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Integral conditions in the inverse problems of heat conduction

Kobilskaya E. B., Lyashenko V. P., Hryhorova T. A.

Kremenchuk Mykhailo Ostrohradskyi National University,
20 Pershotravneva Str., 39600, Kremenchuk, Ukraine

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Thermal processes of new technological methods of heat treatment (thermocyclic, electropulse) of metals and alloys are considered in the paper. Mathematical models of the temperature field in a moving tape and a wire with cyclically acting pulsed heat sources are considered. Based on these models, the formulation of inverse problems for homogeneous and inhomogeneous thermal conductivity equations is proposed. For each case (internal, external heat source or a combination), the appropriate method for solving the inverse problem is proposed. The integral condition of heat balance is used to construct the solutions of the inverse problems. An integral condition of energy balance is used to construct a quadratic residual functional in an extreme problem. The inverse problem in the case when we have a combination of internal and external periodic heat sources is solved using the search method, where the integral condition was used to find the deviations and further refinements of the desired function of the source.

Keywords: *mathematical model, inverse problem, heat sources, integral condition.*

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1. Introduction

The processes of heat treatment of moving and stationary objects are widely used in the powder metallurgy, in the production of wire, as well as other products. In this paper, the heat treatment is used as a separate operation, or in combination with plastic deformation [1–3]. This is due to the requirements to the quality for the final product. Therefore, along with plastic deformation there are various types of heat treatment by both external and internal heat sources [1–3]. It is important that the heat treatment allows the creation of physical and mechanical properties of metals and composite materials. Thermocyclic and pulse treatment using the technology of electro-plastic deformation [1–3] is especially effective during the production of ultra-fine wire.

2. Analysis of publications on the subject of the study

The papers [4–7] contain studies of the thermal processes that occur during sintering of powdered materials, various kinds of heat treatment and thermal processes that occur during wire drawing. Mathematical models describe temperature distribution during processing of moving and stationary wire and other products of cylindrical shape.

Initial boundary value problems for linear and quasi-linear heat equation in cylindrical coordinate system (r, z, ϕ, t) are considered as mathematical models.

From a mathematical point of view, the study of the temperature distribution in moving and stationary axially symmetric objects can be carried out through considering different initial boundary value problems for linear and nonlinear heat equations by introducing certain restrictions on the equation and attracting the appropriate boundary conditions that characterize the heating process. Since most of the temperature distributions investigated by heating the product of cylindrical shape do not depend on the coordinates ϕ, the partial derivative with respect to this variable in the heat equation can be neglected. Wires and other cylindrical shaped product are considered in the form of a moving or stationary cylindrical isotropic medium with constant parameters and thermal characteristics with the heating zone of the length L. In this case mathematical models of temperature fields, in which there are both external and internal sources of heat have been already studied [5, 6]. Internal heat sources $W(z,t,T)$ are caused by an electric current, which flows through the medium. External heat sources are caused by a heat exchange of the product with the environment and are governed by the Newton and Stefan–Boltzmann laws.

The internal heat sources in the mathematical model are represented in the form of a finite function $W(z,t,T)$ in the equation, and external heat sources are represented as boundary conditions of the first, second or third kind.

Influence of non-linear components in the equation and the boundary conditions on the temperature distributions is considered in the mathematical models.

3. The solution of the boundary problem for the heat equation with an external periodically operating heat sources

In the mathematical model, the external heat sources are represented in the form of boundary conditions of the first, second, or third kind. The mathematical model of the temperature field during pulsed treatment of a sample of the cylindrical shape, has the form [7]

$$\lambda\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\lambda\frac{\partial^2 T}{\partial z^2}-vc\rho_n\frac{\partial T}{\partial z}-c\rho_n\frac{\partial T}{\partial t}=0,\quad (r,z)\in\Omega_t, \tag{1}$$

$$T(r,z,0)=T_0, \tag{2}$$

$$T(r,0,t)=T_0,\quad T(r,l,t)=T_l, \tag{3}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0,\quad \lambda\left.\frac{\partial T}{\partial r}\right|_{r_0=0}=f_{12}(t)\left[-\alpha(T)(T-T_c)-\varepsilon\sigma\left(T^4-T_c^4\right)\right], \tag{4}$$

where λ, c, ρ_n are thermal conductivity, heat capacity and density, T_c is ambient temperature, $\alpha,\varepsilon,\sigma$ are the convective heat transfer coefficient from the surface, the emissivity factor and the Stefan–Boltzmann constant. By using the relation [8]

$$u(z,t)=\frac{2}{r_0^2}\int_0^{r_0}T(r,z,t)\,r\,dr \tag{5}$$

and the boundary condition (3), we obtain a problem for the determination of the average temperature along the radius in the area $Q_T=\{(z,t)|z\in(0,l),t\in(0,t_0)\}$

$$\lambda\frac{\partial^2 u}{\partial^2 z}-v(t)c\rho_n\frac{\partial u}{\partial z}-c\rho_n\frac{\partial u}{\partial t}=\frac{2\varepsilon\sigma}{r_0}\left(T_c^4-u^4\right)+f_{12}(t)\frac{2}{r_0}\alpha(T_c-u), \tag{6}$$

$$u(z,0)=T_0, \tag{7}$$

$$u(0,t)=T_0,\quad u(l,t)=T_l. \tag{8}$$

In the latter formula, the heat exchange surface of the moving medium with the environment is kept in mind. The transformation (5) makes it possible to reduce the dimension of the problem and reduce it to a solution of the first boundary value problem for the quasilinear heat equation. Substituting the variables (9) in problem (6)–(8), we reduce Eq. (6) to the canonical form

$$u(z,t)=U(z,t)e^{\mu(t)z+\eta(t)t} \tag{9}$$

and we obtain the following problem in the area $\overline{Q_T}=\{(z,t)|z\in(0,l),t\in(0,t_0)\}$

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial^2 z} + f(z,t), \quad z(t) \in Q_T, \tag{10}$$

$$U(z,0) = T_z = T_0 e^{-\mu(0)z}, \quad 0 \leqslant z \leqslant l, \tag{11}$$

$$U(0,t) = T(t) = T_0 e^{-\eta(t)t}, \quad T(z) = T(t), \quad U(l,t) = T_1(t), \quad 0 \leqslant t \leqslant t_0, \tag{12}$$

where

$$f(z,t) = f_{12}(t)\left(\frac{-2\alpha T_c \pi^2}{\pi^2 r_0 c\rho_n}\right) e^{-\mu z - \eta(t)t}, \quad \mu = \frac{vc\rho_n}{2\lambda}, \quad \eta(t) = \frac{-v^2 c\rho_n}{4\lambda} + f_{12}(t)\frac{2\alpha}{r_0 c\rho_n}.$$

We have a correctly formulated problem, because the solution of problem (10)–(12) exists, it is unique and stable with respect to small perturbations $f(z,t)$, $T(z)$, $T_1(z)$.

Problem of reconstructing the heat source. In the case where the heat source is a known function, we come to the inverse problem.

Such problems arise during control of temperature fields. Here, it is necessary to find temperature distribution in the axisymmetric environment. Let the thermal characteristics of the medium be constant. After applying averaging (6) and further transformations the problem (1)–(4) is transformed into (10)–(12). Here, the function $f(z,t)$ can be determined fully only when we know the temperature distribution throughout the heating area. Therefore, during assigning the heat sources we assume the amount of energy that turns into heating of the area and loss from the surface to be a known. The condition of the heat balance in the case of an external source takes the form

$$\int_{0+}^{t_0} \iint_G \alpha(T) \frac{T(r,z,t) - T_c}{v}\, dg\, dt = c\rho_n \int_{0+}^{t_0} \int_0^{r_0} \int_0^l f_{12}(t)\big(T(r,z,t) - T_0\big)\, dz\, dr\, dt. \tag{13}$$

After averaging (5) and applying (9) to the condition (13), we have

$$\int_{0+}^{t_0} \int_0^l \alpha(T) \frac{U(z,t)e^{\mu z + \eta(t)t} - T_c}{v}\, dz\, dt = c\rho_n \int_{0+}^{t_0} \int_0^l f_{12}(t)\left(U(z,t)e^{\mu z + \eta(t)t} - T_0\right) dz\, dt. \tag{14}$$

Based on Eq. (14) we introduce the quadratic residual functional [9, 10]

$$J(f) = \int_{\varepsilon+}^{t_0} \int_0^l \left[W_1(U,f) - W_2(U)\right]^2 dz\, dt,$$

$$W_1(U,f) = \alpha(T) \frac{U(z,t)e^{\mu z + \eta(t)t} - T_c}{v}, \quad W_2(U) = f_{12}(t)\left(U(z,t)e^{\mu z + \eta(t)t} - T_0\right).$$

We complete the statement of an inverse problem by adding initial and boundary conditions to the heat equation. We have the inverse problem (10)–(12) in the area $\overline{Q_T} = \{(z,t) | z \in (0,l), t \in (0,t_0)\}$. The function $f(z,t)$ in (10) is found under the condition of a minimum of quadratic residual functional and restriction $J(f) \geqslant \delta^2$,

$$\delta^2 = \int_{0+}^{t_0} \int_0^l \sigma^2 dz\, dt,$$

where σ^2 is dispersion of the function $W_2(v)$ [9, 10].

The iterative process is built in the space $L_2(Q)$, $Q = \Omega \times [0, \tau_m]$ using a conjugate gradient method [9–11]

$$f^{k+1} = f^k - b_k S^k, \qquad k = 0, 1, \ldots, \overline{k}, \tag{15}$$

where $S^k = J_f'^k + \gamma_k S^{k-1}$, $\gamma_k = \|J_f'^k\|^2 / \|J_f'^{k-1}\|^2$, $\beta_k = \frac{U^k - f, V^k}{\|V_k\|^2}$, $U^k = U(f^k, z, \tau)$ is the temperature field at the kth iteration, $V^k = V(\Delta f^k, z, \tau)$ the gradient of the temperature field at the kth iteration, when the source varies by amount Δf^k; $(u,w) = \iint_Q u(z,t)w(z,t)\, dz\, dt$, $\|u\| = \sqrt{(u,u)}$ scalar product of elements $u(z,t)$, $w(z,t)$ and the norm of element u in space $L_2(Q)$. The gradient of temperature field

V^k is obtained from the solution of the homogeneous boundary value problem for the inhomogeneous equation

$$\frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial^2 z} + \Delta f(z,t), \quad (z,t) \in Q_T, \tag{16}$$

$$V(z,0) = 0, \quad 0 \leqslant z \leqslant l, \tag{17}$$

$$V(0,t) = 0, \quad V(l,t) = 0, \quad 0 \leqslant t \leqslant t_0. \tag{18}$$

We get the objective functional gradient using the conjugate variable $\psi(z,t)$. Identity $(Lv,\psi) = (v, L*\psi)$ allows us to write the conditions of the conjugated problem:

$$L*\psi = \zeta, \quad (z,t) \in Q_T, \tag{19}$$

$$\psi(z,t_m) = 0, \tag{20}$$

$$\psi|_{\partial\Omega} = 0, \tag{21}$$

where $L*\psi = \psi_t + a^2\psi_{zz}$, $\zeta = \zeta(z,t)$ some function. The formula for the gradient can be written as $J'_q = \psi$, $(z,t) \in Q$.

For the organization of the iterative process (15) at each step we calculate the temperature, temperature gradient and the conjugate variable. To find $U(z,t)$, $V(z,t)$, $\psi(z,t)$ we need to solve all three problems (10)–(12), (16)–(18), and (19)–(21). To this end we use a six-point difference scheme [10]. Let us introduce the grid $\overline{\omega_h} = \{z_i = ih, i = 0,1,\ldots,N\}$, $\omega_\tau = \{t_j = j\tau, j = 0,1,\ldots,j_0\}$ and the grid $\varpi_{h\tau} = \varpi \times \omega_\tau = \{(ih, j\tau), i = 0,1,\ldots,N, j = 0,1,\ldots,j_0\}$ with steps $h = \frac{1}{N}$, $\tau = \frac{t_0}{j_0}$. Let us denote by y_i^j the values in nodes (z_i,t_j) of grid function U, which is defined on $\varpi_{h\tau}$. We replace the derivative $\frac{\partial U}{\partial t}$ by a first difference derivative, and the derivative $\frac{\partial^2 U}{\partial^2 z}$ by a second difference derivative $U_{\overline{z}z}$. Then we will enter arbitrary real parameter σ and consider one-parameter family of difference schemes, i.e.,

$$\frac{y_i^{j+1} - y_i^j}{\tau} = \Lambda(\sigma y_i^{j+1} + (1-\sigma) y_i^j) + \varphi_i^j, \quad 0 < i < N, \quad 0 \leqslant j < j_0, \tag{22}$$

$$y_0^j = T_1^j, \quad y_N^j = T_2^j, \quad y_i^0 = y(z_i,0) = T(z_i), \quad \varphi_i^j = f(z_i, t_{j+0,5}),$$

$$t_{j+0.5} = t_j + 0.5\tau,$$

or

$$\varphi_i^j = 0.5(\overline{f} + f), \quad \overline{f} = f(z_i, t_{j+0.5}), \quad \Lambda y_i = y_{\overline{z}z,i} = (y_{i-1} - 2y_i + y_{i+1})/h^2.$$

The difference scheme (22) is written in a six-point pattern. At $\sigma = 0.5$ it is absolutely stable with respect to the initial data and the right hand side [10]. Then, the sweep method is used for solving the system of equations.

4. Solution to the heat conduction problem with internal heat sources

Consider a general formulation of the problem of determining the temperature field in a moving medium, which is heated by internal heat sources. The heating process occurs in two stages. In the first stage the medium is heated with variable speed $v(t)$. This is an unsteady transition process. During the transition process, which occurs during the time $0 \leqslant t \leqslant t_0$, the medium speed $v(t)$ varies within the range $0 \leqslant v(t) = \text{const}$.

If during the transition process the heat sources with unchanged parameters act in the heating zone, then the temperature field will be of nonsteady nature. Along with the question of determining the temperature distribution during the transition process there appears a technical problem how to define the control parameters of heating process, in order for the temperature of the field during medium movement with a variable $v(t)$ speed to be steady. This can be achieved if we choose the appropriate heat source density.

Starting from the time t_0, for $t \geqslant t_0$ the speed of the wire becomes constant $v = \text{const}$. The choice of control parameters allows maintaining a processing rate required from a technological point of view in the area of the EPD. The density of the sources can be chosen in such a way that during the analyzed period of time the temperature should remain near the predetermined one.

The definition of nonstationary temperature distribution during the transition process leads to the solution of the following initial boundary value problem for the heat equation in area Ω: $\{(z,r,t)|0 < z < l, 0 < r < r_0, 0 < t \leqslant t_0)\}$ [4–7]

$$\lambda\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \lambda\frac{\partial^2 T}{\partial z^2} - v(t)c\rho_n\frac{\partial T}{\partial z} - c\rho_n\frac{\partial T}{\partial t} = -W(z,t,T), \quad (r,z,t) \in \Omega_t,$$

$$T(r,z,0) = T_0,$$

$$T(r,0,t) = T_0, \quad T(r,l,t) = T_l,$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0, \quad \lambda\left.\frac{\partial T}{\partial r}\right|_{r_0=0} = f_{12}(t)\left[-\alpha(T)(T-T_c) - \varepsilon\sigma\left(T^4 - T_c^4\right)\right], \tag{23}$$

where λ, c, ρ_n are thermal conductivity, heat capacity and density, T_c is ambient temperature, $\alpha, \varepsilon, \sigma$ are the convective heat transfer coefficient from the surface, the emissivity factor and the Stefan–Boltzmann constant. The heat sources function $W(z,t,T)$ when depends on the coordinates and time takes the form [4]

$$W(z,t,T) = f_{11}(z)f_2(T), \quad 0 < t \leqslant t_0, \tag{24}$$

$$W(z,t,T) = f_{12}(t)f_2(T), \quad t > t_0,$$

$f_2(T) = \frac{I^2\rho_0(1+\beta T)}{\pi^2 r_0^4}$, ρ_0, β are resistivity and temperature coefficient of resistance of the wire.

Using the relation (5) and the boundary condition (23), we obtain the problem of determining the average temperature along the radius in the area $\overline{Q_T} = \{(z,t)|z \in (0,l), t \in (0,t_0)\}$

$$\lambda\frac{\partial^2 u}{\partial^2 z} - v(t)c\rho_n\frac{\partial u}{\partial z} - c\rho_n\frac{\partial u}{\partial t} + f_{12}(t)\left(\frac{2}{r_0}\alpha + \frac{\beta\rho_0 I^2}{\pi^2 r_0^4}\right) = \frac{2\varepsilon\sigma}{r_0}(T_c^4 - u^4) + f_{12}(t)\frac{\rho_0 I^2}{\pi^2 r_0^4} + f_{12}(t)\frac{2}{r_0}\alpha T_c, \tag{25}$$

$$u(z,0) = T_0, \tag{26}$$

$$u(0,t) = T_0, \quad u(l,t) = T_l. \tag{27}$$

Consider the task of restoring the internal heat source. In this case, we assume the amount of energy that transforms into heating the environment and loss from the surface is should to be known,

$$\int_{\varepsilon+}^{t_0}\int_{0+}^{r_0}\int_0^l f_{12}(t)\frac{I(t)^2\rho_0 l + \beta I(t)^2\rho_0 l T(r,z,t)}{v(t)r_0^4\pi^2}\,dz\,dr\,dt = c\rho_n\int_{\varepsilon+}^{t}\iint_G \big(T(r,z,t) - T_0\big)dg\,dt$$
$$+ \frac{\alpha l}{r_0}\int_{\varepsilon+}^{t_0}\int_{0+}^{r_0}\int_0^l f_{12}(t)\frac{T(r,z,t) - T_c}{v(t)}\,dz\,dr\,dt. \tag{28}$$

We have the inverse problem (25)–(28) in the area $\overline{Q_T} = \{(z,t)|z \in (0,l), t \in (0,t_0)\}$. After averaging over the radius and assuming that the temperature field is independent of the angular coordinate, the condition, Eq. (28) takes the form [4–7]

$$\int_{\varepsilon+}^{t_0}\int_0^l f_{12}(t)\frac{I(t)^2\rho_0 l + \beta I(t)^2\rho_0 u^*(z,t)l}{v(t)r_0^4\pi^2}\,dz\,dt = c\rho_n\int_{\varepsilon+}^{t}\int_0^l (u^*(z,t) - T_0)\,dz\,dt$$
$$+ \frac{\alpha l}{r_0}\int_{\varepsilon+}^{t_0}\oint_l f_{12}(t)\frac{u^*(z,t) - T_c}{v(t)}\,dz\,dr\,dt. \tag{29}$$

Based on Eq. (29) we introduce the quadratic residual functional [9–11]

$$J(f)=\int_{\varepsilon+}^{t_0}\int_0^l \left[W_1(U,f)-W_2(U)\right]^2 dz\,dt,$$

$$W_1(U,f)=f_{12}(t)\frac{I(t)^2\rho_0 l+\beta I(t)^2\rho_0 u^*(z,t)l}{v(t)r_0^4\pi^2},$$

$$W_2(U)=c\rho_n\left(u^*(z,t)-T_0\right)+\frac{\alpha l}{r_0}f_{12}(t)\frac{u^*(z,t)-T_c}{v(t)}.$$

The further treatment of the problem with internal heat sources is similar to the solution of the problem with an external heat source.

5. Boundary value problems of heat conduction with a combination of internal and external heat sources

We consider a mathematical model of the thermal process with a combination of internal and external periodically-operating heat sources, i.e., the heat sources are reflected by both the equation and the boundary conditions,

$$\lambda\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\lambda\frac{\partial^2 T}{\partial z^2}-vc\rho_n\frac{\partial T}{\partial z}-c\rho_n\frac{\partial T}{\partial t}=-W(z,t,T),\quad (r,z,t)\in\Omega_t,$$

$$T(r,z,0)=T_0,$$

$$\begin{aligned}\lambda\left.\frac{\partial T}{\partial z}\right|_{z=0}&=f_{1i}(t)\left[\alpha(T)(T-T_c)+\varepsilon\sigma\left(T^4-T_c^4\right)\right],\\ \lambda\left.\frac{\partial T}{\partial z}\right|_{z=L}&=f_{1i}(t)\left[\alpha(T)(T-T_c)+\varepsilon\sigma\left(T^4-T_c^4\right)\right],\end{aligned}\tag{30}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0,\quad \lambda\left.\frac{\partial T}{\partial r}\right|_{r_0=0}=f_{1i}(t)\left[-\alpha(T)(T-T_c)-\varepsilon\sigma\left(T^4-T_c^4\right)\right],\tag{31}$$

where $W(z,t,T)$ in the case of dependent on the coordinates and time heat sources get the form (24). After averaging of Eq. (5) we attain to the problem in the area $\overline{Q_T}=\{(z,t)|z\in(0,l),t\in(0,t_0)\}$

$$\lambda\frac{\partial^2 u}{\partial^2 z}-vc\rho_n\frac{\partial u}{\partial z}-c\rho_n\frac{\partial u}{\partial t}=F(T,t),\tag{32}$$

$$u(z,0)=T_0,\tag{33}$$

$$\begin{aligned}\lambda\left.\frac{\partial T}{\partial z}\right|_{z=0}&=\pm f_{12}(t)\left[\alpha(T)(T_c-u)+\varepsilon\sigma\alpha_T\left(T_c^4-u^4\right)\right],\\ \lambda\left.\frac{\partial T}{\partial z}\right|_{z=L}&=\pm f_{12}(t)\left[\alpha(T)(u-T_c)+\varepsilon\sigma\alpha_T\left(u^4-T_c^4\right)\right],\end{aligned}\tag{34}$$

$$u(z,0)=T_0(t),\tag{35}$$

$$F(T,t)=\frac{2\varepsilon\sigma}{r_0}\left(T_c^4-u^4\right)+f_{12}(t)\frac{\rho_0 I^2}{\pi^2 r_0^4}(1+\beta u)-f_{12}(t)\frac{2}{r_0}\alpha u+f_{12}(t)\frac{2}{r_0}\alpha T_c.$$

We need to find $I(u)$, which is the function of temperature

$$\begin{aligned}\int_{\varepsilon+}^{t_0}\int_{0+}^{r_0}\int_0^l f_{12}(t)\frac{I(t)^2\rho_0 l+\beta I(t)^2\rho_0 lT(r,z,t)}{v(t)r_0^4\pi^2}\,dz\,dr\,dt&=c\rho_n\int_{\varepsilon+}^{t_0}\iint_G\big(T(r,z,t)-T_0\big)dg\,dt\\&+\frac{\alpha l}{r_0}\int_{\varepsilon+}^{t_0}\int_{0+}^{r_0}\int_0^l f_{12}(t)\frac{T(r,z,t)-T_c}{v(t)}\,dz\,dr\,dt.\end{aligned}$$

We start the solution process with a priori setting the value of $I(u)$, which can be determined from the integral equation (34), where the function $u^*(z,t)$ is taken from the solution of the simplified problem. Then we will find a solution to problem (30)–(33). This is a correctly formulated direct heat conduction problem and can be solved numerically with the help of a difference scheme. In the problem (30)–(33) the right side of the equation, the boundary conditions contain piecewise continuous functions. To obtain the solution, we will build a conservative difference scheme using the balance method. Conservative schemes can be obtained by the balance method (integro-interpolation method) [10].

Deviation of the model function $u_\beta(z,t)$ that is found in the approximation β and given by the condition (34) is used as an error signal for further clarification $I(u)$. To determine the variation of the function $I(u)$, which reduces the residual $\eta = u_M(0,t) - T_0(t)$ by combining method [11–14], we will decompose $I(u)$ into Taylor series in powers u and consider only finite number of terms

$$I(u) = I_0 + I_1 u + I_2 u^2 + I_3 u^3 + \ldots + I_j u^j.$$

This polynomial contains the $j+1$ of variable parameters I_j. If the function $T_0(t)$ varies monotonically, then we will present Eq. (35) in the form

$$I(u) = a_0 + a_1(\overline{u} - \overline{T}_{0k}) + a_2\overline{u}(\overline{u} - \overline{T}_{0k}) + a_3\overline{u}(\overline{u} - \overline{T}_{0k})^2 + a_4\overline{u}^2(\overline{u} - \overline{T}_{0k})^2 + \ldots,$$

$$\overline{u} = u - T_0(0),$$

$$\overline{T}_{0k} = T_0(t_k) - T_0(0).$$

On the right end of the interval $[0,t_k]$, at $\overline{u} = \overline{T}_{0k}$, the unknown coefficient $a_j(I_j)$ depends only on a_0. To find the parameters $a_j(I_j)$ $(j = 1,2,\ldots,J)$ to each of them is assigned a function value $T_0(t_j)$ at a certain point t_j of the interval $[0,t_k]$. Thus, the function value $T_0(t_k)$ at time t_k corresponds to the value a_0. Search for optimal parameter values $a_j(I_j)$ which provide for the residual $\xi_j = u_m(t_j) - T_0(t_j)$ condition $|\xi_j| \leqslant \delta$ is based on a method of consistent residuals minimization by the following algorithm.

To each parameter a_j a certain residual ξ_j $(j = 0,1,2,\ldots,J)$ is assigned. The test step Δa_1 is performed by the first parameter a_1 and the solution (direct problem) is found, which results in determination of the increment of the corresponding residuals $\Delta\xi_1$ and the approximate value of the derivative $\frac{\partial\xi_1}{\partial a_1} = \frac{\Delta\xi_1}{\Delta a_1}$ is calculated. Next, several working steps on the parameter a_1 at a fixed value of other parameters is done until the condition $|\xi_j| \leqslant \delta$ for the residual ξ_j $(j = 1)$ is satisfied. The value of working steps is determined according to the following formula at $A_j = 1$

$$\Delta a_j = -A_j \xi_j \frac{\partial a_1}{\partial \xi_1}. \tag{36}$$

Thus, the derivative $\frac{\partial a_1}{\partial \xi_1}$ can be refined after each βth working step, i.e.,

$$\left(\frac{\partial a_j}{\partial \eta_j}\right)^{(\beta)} = \frac{a_j^{(\beta)} - a_j^{(\beta-1)}}{\xi_j^{(\beta)} - \xi_j^{(\beta-1)}}.$$

Suppose that for the residuals ξ_j $(j = 1,2,\ldots,s, s < J)$, which correspond to the parameters $a_1, a_2, \ldots, a_s$, condition $|\xi_j| \leqslant \delta$ is satisfied. In this case, the values of the parameters $a_1, a_2, \ldots, a_{s+1}$, which are subject to the inequality $|\xi_j| \leqslant \delta$, ξ_j $(j = 1,2,\ldots,s+1)$ are determined. In the beginning, we make a test step Δa_{s+1} on the parameter a_{s+1} and then we calculate the derivative $\frac{\partial\xi_{s+1}}{\partial a_{s+1}} = \frac{\Delta\xi_{s+1}}{\Delta a_{s+1}}$. Then the first working step is performed on this parameter, the step size is determined in accordance with (36). Each following step on the parameter a_{s+1} occurs after calculation cycle that is associated with a change of the parameters $a_1, a_2, \ldots, a_{s+1}$ untill condition $|\xi_j| < \delta$ $(j = 1,2,\ldots,s)$ is satisfied. In this case, we do not implement test steps to find the derivative $\frac{\partial\xi_j}{\partial a_j}$ $(j = 1,2,\ldots,s)$, in contrast to the case of the calculation cycle, which preceded the change of parameter a_{s+1}. The derivative $\frac{\partial\xi_j}{\partial a_j}$ for the first working step is taken from the previous cycle, and, in this case, the increment Δa_j is calculated according to the formula (32) at $0 < A_j \leqslant 1$. The calculation is terminated when all the components ξ_j $(j = 1,2,\ldots,s, s < J)$ of the residual vector satisfy the condition $|\xi_j| \leqslant \delta$.

6. Conclusions

A mathematical model of thermal processes in a cylindrical area with the existing internal and external heat sources is considered. We have investigated the methods for solving the inverse boundary value problems of heat conduction. From the physical point of view, the problem describes thermal processes during thermal cycling and electric pulse treatment of cylindrically shaped samples.

Depending on the type of the inverse problem and the known parameters of the process describing the mathematical model, the appropriate method of its solution is proposed. In particular, the problem of restoration of the right side of a parabolic equation is considered. We have shown that the problem of finding the temperature field and concentration distribution of both internal and external heat sources can be reduced to external ones and solved by one algorithm.

[1] Maeno T., Mori K., Nagai T. Improvement in formability by control of temperature in hot stamping of ultra-high strength steel parts. CIRP Ann. Manuf. Technol. **63** (1), 301–304 (2014).

[2] Spitsin V. I., Troitsky O. A. Electroplastic Deformation of Metals. Nauka, Moscow (1985), (in Russian).

[3] Holzweissig M. J., Lackmann J., Konrad S., Schaper M., Niendorf T. Influence of Short Austenitization Treatments on the Mechanical Properties of Low-Alloy Steels for Hot Forming Applications. Metall. Mater. Trans. A. **46** (7), 3199–3207 (2015).

[4] Troitsky O. A., Stashenko V. I., Ryzhkov V. G., Lyashenko V. P., and Kobilskaya E. B. Electroplastic wire drawing and new technologydevelopment light wire. Problems of Atomic Science and Technology. **4**, 111–117 (2011).

[5] Lyashenko V., Kobilskaya E. Control of heat source in a heat conduction problem. AIP Conference Proceedings. **1629** (1), 94–101 (2014).

[6] Lyashenko V., Hryhorova T. Generalized mathematical model of thermal diffusion in powder metallurgy. AIP Conference Proceedings. **1629** (1), 85–93 (2014).

[7] Lyashenko V., Hryhorova T. Investigation of the temperature field of a two-layer cylinder with different thermophysical characteristics. Visnyk of V. N. Karazin Kharkiv National University. **890**, 47–52 (2010).

[8] Lykov A. V. Theory of Heat Conductivity. Vyshcha shkola, Moscow (1967), (in Russian).

[9] Alifanov O. M. Inverse Problems of Heat Transfer. Machinery, Moscow (1988), (in Russian).

[10] Samarsky A. A., Vabishchevich P. N. Computational Heat Transfer. Editorial URSS, Moscow (2003).

[11] Nikitenko N. I. The theory of Heat and Mass Transfer. Naukova Dumka, Kiev (1983), (in Russian).

[12] Chapko R. On the numerical solution of the first initial boundary value problems for heat equation in the torus case. Journal of Engineering Mathematics. **43**, 75–87 (2002).

[13] Jonas P., Louis A. K. Approximate Inverse for a one dimensional Inverse Heat Conduction Problem. Inverse Problems. **16**, 175–185 (2000).

[14] Wang P., Zheng K. Reconstruction of spatial heat sources in heat conduction problems. Applicable Analysis. **85** (5), 459–465 (2006).

Інтегральні умови в обернених задачах теплопровідності

Кобильська О. Б., Ляшенко В. П., Григорова Т. А.

Кременчуцький національний університет ім. М. Остроградського, вул. Першотравнева, 20, 61046, Кременчук, Україна

У роботі розглянуто теплові процеси під час нових технологічних методів обробки металів і сплавів (термоциклічної, електроімпульсної). Побудовано математичні моделі температурного поля в рухомій стрічці та дроті з циклічно діючими імпульсними джерелами тепла. На основі цих моделей запропоновано формулювання обернених задач для однорідних та неоднорідних рівнянь теплопровідності. Для кожного випадку (внутрішнього, зовнішнього джерела тепла або їх комбінації) запропоновано відповідний метод розв'язування оберненої задачі. Інтегральна умова балансу тепла використовується для побудови розв'язків обернених задач. Зокрема, інтегральна умова балансу тепла використовується для побудови квадратичного функціоналу якості в екстремальній задачі. Обернену задачу у випадку, коли є комбінація внутрішніх та зовнішніх періодично діючих джерел тепла, розв'язано за допомогою пошукового методу, де інтегральна умова використана для пошуку відхилень та подальшого уточнення потрібної функції джерела.

Ключові слова: *математична модель, обернена задача, джерело тепла, інтегральна умова.*