \lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\left(Q_2^E(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-\xi_1))-Q_2^E(x_0(\cdot))\right)\\
&= \lim_{\alpha_0\to 0}\frac{1}{\alpha_0}\left(F(x_0(\cdot)+\alpha_0 v_0(\cdot)H(\cdot-\xi_1))-F(x_0(\cdot))\right)=F'(x_0(\cdot))v_0(\cdot)H(\cdot-\xi_1).
\end{aligned}$$

The second condition also holds. Now we will find a form of interpolation ICF if $x_0(z)\equiv x_0=\text{const}$, $x_2(z)\equiv x_2=\text{const}$, $x(z)\equiv x=\text{const}$. From [5], we obtain that for $n=2$ the interpolation ICF has the form

$$Q_2(x(\cdot))=F(x(\cdot))+\frac{(x-x_0)F_{01}}{1-\prod_{i=1}^{2}\frac{x-x_{i-1}}{x_2-x_{i-1}}\left(1-\frac{F_{01}}{F_{02}}\right)},$$

where $F_{ok}=\frac{F(x_k)-F(x_0)}{x_k-x_0}$, $k=1,2$. The sequence of nods x_0, x_1, x_2 we will represent in the form x_0, $x_1=x_0+\alpha_0$, x_2, where $\alpha_0\in\mathbb{R}$. After some transmutations and if $\alpha_0\to 0$ we obtain

$$Q_{n+1}^E(x(\cdot))=F(x_0)+\frac{q_1^E(x)|}{|1}+\frac{q_2^E(x)|}{|1},$$

where $q_1^E(x)=(x-x_0)F'(x_0)$, …, $q_2^E(x)=-\left(\frac{x-x_0}{x_2-x_0}\right)\left(1-\frac{F'(x_0)}{F_{02}}\right)$.

Example 2. Let us consider the functional of the following form $F(x(\cdot)) = \left(\int_0^1 x(t)dt\right)^3$, $x(t) \in Q[0,1]$. If we have the double node $x_0(t)$ and the single $x_0(\cdot) + (x_2(\cdot) - x_0(\cdot))H(\cdot - \xi_2)$ then

$$Q_2^E(x(\cdot)) = \left(\int_0^1 x(t)\,dt\right)^3$$
$$+\frac{3\left(\int_0^1 x(t)dt\right)^2 \int_0^1 (x(z_1) - x_0(z_1))\,dz_1}{1 - 9\left(\int_0^1 x(t)\,dt\right)^2 \int_0^1 \int_{z_1}^1 \prod_{i=1}^{2}(x(z_i) - x_0(z_i)) \dfrac{\left(\int_0^{z_2} x_0(s)ds + \int_{z_2}^1 x_2(s)ds\right)}{\left(\left(\int_0^{z_2} x_0(t)dt + \int_{z_2}^1 x_2(t)dt\right)^3 - \left(\int_0^1 x_0(t)dt\right)^3\right)^3}} B\,dz^2,$$

where

$$B = \left(\int_0^{z_2} x_0(t)dt + \int_{z_2}^1 x_2(t)dt\right)\left(\left(\int_0^{z_2} x_0(t)dt + \int_{z_2}^1 x_2(t)dt\right)^3 - \left(\int_0^1 x_0(t)dt\right)^3\right)$$
$$- 2\int_{z_2}^1 (x_2(z_1) - x_0(z_1))dz_1 \left(2\left(\int_0^{z_2} x_0(t)dt + \int_{z_2}^1 x_2(t)dt\right)^3 + \left(\int_0^1 x_0(t)dt\right)^3\right).$$

3. The general solution

Let us denote by $D_{z^j} = \frac{\partial^j}{\partial z_1 \partial z_2 \ldots \partial z_j}$; $dz^j = dz_j dz_{j-1}\ldots dz_1$;

$$f_k(x^n(\cdot, z^n)) = \mathop{\mathrm{D}}_{i=1}^{k} \frac{q_{k-1}(x^n(\cdot, z^n))}{-1}; \quad q_0(x^n(\cdot, z^n)) = F(x_0(\cdot)) - F(x^n(\cdot, z^n)) + 1; \quad k = 1, \ldots, n.$$

Let us consider the interpolants (2), (3) of degree $m+1$. We represent the sequence of functions $x_0(t)$, $x_1(t)$, …, $x_{m+1}(t)$ in the form

$$x_0(t), x_1(t), \ldots, x_k(t), x_{k+1}(t) = x_k(t) + \alpha_k v_k(t), x_{k+2}(t), \ldots, x_{m+1}(t), \tag{11}$$

where $\alpha_k \in \mathbb{R}, v_k(t) \in Q[0,1]$. Then

$$x^{m+1}(\cdot, \xi^{m+1}) = x_0(\cdot) + \sum_{\substack{i=1\\ i\neq k+1}}^{m+1} H(\cdot - \xi_i)(x_i(\cdot) - x_{i-1}(\cdot)) + \alpha_k v_k H(\cdot - \xi_{k+1}),$$
$$\xi^{m+1} = (\xi_1, \xi_2, \ldots, \xi_{m+1}) \in \Omega_{z^{m+1}} = \{z^{m+1} \colon 0 \leqslant z_1 \leqslant \ldots z_{m+1} \leqslant 1\}.$$

Let us substitute the expression (11) into formulas (2), (3) for $n = m+1$ and find a limit if $\alpha_k \to 0$. Let us suppose that the respective Gato derivative exists and the substitution rule (4) holds. Having made necessary calculations, we obtain the ICF $Q_{m+1}^E(x(\cdot))$ of the form

$$Q_{m+1}^E(x(\cdot)) = F(x_0(\cdot)) + \mathop{\mathrm{D}}_{l=1}^{m+1} \frac{q_l^E(x(\cdot))}{1}, \tag{12}$$

where

$$\begin{aligned} &1.\ q_l^E(x(\cdot)) = q_l(x(\cdot)) \ \text{ for } \ l \leqslant k; \\ &2.\ q_l^E(x(\cdot)) = (-1)^l \int_0^1 \int_{z_1}^1 \ldots \int_{z_{k+1}}^1 (x(z_{k+1}) - x_k(z_{k+1}))\ldots \int_{z_{l-1}}^1 \\ &\quad D_{z^l}\left(f_l'(x^l(\cdot, z^l))F'(x^l(\cdot, z^l))H(\cdot - z_{k+1})\right) \prod_{i=1; i\neq k+1}^{l} \frac{x(z_i) - x_{i-1}(z_i)}{x_i(z_i) - x_{i-1}(z_i)} dz^l. \end{aligned} \tag{13}$$

Here if $l = 2p$

$$q_0^E\left(x_0(\cdot)+\sum_{\substack{i=1\\ i\neq k+1}}^{2p} H(\cdot-\xi_i)(x_i(\cdot)-x_{i-1}(\cdot))\right)$$
$$= -\frac{\left(F\left(x_0(\cdot)+\sum\limits_{\substack{i=1\\ i\neq k+1}}^{2p} H(\cdot-\xi_i)(x_i(\cdot)-x_{i-1}(\cdot))\right)-F(x_0(\cdot))\right)^2}{F'\left(x_0(\cdot)+\sum\limits_{\substack{i=1\\ i\neq k+1}}^{2p} H(\cdot-\xi_i)(x_i(\cdot)-x_{i-1}(\cdot))\right)v_k(\cdot)H(\cdot-z_{k+1})}$$

and if $l = 2p+1$

$$q_0^E\left(x_0(\cdot)+\sum_{\substack{i=1\\ i\neq k+1}}^{2p+1} H(\cdot-\xi_i)(x_i(\cdot)-x_{i-1}(\cdot))\right)$$
$$= F'\left(x_0(\cdot)+\sum_{\substack{i=1\\ i\neq k+1}}^{2p+1} H(\cdot-\xi_i)(x_i(\cdot)-x_{i-1}(\cdot))\right)v_k(\cdot)H(\cdot-z_{k+1}).$$

Theorem 3. *Let the substitution rule* (4) *takes place and integrals* (13) *exist on* $Q[0,1]$. *Then the continual interpolation conditions* (6), (7) *take place for interpolant of the Hermitian type and it does not depend on direction of differentiation.*

Proof. According to [4] equality $Q_{m+1}\left(x^{m+1}(\cdot,\xi^{m+1})\right) = F\left(x^{m+1}(\cdot,\xi^{m+1})\right)$ takes place. Taking into consideration (11) we have that $Q_{m+1}\left(\hat{x}^{m+1}(\cdot,\xi^{m+1})\right) = F\left(\hat{x}^{m+1}(\cdot,\xi^{m+1})\right)$. In the last equality we consider limit if $\alpha_k \to 0$ and obtain (6).

Now we prove interpolativity of derivatives. By definition of Gato derivative we obtain

$$\begin{aligned}
&{Q_{m+1}^E}'\left(x_0(\cdot)+\sum_{l=1}^{k}(x_l(\cdot)-x_{l-1}(\cdot))H(\cdot-\xi_l)\right)v_kH(\cdot-\xi_{k+1})\\
&= \lim_{\alpha_k\to 0}\left(Q_{m+1}^E\left(x_0(\cdot)+\sum_{l=1}^{k}(x_l(\cdot)-x_{l-1}(\cdot))H(\cdot-\xi_l)+\alpha_kv_k(\cdot)H(\cdot-\xi_{k+1})\right)\right.\\
&\quad\left. - Q_{m+1}^E\left(x_0(\cdot)+\sum_{l=1}^{k}(x_l(\cdot)-x_{l-1}(\cdot))H(\cdot-\xi_l)\right)\right)\\
&= \lim_{\alpha_k\to 0}\left(F\left(x_0(\cdot)+\sum_{l=1}^{k}(x_l(\cdot)-x_{l-1}(\cdot))H(\cdot-\xi_l)+\alpha_kv_k(\cdot)H(\cdot-\xi_{k+1})\right)\right.\\
&\quad\left. - F\left(x_0(\cdot)+\sum_{l=1}^{k}(x_l(\cdot)-x_{l-1}(\cdot))H(\cdot-\xi_l)\right)\right)\\
&= F'\left(x_0(\cdot)+\sum_{l=1}^{k}(x_l(\cdot)-x_{l-1}(\cdot))H(\cdot-\xi_l)\right)v_kH(\cdot-\xi_{k+1}).
\end{aligned}$$

Therefore the condition (7) takes place. For construction of the function $q_l^E(x(\cdot))$, we use $v_k(z_{k+1})$ in the kernel $\frac{x(z_{k+1})-x_k(z_{k+1})}{v_k(z_{k+1})}$ and

$$F'\left(x_0(\cdot)+\sum_{l=1}^{k}\big(x_l(\cdot)-x_{l-1}(\cdot)\big)H(\cdot-\xi_l)\right)v_kH(\cdot-\xi_{k+1})$$

if $k+1 \leqslant l \leqslant m+1$. So the function

$$q_l^E(x(\cdot)) = (-1)^l \int_0^1 \int_{z_1}^1 \dots \int_{z_{k+1}}^1 (x(z_{k+1}) - x_k(z_{k+1})) \dots \int_{z_{l-1}}^1 D_{z^l}\left(f_l'(x^l(\cdot,z^l))F'(x^l(\cdot,z^l))H(\cdot - z_{k+1})\right) \prod_{\substack{i=1 \\ i\neq k+1}}^{l} \frac{x(z_i) - x_{i-1}(z_i)}{x_i(z_i) - x_{i-1}(z_i)} dz^l$$

does not depend on direction of differentiation. Here if $l = 2p$ then

$$q_0^E\left(x_0(\cdot) + \sum_{\substack{i=1 \\ i\neq k+1}}^{2p} H(\cdot - \xi_i)(x_i(\cdot) - x_{i-1}(\cdot))\right) = -\left(F\left(x_0(\cdot) + \sum_{\substack{i=1 \\ i\neq k+1}}^{2p} H(\cdot - \xi_i)(x_i(\cdot) - x_{i-1}(\cdot))\right) - F(x_0(\cdot))\right)^2$$

and if $l = 2p+1$

$$q_0^E\left(x_0(\cdot) + \sum_{\substack{i=1 \\ i\neq k+1}}^{2p+1} H(\cdot - \xi_i)(x_i(\cdot) - x_{i-1}(\cdot))\right) = 1.$$

The theorem is proved. ■

Case of functions of a single variable

Let us find a representation of a interpolation ICF (12), (13) if $x_i(z) \equiv x_i = \text{const}$, $i = 0,\dots,m+1$, $x(z) \equiv x = \text{const}$. The next theorem takes place.

Theorem 4. *The continued fraction*

$$Q_{m+1}(x) = F(x_0) + \mathop{\mathrm{D}}_{i=1}^{m+1} \frac{q_i(x)}{1} \tag{14}$$

is a interpolation fraction for the function $F(x)\colon [a,b] \to \mathbb{R}$ with nods $x_i \in [a,b]$, $i = 0,\dots,m+1$, where

$$q_1(x) = (x - x_0)F_{01}, \quad q_2(x) = -\prod_{i=1}^{2} \frac{x - x_{i-1}}{x_2 - x_{i-1}}(1 - f_2(x_2)), \dots,$$

$$q_l(x) = -\prod_{i=1}^{l} \frac{x - x_{i-1}}{x_l - x_{i-1}}(1 - f_l(x_l));$$

$$f_2(x) = \frac{F_{01}}{F_{02}}; \quad f_l(x) = \mathop{\mathrm{D}}_{i=1}^{l} \frac{q_{l-i}(x)}{1}, \quad q_0(x) = F(x_0) - F(x) - 1; \quad l = 3,4,\dots,m+1, \tag{15}$$

$$F_{0l} = \frac{F(x_l) - F(x_0)}{x_l - x_0}, \quad l = 1,2,\dots.$$

We will represent a sequence of the nods $x_0, x_1, \dots, x_{m+1}$ in the form $x_0, x_1, \dots, x_k, x_{k+1} = x_k + \alpha_k, x_{k+2}, \dots, x_{m+1}$, where $\alpha_i \in \mathbb{R}$.

After some transformations and if $\alpha_k \to 0$ we have

$$
\begin{aligned}
q_1^E(x) &= q_1(x) = (x-x_0)F_{01}; \quad q_2^E(x) = q_2(x) = -\prod_{i=1}^{2}\frac{x-x_{i-1}}{x_2-x_{i-1}}(1-f_2(x_2)); \quad \ldots;\\
q_k^E(x) &= q_k(x) = -\prod_{i=1}^{k}\frac{x-x_{i-1}}{x_k-x_{i-1}}(1-f_k(x_k)); \quad f_2^E(x)=f_2(x); \quad f_k^E(x)=f_k(x);\\
q_{k+1}^E(x) &= \lim_{\alpha_k\to 0}\frac{x-x_k}{x_{k+1}-x_k}\prod_{i=1}^{k}\frac{x-x_{i-1}}{x_{k+1}-x_{i-1}}(1-f_{k+1}^E(x_{k+1}))\\
&= \lim_{\alpha_k\to 0}\frac{x-x_k}{\alpha_k}\prod_{i=1}^{k}\frac{x-x_{i-1}}{x_{k+1}+\alpha_k-x_{i-1}}\left(\frac{q_k^E(x_k+\alpha_k)}{f_k^E(x_k+\alpha_k)-1}-1\right)\\
&= \lim_{\alpha_k\to 0}\frac{x-x_k}{\alpha_k}\prod_{i=1}^{k}\frac{(x-x_{i-1})((q_k^E(x_k+\alpha_k)-q_k^E(x_k))-(f_k^E(x_k+\alpha_k)-f_k^E(x_k)))}{(x_{k+1}+\alpha_k-x_{i-1})(f_k^E(x_k+\alpha_k)-1)}\\
&= (x-x_k)\prod_{i=1}^{k}\frac{x-x_{i-1}}{x_k-x_{i-1}}\frac{q_k^{E\prime}(x_k)-f_k^{E\prime}(x_k)}{f_k^E(x_k)-1};\\
q_{k+2}^E(x) &= -\frac{(x-x_k)^2}{(x_{k+2}-x_k)^2}\prod_{i=1}^{k}\frac{x-x_{i-1}}{x_{k+2}-x_{i-1}}(1-f_{k+2}^E(x_{k+2})); \quad \ldots;\\
q_{n+1}^E(x) &= -\prod_{j=k+2}^{n+1}\frac{x-x_j}{x_{n+1}-x_j}\left(\frac{x-x_k}{x_{n+1}-x_k}\right)^2\prod_{i=1}^{k}\frac{x-x_{i-1}}{x_{n+1}-x_{i-1}}(1-f_{n+1}^E(x_{n+1})),\\
f_p(x) &= \mathop{\mathrm{D}}_{i=1}^{p}\frac{q_{p-i}^E(x)}{-1}, \quad q_0(x)=F(x_0)-F(x)-1,\\
Q_{n+1}^E(x) &= F(x_0)+\mathop{\mathrm{D}}_{i=1}^{n+1}\frac{q_i^E(x)}{1}, \quad p=k+2,\ldots,n+1.
\end{aligned}
\tag{16}
$$

Theorem 5. *The continued fraction (16) is the interpolation fraction for the function $F(x)\colon [a,b]\to\mathbb{R}$ with nods $x_i\in[a,b]$, $i=0,\ldots,n$, and satisfy the interpolation conditions*

$$
Q_{n+1}^E(x_i)=F(x_i), \quad {Q_{n+1}^E}'(x_k)=F'(x_k), \quad x_{k+1}=x_k, \quad 1\leqslant k\leqslant n+1, \quad i=0,\ldots,n+1.
$$

Proof. According to Theorem 4, we have $Q_{n+1}^E(x_i)=F(x_i)$, $x_{k+1}=x_k$, $i=0,\ldots,n+1$. Let us check interpolativity of derivative for x_k

$$
\begin{aligned}
{Q_{n+1}^E}'(x_k) &= \lim_{\alpha_k\to 0}\frac{1}{\alpha_k}(Q_{n+1}^E(x_k+\alpha_k)-Q_{n+1}^E(x_k)) = \lim_{\alpha_k\to 0}\frac{1}{\alpha_k}(Q_{k+1}^E(x_k+\alpha_k)-Q_{k+1}^E(x_k))\\
&= \lim_{\alpha_k\to 0}\frac{1}{\alpha_k}(F(x_k+\alpha_k)-F(x_k)) = F'(x_k).
\end{aligned}
$$

■

Let us consider a partial case of a double node. Let x_0 is a single node and x_1 is a double node. From (16) we have that

$$
\begin{aligned}
Q_2^E(x(\cdot)) &= F(x_0)+\frac{q_1^E(x)|}{|1}+\frac{q_2^E(x)|}{|1},\\
q_2^E(x(\cdot)) &= -\frac{(x-x_0)(x-x_1)}{x_1-x_0}\frac{(x_1-x_0)F'(x_1)-(F(x_1)-F(x_0))}{(x_1-x_0)(F(x_1)-F(x_0))},
\end{aligned}
\tag{17}
$$

where $q_1^E(x(\cdot))=\frac{(x-x_0)}{x_1-x_0}(F(x_1)-F(x_0))$.

The fraction (17) is the interpolation fraction for the function $F(x) : [a, b] \to \mathbb{R}$ with nods x_0, x_1 and satisfy interpolation conditions

$$Q_2^E(x_0) = F(x_0), \quad Q_2^E(x_1) = F(x_1), \quad {Q_2^E}'(x_1) = F'(x_1). \tag{18}$$

Interpolativity of a function in the nodes x_0, x_1 it is easy to prove by checking conditions. The third interpolativity condition (18) follows from following equalities

$$\begin{aligned} {Q_2^E}'(x_1) &= \lim_{\alpha_2 \to 0} \frac{Q_2^E(x_1+\alpha_2) - Q_2^E(x_1)}{\alpha_2} \\ &= \lim_{\alpha_2 \to 0} \frac{1}{\alpha_2} \frac{\alpha_2 F_{01} + \alpha_2 (x_1 + \alpha_2 - x_0) \frac{(x_1-x_0)F'(x_1) - (F(x_1)-F(x_0))}{(x_1-x_0)^2}}{1 + q_2^E(x_1+\alpha_2)} = F'(x_1). \end{aligned}$$

4. Conclusion

The problem of construction of an interpolation ICF is solved. The constructed ICF satisfies the interpolation conditions. We have shown that the constructed interpolants of the Hermitian type do not depend on a direction of differentiation, are unique and have property of the ICF of a corresponding degree. Also it is proved that the obtained interpolation ICF is the logical generalisation of interpolative continued fractions for a function of a single variable. The corresponding proofs were given in examples.

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Інтерполяційний інтегральний ланцюговий дріб з двократним вузлом

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Для функціонала, заданого на континуальній множині вузлів на підставі раніше побудованого інтерполяційного інтегрального ланцюгового дробу типу Ньютона, побудовано та досліджено інтерполянт з k-им двократним вузлом. Доведено, що побудований інтегральний ланцюговий дріб буде інтерполянтом типу Ерміта.

Ключові слова: *неперервний інтерполяційний дріб Ньютона, двократний вузол неперервного дробу, Ермітовий інтерполянт.*

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