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**Vasyl Olshanskiy [1], Volodymyr Burlaka [2], Maksym Slipchenko [3]**

1. Department of Physics and Theoretical Mechanics, Kharkiv Petro Vasylenko National Technical University of Agriculture, Ukraine, Kharkiv, Alchevskikh Street, 44, E-mail: OlshanskiyVP@gmail.com
2. Department of Physics and Theoretical Mechanics, Kharkiv Petro Vasylenko National Technical University of Agriculture, Ukraine, Kharkiv, Alchevskikh Street, 44, E-mail: Burlaka2V@ukr.net
3. Department of Physics and Theoretical Mechanics, Kharkiv Petro Vasylenko National Technical University of Agriculture, Ukraine, Kharkiv, Alchevskikh Street, 44, E-mail: Slipchenko_M@ukr.net

# SOLUTION OF THE EQUATION OF FORCE OF IMPACT OF SOLIDS EXPRESSED BY THE ATEB-SINE

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**Abstract.** A nonlinear differential equation of the force of direct central quasistatic impact of elastic bodies bounded in the area of their contact by rotation surfaces is compiled. To determine the coefficients of the equation and the order of its degree nonlinear force, we used the well-known solution of the axisymmetric contact problem of elasticity theory, constructed in due time by I. Ya. Shtaerman, for the case of dense static contact of bodies, when the order of their boundary surfaces is not lower than the second. In the case of the second order, it goes into the well-known static solution of G. Hertz, whose assumption in the theory of shock is also taken here in the formulation of the dynamics problem. A closed analytic solution of the composite differential equation with respect to the force of impact as a function of time is constructed. It is expressed through Ateb-sine. This function also describes the process of motion of the centers of mass of bodies in the stages of their compression and expansion. Compact formulas are derived for calculating the maximum values of the impact force, the approach of the centers of mass of the bodies and the duration of the impact. Thanks to the use of the Ateb-sine and its approximation by elementary functions, it was possible to obtain a fairly simple scan of the fleeting process of mechanical shock in time. It is shown that well-known dependencies that describe the impact of elastic balls follow from the derived formulas. Examples of calculations are given in which the influence of various factors on the main characteristics of the impact is investigated. It is noted that the theory set forth concerns only the impact of bodies with low velocities, when plastic deformation does not occur during their dynamic compression. To extend the theory beyond the limits of elasticity, it is necessary to determine a constant for the stage of compression in the impact force equation not by calculation, but by experimental method. Then, during compression and decompression of bodies, the impact force will be described by different analytical expressions, and the speed recovery coefficient will become less than one, which is consistent with practice.

**Keywords:** impact force equation, analytical solution, Ateb-sine, rapprochement of centers of mass, duration of impact.

## Introduction

There are four main variants of mechanical impact theory. The first of these is the stereo mechanical shock theory, which is based on Newton's hypotheses. It is traditionally covered in courses in theoretical mechanics [1]. Much later, the second and third variants appeared. This is the corresponding wave theory of shock proposed by Saint-Venan [2], which does not take into account local deformations and quasi-static Hertz theory [3], which does not take into account wave processes in dynamic compression of bodies. The fourth option includes hybrid theories, where the aforementioned, for example, wave and quasistatic theories are combined [4].

## Problem Statement

When implementing hybrid theories traditionally, starting with S. P. Tymoshenko [5], make up the nonlinear integral equation of the impact force, and then solve it by numerical methods, replacing the integrals with known sums [6]–[8]. Thus, a graph of the process of changing the force of impact over time is obtained, and then other characteristics are calculated. But today there are few analytical solutions in which the scan of the impact force and other parameters in time is described by closed formulas. Such formulas were obtained only in [3] by an approximate solution of the integral equation of impact force. The use of Ateb-functions opens the possibility to obtain accurate analytical solutions of differential equations of impact force. This is the opportunity that is implemented in this article.

## Analysis of modern information sources

The theory of Ateb-functions has been developed and has become widespread in solving applied problems thanks to the work of mathematicians and mechanics of the Lviv School [9]–[13]. Their use has made it possible to solve analytically many complex problems in the theory of oscillations of nonlinear mechanical systems [14]–[17]. Proceeding from this, in [18] even the study of these functions in an advanced course of theoretical mechanics was proposed. Ateb-functions continue to be used in theoretical studies by foreign authors [19], [20]. Recently, they have become an effective means of solving the problems of quasistatic impact of solids [21]–[23], where they analytically describe fleeting processes. To further illustrate the capabilities of these functions in shock theory, the goal of this work was set.

## Statement of purpose and tasks of research

The purpose of the article is to analytically solve the nonlinear differential equation of the impact force of elastic bodies bounded in the field of their contact by high order rotation surfaces and to use the solution to analyze the influence of various factors on the dynamic interaction process.

## Main material presentation

When setting the impact problem here we use the Hertz theory, but we build the solution of the problem not in displacements, but in efforts. After determining the impact force as a function of time, it is now quite simple to determine the other characteristics of the process. In this case, the static dependences of the strain on the force obtained by I. Ya. Staerman for the case of tight contact of bodies of revolution [24] are important. As a result, they lead to a generalization of Hertz's results in impact theory.

## 1. Differential equations of impact force

According to the quasistatic theory of impact, the approach of the centers of mass of bodies $x(t)$, after their collision, is described by the equation:

$$M\ddot{x} = -P \ , \tag{1}$$

in which $M = \dfrac{m_1 m_2}{m_1 + m_2}$ – equivalent mass; $m_1$, $m_2$ – masses of bodies involved in impact; $P = P(t)$ – force of shock interaction; the dot denotes the derivative of $x$ with respect to $t$.

In order to have an additional dependence of $x$ on $P$, we assume that in the contact area of the body are bounded by the surfaces of rotation $z_1 = f_1(r)$ and $z_2 = -f_2(r)$, that the axis of symmetry $0z$ passes along the line of action of dynamic compressive forces (Fig. 1).

We assume that, when $r = 0$ non-zero, derivatives of functions $f_1(r)$ and $f_2(r)$, since their $2n$-th order, that is:

$$f_1^{(j)}(0) = f_2^{(j)}(0) = 0 \text{ , when } j <= 2n \ ,$$

$$f_1^{(2n)}(0) + f_2^{(2n)}(0) = (2n)!A,$$

where $A$ – positive constant; $n = 1, 2, ...$

Then according to the I. Ya. Staerman's decision [24]:

$$x = kP^{\lambda}\ , \tag{2}$$

moreover:

$$\lambda = \frac{2n}{2n+1}\ ;\ k = \left[\frac{(2n+1)(Q_1+Q_2)}{4n}\right]^{\frac{2n}{2n+1}} \left[\frac{(2n)!!A}{(2n-1)!!}\right]^{\frac{1}{2n+1}}\ ;\ Q_1 = \frac{1-\nu_1^2}{E_1}\ ;\ Q_2 = \frac{1-\nu_2^2}{E_2}\ , \tag{3}$$

where $E_1$, $E_2$, $\nu_1$, $\nu_2$ – respectively, the modulus of elasticity and the Poisson coefficients of the materials of the bodies.

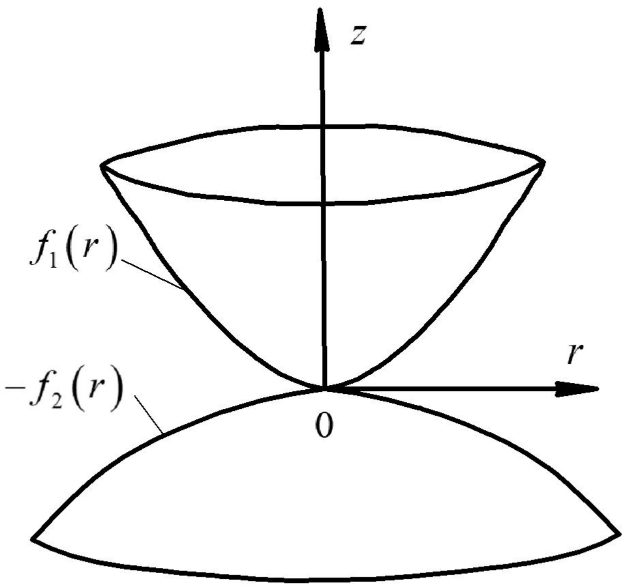


**Fig. 1.** Contact diagram of rotation bodies

Note that in the case of $n=1$ dependencies (2), (3) correspond to the contact interaction of balls' radiuses $R_1$ and $R_2$.

For this $n$:

$$A = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right);\ \lambda = \frac{2}{3}\ ;\ k = \left(\frac{3}{4}(Q_1+Q_2)\sqrt{\frac{R_1+R_2}{R_1\cdot R_2}}\right)^{2/3}\ ,$$

which is a well-known result [3, 4].

Taking two derivatives of $t$ from expression (2), we obtain:

$$\dot{x} = k\lambda P^{\lambda-1}\dot{P}\ ,\ \ddot{x} = k\lambda\left[(\lambda-1)P^{\lambda-2}\cdot\dot{P}^2 + P^{\lambda-1}\ddot{P}\right].$$

Substituting these dependencies in (1), we arrive at Eq.:

$$\ddot{P} + (\lambda-1)\frac{\dot{P}^2}{P} + \frac{P^{2-\lambda}}{kM\lambda} = 0\ . \tag{4}$$

Given the absence $t$, by replacement $\ddot{P} = \dot{P}\dfrac{d\dot{P}}{dP}$, instead of (4), we obtain the Bernoulli equation:

$$\frac{d\dot{P}}{dP} + (\lambda-1)\frac{\dot{P}}{P} + \frac{P^{2-\lambda}}{kM\lambda\dot{P}} = 0\ , \tag{5}$$

which determines the force of impact.

## 2. Constructing the equation of impact force

We look for the solution of equation (5) as the product of two unknown functions:

$$\dot{P} = F(P)\cdot G(P)\ . \tag{6}$$

Then $\dfrac{d\dot{P}}{dP} = \dfrac{dF}{dP}\cdot G + F\dfrac{dG}{dP}$ and (5) we reduce to two first order differential equations:

$$\frac{dF}{dP}=\left(1-\lambda\right)\frac{F}{P};\ \frac{dG}{dP}=-\frac{P^{2-\lambda}}{kM\lambda GF^{2}}.$$

Consistent integration of these results is accurate to constant $C$:

$$F\left(P\right)=P^{1-\lambda};\ G\left(P\right)=\sqrt{C-\frac{2P^{\lambda+1}}{kM\lambda\left(\lambda+1\right)}}. \tag{7}$$

Let $P_c$ denote the maximum force value at the end of the compression process, when $\dot{P}=0$. Then the constant $C$ gets value $C=\frac{2P_c^{\lambda+1}}{kM\lambda\left(\lambda+1\right)}$, and the solution (6), given (7), takes the form:

$$\dot{P}=\frac{dP}{dt}=\sqrt{\frac{2}{kM\lambda\left(\lambda+1\right)}}\sqrt{P^{2(1-\lambda)}\left(P_c^{\lambda+1}-P^{\lambda+1}\right)}.$$

Integrating this expression gives the relation:

$$\int_0^P\frac{dP}{\sqrt{P^{2(1-\lambda)}\left(P_c^{\lambda+1}-P^{\lambda+1}\right)}}=\sqrt{\frac{2}{kM\lambda\left(\lambda+1\right)}}t.$$

Moving to a new integration variable $P=P_cu$ reduces it to the equation:

$$\int_0^{P/P_c}\frac{du}{\sqrt{u^{2(1-\lambda)}\left(1-u^{\lambda+1}\right)}}=\sqrt{\frac{2P_c^{1-\lambda}}{kM\lambda\left(\lambda+1\right)}}t.$$

Further replacement $\xi=u^{\lambda}$ gives it a more compact shape:

$$\int_0^{\left(P/P_c\right)^{\lambda}}\frac{d\xi}{\sqrt{1-\xi^{\frac{\lambda+1}{\lambda}}}}=\gamma t, \tag{8}$$

where

$$\gamma=\sqrt{\frac{2\lambda P_c^{1-\lambda}}{kM\left(\lambda+1\right)}}. \tag{9}$$

Using the integral representation of the Ateb-sine [13, 16], from (8) we obtain the solution of the differential equation (5):

$$P\left(t\right)=P_c\left[\mathrm{Sa}\left(\frac{1}{\lambda},1,\frac{\lambda+1}{2\lambda}\gamma t\right)\right]^{1/\lambda}. \tag{10}$$

Denote by $t_c$ the duration of the compression process, at the end of which $P\left(t_c\right)=P_c$. Then, according to (10), equality is satisfied:

$$\mathrm{Sa}\left(\frac{1}{\lambda},1,\frac{\lambda+1}{2\lambda}\gamma t_c\right)=1,$$

or:

$$\gamma t_c=\mathrm{I}=\int_0^1\frac{d\xi}{\sqrt{1-\xi^{\frac{\lambda+1}{\lambda}}}}. \tag{11}$$

Since this integral is expressed in terms of gamma functions [25], then:

$$\mathrm{I}=\frac{\sqrt{\pi}\lambda}{\lambda+1}\cdot\frac{\Gamma\left(\frac{\lambda}{\lambda+1}\right)}{\Gamma\left(\frac{1+3\lambda}{2+2\lambda}\right)}. \tag{12}$$

Further, according to the theorem about change of amount of motion, we have:

$$M\upsilon_0 = P_c \int_0^{t_c} \left[ \mathrm{Sa}\left( \frac{1}{\lambda}, 1, \frac{\lambda+1}{2\lambda} \gamma t \right) \right]^{1/\lambda} dt . \quad (13)$$

By substituting $\gamma t = \eta$ for expression (13) we give the form:

$$M\upsilon_0 = \frac{P_c}{\gamma} \int_0^{\mathrm{I}} \left[ \mathrm{Sa}\left( \frac{1}{\lambda}, 1, \frac{\lambda+1}{2\lambda} \eta \right) \right]^{1/\lambda} d\eta = \frac{P_c}{\gamma} \cdot \frac{2\lambda}{\lambda+1} .$$

Then:

$$\gamma = \frac{P_c}{M\upsilon_0} \cdot \frac{2\lambda}{\lambda+1} = \sqrt{\frac{2\lambda P_c^{1-\lambda}}{kM(\lambda+1)}} . \quad (14)$$

The dependence (9) is taken into account here.

It follows from (14) that:

$$P_c = \left( \frac{M\upsilon_0^2}{k} \cdot \frac{1+\lambda}{2\lambda} \right)^{\frac{1}{1+\lambda}}, \ \gamma = \left( \frac{\upsilon_0}{k} \right)^{\frac{1}{1+\lambda}} \left[ \frac{2\lambda}{(1+\lambda) M\upsilon_0} \right]^{\frac{\lambda}{1+\lambda}} . \quad (15)$$

Formulas (10) and (15) determine the maximum impact force and the change in force over time. Using these dependences and (2), the approach of the centers of mass of the bodies was found. It is expressed by:

$$x(t) = x_c \cdot \mathrm{Sa}\left( \frac{1}{\lambda}, 1, \frac{\lambda+1}{2\lambda} \gamma t \right) . \quad (16)$$

The maximum approach is:

$$x_c = kP_c^{\lambda} = k \left( \frac{M\upsilon_0^2}{2k} \cdot \frac{1+\lambda}{\lambda} \right)^{\frac{\lambda}{1+\lambda}} = \frac{\upsilon_0}{\gamma} . \quad (17)$$

The duration of the collision process, according to (11), (12), is equal:

$$t_c = \frac{\mathrm{I}}{\gamma} = \frac{x_c}{\upsilon_0} \cdot \frac{\sqrt{\pi}\lambda}{\lambda+1} \cdot \frac{\Gamma\left( \frac{\lambda}{\lambda+1} \right)}{\Gamma\left( \frac{1+3\lambda}{2+2\lambda} \right)} ,$$

which together with (15) and (17) determines the basic parameters of impact.

The second stage of impact (body stretching) is symmetrical in time relative to the vertical $t = t_c$. Therefore, formulas (10) and (16) remain valid at this stage, but it is advisable to replace them for convenience of calculations:

$$\mathrm{Sa}\left( \frac{1}{\lambda}, 1, \frac{\lambda+1}{2\lambda} \gamma t \right) \text{ on } \mathrm{Sa}\left( \frac{1}{\lambda}, 1, \frac{\lambda+1}{2\lambda} (2\mathrm{I} - \gamma t) \right).$$

Then you can get result by calculating the values of the Ateb-sine in the first quarter of its period, when $\gamma t \in [0; \mathrm{I}]$.

One-act impact ends at $t = 2t_c$.

In the case of $n = 1$, the resulting solutions take the form:

$$P(t) = P_c \left[ \mathrm{Sa}\left( \frac{3}{2}, 1, \frac{5}{4} \cdot \frac{\upsilon_0}{x_c} t \right) \right]^{3/2}; \ x(t) = x_c \cdot \mathrm{Sa}\left( \frac{3}{2}, 1, \frac{5}{4} \cdot \frac{\upsilon_0}{x_c} t \right);$$

$$P_c = \left( \frac{5M\upsilon_0^2}{4k} \right)^{3/5}; \ x_c = k^{3/5} \left( \frac{5M\upsilon_0^2}{4} \right)^{2/5} .$$

These results differ only in designation from those obtained in [21].

The case of large $n$ is also noteworthy, when the limiting surfaces of the bodies of revolution approach the planes. When $n \gg 1$, $\lambda \approx 1$, $\mathrm{Sa}\left(\frac{1}{\lambda},1,\frac{\lambda+1}{2\lambda}\gamma t\right) \approx \sin\frac{t}{\sqrt{kM}}$, $\gamma \approx \frac{1}{\sqrt{kM}}$, $P(t) \approx P_c \sin\frac{t}{\sqrt{kM}}$, $x(t) \approx x_c \sin\frac{t}{\sqrt{kM}}$, $P_c \approx \left(\frac{M\upsilon_0^2}{k}\right)^{1/2}$, $x_c \approx \upsilon_0\sqrt{kM}$.

Impact duration is $2t_c = \frac{2\mathrm{I}}{\gamma} = \frac{x_c}{\upsilon_0} \cdot \frac{\sqrt{\pi}\Gamma(1/2)}{\Gamma(1)} = \pi\sqrt{kM}$, because $\Gamma(1/2) = \sqrt{\pi}$.

For flat limiting surfaces, the problem becomes linear and the force of impact and approach of the center of mass of bodies varies according to the law of the trigonometric sine.

## Numerical results

Consider the impact of two elastic bodies with masses $m_1 = m_2 = 0.5$ kg with velocity $\upsilon_0 = 4$ m/s. The material of the bodies we take the rubber in which $E_1 = E_2 = 4 \cdot 10^6$ Pa ; $\nu_1 = \nu_2 = 0.5$. Suppose that the boundary surfaces of rotation are of the fourth order ($n = 2$), moreover $A = 10^3$ m$^{-3}$. For received numeric data $k = 2.405 \cdot 10^{-5}$ m $\cdot$ N$^{-0,8}$; $\lambda = 0.8$; $P_c = 849.069$ N; $x_c \approx 5.3 \cdot 10^{-3}$ m. To find the duration of the impact, let us consider that $\Gamma(4/9) \approx 1.99289$; $\Gamma(17/18) \approx 1.03529$. Then $t_s = 2t_c \approx 4.018 \cdot 10^{-3}$ s. Changes in the time of the force of impact and the approximation of mass centers are described by expressions:

$$\frac{P(t)}{P_c} = \left[\mathrm{Sa}\left(\frac{5}{4},1,\frac{9}{8} \cdot \frac{\upsilon_0}{x_c}t\right)\right]^{5/4}; \quad \frac{x(t)}{x_c} = \mathrm{Sa}\left(\frac{5}{4},1,\frac{9}{8} \cdot \frac{\upsilon_0}{x_c}t\right). \tag{18}$$

Therefore, for further calculations, we use the following approximation of the Ateb-sine in the first quarter of its period:

$$\mathrm{Sa}\left(\frac{5}{4}, 1, \frac{9}{8}\eta\right) \approx \begin{cases} \eta & 0 \le \eta < 0,2; \\ 0.1999 + 1.0133(\eta - 0.2) - 0.2213(\eta - 0.2)^2 \text{ when } & 0.2 \le \eta \le 0.8; \\ 1 - 1.6\sin^2\left[0.5929(\mathrm{I} - \eta)\right] & 0.8 < \eta \le \mathrm{I}, \end{cases}$$

where $\mathrm{I} \approx 1.5164$.

The results of calculations are recorded in Table 1.

*Table 1*

***The values of $P(t)$ and $x(t)$ calculated by the formulas (18)***

| $t/t_c$ | $\upsilon_0 t/x_c$ | $P(t)/P_c$ | $x(t)/x_c$ |
|---|---|---|---|
| 0.125 | 0.1895 | 0.125 | 0.189 |
| 0.250 | 0.3791 | 0.293 | 0.374 |
| 0.375 | 0.5686 | 0.466 | 0.543 |
| 0.500 | 0.7581 | 0.636 | 0.697 |
| 0.625 | 0.9476 | 0.786 | 0.825 |
| 0.750 | 1.1372 | 0.907 | 0.921 |
| 0.875 | 1.3267 | 0.975 | 0.980 |
| 1.000 | 1.5162 | 1.000 | 1.000 |

In order to compare the numerical results in Table 2 records the values $P(t)$ and $x(t)$, obtained by the numerical computer integration of equation (4), under initial conditions: $P(t_c) = P_c$; $\dot{P}(t_c) = 0$. These conditions are used due to the symmetry of the elastic impact with respect to time $t = t_c$.

*Table 2*

***Results of numerical integration of equation (4)***

| $t/t_c$ | $P(t)/P_c$ | $x(t)/x_c$ | $t/t_c$ | $P(t)/P_c$ | $x(t)/x_c$ |
|---|---|---|---|---|---|
| 0.125 | 0.125 | 0.189 | 0.625 | 0.786 | 0.825 |
| 0.250 | 0.291 | 0.373 | 0.750 | 0.902 | 0.921 |
| 0.375 | 0.468 | 0.544 | 0.875 | 0.975 | 0.980 |
| 0.500 | 0.637 | 0.698 | 1.000 | 1.000 | 1.000 |

We have small differences in the corresponding results in Table 1 and Table 2, which confirms the plausibility of the analytical solutions obtained and the proposed approximation of the Ateb-sine.

To analyze the influence of the order of the boundary surfaces on the impact characteristics, their calculation was performed when $n = 3$, with the preservation of numerical data. At closer contact of the bodies subjected to impact, $k = 6.196 \cdot 10^{-6}\ \text{m} \cdot \text{N}^{-6/7}$; $\lambda = 6/7$; $P_c = 1403{,}304$ N; $x_c \approx 3{,}088 \cdot 10^{-3}$ m; $t_s \approx 2{,}431$ s. With increasing $n$ the maximum value of the impact force increased and the maximum convergence of the centers of mass decreased and the duration of the impact process decreased.

## Conclusions

1. The generalized nonlinear differential force of the impact force has a compact analytical solution supplied by the Ateb-sine.

2. This solution gives scan to the impact process over time and the ability to calculate its main characteristics.

3. The use of the solution is greatly simplified by the presence of the Ateb-sine approximation by elementary functions.

4. The numerical results that the formulas lead to are in good agreement with the results of the computer integration of the original differential equation.

5. The theory under consideration applies only to elastic impact, which significantly limits the initial velocity of collisions of bodies.

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