The study of heat transfer and stress-strain state of a material, taking into account its fractal structure

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In the work, on the basis of the apparatus of fractional integro-differentiation, the mathematical models of heat-and-moisture transfer and of deformation-relaxation processes in the medium with "memory" effects and self-organization are constructed. Numerical implementation of the mathematical models of heat-transfer and moisture-transfer is based on the adaptation of the predictor-corrector method. That is why the mathematical models obtained in this work are in a finite-difference form. For the explicit difference scheme, the stability conditions are determined on the basis of the method of conditionally defined known functions as well as by means of the Fourier integral method. An integral representation of the deformation and stress of the fractional-differential rheological model is obtained using the Laplace transform method. Including the numerical and analytical methods of implementation of the constructed models, in this paper, the main results are presented, in particular, identification of fractal parameters for the creep function according to the experimental data.

Keywords: *fractal structure, self-organization, heat-and-mass transfer, fractional-differential apparatus, Kelvin model.*

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1. Introduction

Today, one can observe the development of a fractional integro-differential apparatus and its use for modeling direct and inverse problems in various fields of science, including the processes of visco-elastic deformation and heat-and-mass transfer. Non-integer integro-differentiation was used in [1–5] to model systems characterized by biological variability in rheological properties, "memory" effects, structural heterogeneity, spatial non-locality, deterministic chaos, and self-organization. It should be noted that despite the considerable number of works devoted to the study of non-integer derivatives and integrals, as well as their application to modeling, there are still a number of unresolved and controversial issues. In particular, the ambiguity of the application of the mathematical apparatus of fractional operators of integration and differentiation consists in the absence of an explicit physical, geometric, and probabilistic interpretation of such operations, the imposition of the properties of an integer differentiation apparatus and various approaches to its definition – Riemann-Liouville, Kaputo, Grunwald-Letnikov, Weil, Marshaud, Riesz, Wright. The studies of deformation-relaxation processes in the works of scientists [6, 7] have shown that the use of a fractional integro-differential apparatus for such processes makes it possible to more adequately, based on physical considerations, generalize experimental data to identify model parameters. At the initial stage are the studies devoted to the search for an effective method for identifying fractal parameters of models, in particular, this is partially reflected in [8,9]. In order to calculate heat-and-mass transfer and deformation processes, taking into account the effect of "memory" and self-organization, it is important to solve the problem of identifying fractional differential parameters of models, since taking into account of certain continuum of the found parameters, it

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is possible to investigate the stress-strain state of the material to ensure its required quality, to create new mathematical models based on the existing ones. To solve mathematical models of heat-and-mass transfer and visco-elastic deformation in media with a fractal structure, both analytical and numerical methods of implementation are used. In the works [10–13] the greatest preference for the analysis and derivation of fractional integro-differential equations is given to analytical methods, for example, the integral transform methods of Laplace, Mellin, Fourier, Green's functions, and the spectral method using Laguerre polynomials. However, since analytical methods are limited in application, numerical methods [14–16], such as the finite element method [17] and the finite difference method, are more efficient and easier to apply. The finite-difference approximations for the implementation of mathematical models, which provides accounting for the eridarity and self-organization of the material, were used in the works [2, 18]. Thus, as a result of the analysis, it can be noted that the use of non-integer integro-differentiation for modeling deformation and heat-and-mass transfer processes makes it possible to describe new properties of material, such as the effects of "memory", self-organization, heterogeneity of structure, and its self-similarity. That is why the purpose of this work is the construction and study of mathematical models of deformation-relaxation, moisture-and-heat transfer processes of capillary-porous materials, taking into account their fractal structure, as well as the development of finite-difference schemes for approximating these mathematical models.

2. Mathematical model of heat and moisture transfer taking into account the effect of "memory" and self-organization and its numerical realization

A two-dimensional mathematical model of heat-and-moisture transfer of capillaryporous materials can be described by an interconnected system of partial differential equations of fractional order in time t and spatial coordinates x_1 and x_2:

$$c\rho\frac{\partial^{\alpha}T\left(t,x_1,x_2\right)}{\partial t^{\alpha}}=\lambda_1\frac{\partial^{\beta}T\left(t,x_1,x_2\right)}{\partial x_1^{\beta}}+\lambda_2\frac{\partial^{\beta}T\left(t,x_1,x_2\right)}{\partial x_2^{\beta}}+\varepsilon\rho_0 r\frac{\partial^{\alpha}U\left(t,x_1,x_2\right)}{\partial t^{\alpha}},\tag{1}$$

$$\frac{\partial^{\alpha}U\left(t,x_1,x_2\right)}{\partial t^{\alpha}}=a_1\frac{\partial^{\beta}U\left(t,x_1,x_2\right)}{\partial x_1^{\beta}}+a_2\frac{\partial^{\beta}U\left(t,x_1,x_2\right)}{\partial x_2^{\beta}}+a_1\delta\frac{\partial^{\beta}T\left(t,x_1,x_2\right)}{\partial x_1^{\beta}}+a_2\delta\frac{\partial^{\beta}T\left(t,x_1,x_2\right)}{\partial x_2^{\beta}}.\tag{2}$$

We add the boundary conditions of the third kind:

$$\lambda_i\left.\frac{\partial^{\gamma}T}{\partial x_i^{\gamma}}\right|_{x_i=0,l_i}+\rho_0\left(1-\varepsilon\right)\beta^*\big(U|_{x_i=0,l_i}-U_p\big)=\alpha^*\big(\,T|_{x_i=0,l_i}-t_c\big),\tag{3}$$

$$a_i\delta\left.\frac{\partial^{\gamma}T}{\partial x_i^{\gamma}}\right|_{x_i=0,l_i}+a_i\left.\frac{\partial^{\gamma}U}{\partial x_i^{\gamma}}\right|_{x_i=0,l_i}=\beta^*\big(U_p-U|_{x_i=0,l_i}\big).\tag{4}$$

And also the initial conditions:

$$T(0,x_1,x_2)=T_0(x_1,x_2),\quad U(0,x_1,x_2)=U_0.\tag{5}$$

Where $(t,x_1,x_2)\in D$, $D=[0,\tau_{\max}]\times[0,l_1]\times[0,l_2]$, T is temperature, U is moisture content, $c(T,U)$ is specific heat capacity, $\rho(U)$ is density, ρ_0 is base density, ε is phase transition coefficient, r is specific heat of vaporization, $\lambda_i(T,U)$ $(i=1,2)$ are coefficients of thermal conductivity, $a_i\ (T,U)$ $(i=1,2)$ are coefficients of moisture conductivity, $\delta(T,U)$ is thermo-gradient coefficient, t_c is ambient temperature value, $U_p(t_c,\varphi)$ is equilibrium moisture content, φ is relative moisture content of the drying agent, $\alpha^*(t_c,\upsilon)$ is coefficient of heat exchange, υ is the speed of drying agent movement, $\beta^*(t_c,\phi,\upsilon)$ is coefficient of moisture exchange, α is fractional order of time derivative, $(0<\alpha\leqslant 1)$, β, γ are fractional indices of the derivative by spatial coordinates $(1<\beta\leqslant 2)$, $(0<\gamma\leqslant 1)$.

Based on the use of fractional integro-differential apparatus finite-difference schemes are developed for approximation of the mathematical model (1)–(5). So, the difference approximation of the fractional derivative α in the time interval $[t^k, t^{k+1}]$, taking into account the Riemann–Liouville formula [19], can be written as follows:

$$\left.\frac{\partial^\alpha u}{\partial t^\alpha}\right|_{t^k} \approx \frac{u^{k+1} - \alpha u^k}{\Gamma(2-\alpha)\,\Delta t^\alpha}, \quad \left(\Delta t = t^{k+1} - t^k\right). \tag{6}$$

Using the Grunwald–Letnikov formula [19], the difference approximations of the fractional derivative β by spatial coordinates x_1 and x_2 are written as follows:

$$\left.\frac{\partial^\beta u}{\partial x_i^\beta}\right|_{x_{i(ni)}} \approx \frac{1}{h_i^\beta}\sum_{j=0}^{m} q_j u_{ni-j+1}, \tag{7}$$

$$i = 1,2; \quad h_i = x_{i(ni+1)} - x_{i(ni)}; \quad q_0 = 1, \quad q_j = (-1)^j\,\frac{\beta(\beta-1)\ldots(\beta-j+1)}{j!}.$$

Adapted is the method of predictor-corrector for the numerical implementation of a two-dimensional mathematical model of non-isothermal moisture transfer, taking into account the fractal structure of materials. The difference scheme for the numerical realization of the system of differential equations (1) and (2) is obtained, taking into account the expressions of the approximations (6) and (7):

$$c\rho\frac{T_{n,m}^{k+1} - \alpha T_{n,m}^{k}}{\Gamma(2-\alpha)\,\Delta t^\alpha} = \frac{\lambda_1}{h_1^\beta}\sum_{j=0}^{n} q_j T_{n-j+1,m}^{k+\omega} + \frac{\lambda_2}{h_2^\beta}\sum_{j=0}^{m} q_j T_{n,m-j+1}^{k+\omega} + \varepsilon\rho_0 r\frac{U_{n,m}^{k+1} - \alpha U_{n,m}^{k}}{\Gamma(2-\alpha)\,\Delta t^\alpha}, \tag{8}$$

$$\frac{U_{n,m}^{k+1} - \alpha U_{n,m}^{k}}{\Gamma(2-\alpha)\,\Delta t^\alpha} = \frac{a_1}{h_1^\beta}\sum_{j=0}^{n} q_j U_{n-j+1,m}^{k+\omega} + \frac{a_2}{h_2^\beta}\sum_{j=0}^{m} q_j U_{n,m-j+1}^{k+\omega} + \frac{a_1\delta}{h_1^\beta}\sum_{j=0}^{n} q_j T_{n-j+1,m}^{k+\omega} + \frac{a_2\delta}{h_2^\beta}\sum_{j=0}^{m} q_j T_{n,m-j+1}^{k+\omega}, \tag{9}$$

where $T_{n,m}^k$, $U_{n,m}^k$ are the temperature and the moisture content in a finite-difference form ($x_{1,n} = t(n-1)h_1$, $x_{2,m} = (m-1)h_2, t^k = k\Delta t$), ($n = 1,\ldots,N$, $m = 1,\ldots,M$, $k = 0,\ldots,K$).

In the case where $\omega = 1$, we obtain an implicit finite-difference scheme, and when $\omega = 0$ we obtain an explicit scheme.

The predictor is realized by means of an implicit difference scheme, and the corrector by means of an explicit difference scheme. On the first half of the time interval write an implicit difference scheme, in which we take into account only the derivative of the fractional order β by coordinate x_1:

$$c\rho\frac{T_{n,m}^{k+1/4} - \alpha T_{n,m}^{k}}{\Gamma(2-\alpha)\,(\Delta\tau/2)^\alpha} = \frac{\lambda_1}{h_1^\beta}\sum_{j=0}^{n} q_j T_{n-j+1,m}^{k+1/4} + \varepsilon\rho_0 r\frac{U_{n,m}^{k+1/4} - \alpha U_{n,m}^{k}}{\Gamma(2-\alpha)\,(\Delta\tau/2)^\alpha}, \tag{10}$$

$$\frac{U_{n,m}^{k+1/4} - \alpha U_{n,m}^{k}}{\Gamma(2-\alpha)\,(\Delta\tau/2)^\alpha} = \frac{a_1}{h_1^\beta}\sum_{j=0}^{n} q_j U_{n-j+1,m}^{k+1/4} + \frac{a_1\delta}{h_1^\beta}\sum_{j=0}^{n} q_j T_{n-j+1,m}^{k+1/4}, \tag{11}$$

The equation (10) and boundary conditions converted, and it is wrote in matrix form, such as:

$$U_n^{k+1/4} = AT_n^{k+1/4} + \alpha U_n^k - \frac{\alpha c\rho}{\varepsilon\rho_0 r}T_n^k + \Psi_1, \tag{12}$$

where

$$U_n^{k+1/4} = \left[U_{1,m}^{k+1/4}, U_{2,m}^{k+1/4}, \ldots, U_{N-1,m}^{k+1/4}, U_{N,m}^{k+1/4}\right]^T,$$

$$T_n^{k+1/4} = \left[T_{1,m}^{k+1/4}, T_{2,m}^{k+1/4}, \ldots, T_{N-1,m}^{k+1/4}, T_{N,m}^{k+1/4}\right]^T,$$

$$U_n^k = \left[0, U_{2,m}^k, \dots, U_{N-1,m}^k, 0\right]^T, \quad T_n^k = \left[0, T_{2,m}^k, \dots, T_{N-1,m}^k, 0\right]^T,$$

$$\Psi_1 = \left[U_{p1}^* - \frac{\alpha_1^* t_{c1}^*}{\rho_0 (1-\varepsilon) \beta_1^*}, 0, \dots, 0, U_{p1} - \frac{\alpha_1 t_{c1}}{\rho_0 (1-\varepsilon) \beta_1}\right]^T.$$

Components a_{ij}, $i, j = \overline{1, N}$, matrices A are defined by the expressions:

$$a_{ij} = \begin{cases} 0,\ j \geqslant i+2; & \left(\frac{\alpha_1}{\rho_0(1-\varepsilon)\beta_1} - B_1\right),\ i = j = N; \\ 0,\ i = N, 1 \leqslant j \leqslant N-2; & -B_1^*,\ i = 1, j = 2; \\ \left(\frac{c\rho}{\varepsilon\rho_0 r} - A_1 q_1\right),\ i = j \neq 1 \neq N; & B_1\gamma,\ i = N,\ j = N-1; \\ \left(\frac{\alpha_1^*}{\rho_0(1-\varepsilon)\beta_1^*} + B_1^*\gamma\right),\ i = j = 1; & -A_1 q_{i-j+1},\ \text{other}. \end{cases}$$

Similarly, write in the matrix form (11) and the corresponding boundary conditions:

$$BT_n^{k+1/4} + CU_n^{k+1/4} + \Psi_2 + \alpha U_n^k = 0, \tag{13}$$

$$C = (c_{ij}), \quad B = (b_{ij}), \quad i, j = \overline{1, N},$$

$$c_{ij} = \begin{cases} 0,\ j \geqslant i+2; \\ 0,\ i = N,\ 1 \leqslant j \leqslant N-2; \\ (Z_1 q_1 - 1),\ i = j \neq 1 \neq N; \\ (a_1\gamma - \beta_1^* \Gamma(2-\gamma) h_1^\gamma),\ i = j = 1; \\ -(a_1 + \beta_1 \Gamma(2-\gamma) h_1^\gamma),\ i = j = N; \\ -a_1,\ i = 1,\ j = 2; \\ a_1\gamma,\ i = N,\ j = N-1; \\ Z_1 q_{i-j+1},\ \text{other}. \end{cases} \qquad b_{ij} = \begin{cases} 0,\ j \geqslant i+2; \\ 0,\ i = N,\ 1 \leqslant j \leqslant N-2; \\ Zq_1\delta,\ i = j \neq 1 \neq N; \\ a_1\delta\gamma,\ i = j = 1;\ i = N,\ j = N-1; \\ -a_1\delta,\ i = j = N;\ i = 1,\ j = 2; \\ Zq_{i-j+1}\delta,\ \text{other}. \end{cases}$$

$$Z_1 = \frac{a_1 \Gamma(2-\alpha)(\Delta\tau/2)^\alpha}{h_1^\beta}, \quad \Psi_2 = \left[\beta_1^* \Gamma(2-\gamma) h_1^\gamma U_{p1}^*, 0, \dots, 0, \beta_1 \Gamma(2-\gamma) h_1^\gamma U_{p1}\right]^T.$$

Substitute (12) into (13) and obtain a system of equations, which we solve with respect to the function T:

$$(B + CA) T_n^{k+1/4} - \frac{\alpha c\rho}{\varepsilon\rho_0 r} CT_n^k + (\alpha C + \alpha) U_n^k + \Psi_1 + \Psi_2 = 0. \tag{14}$$

We found from (14) the set of solutions $T_{1,m}^{k+1/4}, T_{2,m}^{k+1/4}, \dots, T_{N-1,m}^{k+1/4}, T_{N,m}^{k+1/4}$, $(k = 0, 1, \dots, K-1)$, then was found from (12) the set of solutions — $U_{1,m}^{k+1/4}, U_{2,m}^{k+1/4}, \dots, U_{N-1,m}^{k+1/4}, U_{N,m}^{k+1/4}$, $(k = 0, 1, \dots, K-1)$.

On the second half of the time interval we write an implicit difference scheme, in which we take into account only the derivative of the fractional order β by coordinate x_2:

$$c\rho \frac{T_{n,m}^{k+1/2} - \alpha T_{n,m}^{k+1/4}}{\Gamma(2-\alpha)(\Delta\tau/2)^\alpha} = \frac{\lambda_2}{h_2^\beta} \sum_{j=0}^{m} q_j T_{n,m-j+1}^{k+1/2} + \varepsilon\rho_0 r \frac{U_{n,m}^{k+1/2} - \alpha U_{n,m}^{k+1/4}}{\Gamma(2-\alpha)(\Delta\tau/2)^\alpha}, \tag{15}$$

$$\frac{U_{n,m}^{k+1/2} - \alpha U_{n,m}^{k+1/4}}{\Gamma(2-\alpha)(\Delta\tau/2)^\alpha} = \frac{a_2}{h_2^\beta} \sum_{j=0}^{m} q_j U_{n,m-j+1}^{k+1/2} + \frac{a_2\delta}{h_2^\beta} \sum_{j=0}^{m} q_j T_{n,m-j+1}^{k+1/2}. \tag{16}$$

Similarly to the first half step, we got two systems in the second half step. As a result of their solutions we get a set of solutions with respect to the function T — $T_{n,1}^{k+1/2}, T_{n,2}^{k+1/2}, \dots, T_{n,M-1}^{k+1/2}, T_{n,M}^{k+1/2}$, $(k = 0, 1, \dots, K-1)$, and U — $U_{n,1}^{k+1/2}, U_{n,2}^{k+1/2}, \dots, U_{n,M-1}^{k+1/2}, U_{n,M}^{k+1/2}$, $(k = 0, 1, \dots, K-1)$.

To find solutions for the entire time interval, we use a corrector, which is implemented on an explicit difference scheme:

$$c\rho\frac{cT_{n,m}^{k+1}-\alpha cT_{n,m}^{k}}{\Gamma\left(2-\alpha\right)\Delta\tau^{\alpha}}=\frac{\lambda_1}{h_1^{\beta}}\sum_{j=0}^{n}q_jT_{n-j+1,m}^{k+1/2}+\frac{\lambda_2}{h_2^{\beta}}\sum_{j=0}^{m}q_jT_{n,m-j+1}^{k+1/2}\varepsilon\rho_0 r+\frac{U_{n,m}^{k+1}-\alpha U_{n,m}^{k}}{\Gamma\left(2-\alpha\right)\Delta\tau^{\alpha}},\qquad(17)$$

$$\frac{U_{n,m}^{k+1}-\alpha U_{n,m}^{k}}{\Gamma\left(2-\alpha\right)\Delta\tau^{\alpha}}=\frac{a_1}{h_1^{\beta}}\sum_{j=0}^{n}q_jU_{n-j+1,m}^{k+1/2}+\frac{a_2}{h_2^{\beta}}\sum_{j=0}^{m}q_jU_{n,m-j+1}^{k+1/2}+\frac{a_1\delta}{h_1^{\beta}}\sum_{j=0}^{n}q_jT_{n-j+1,m}^{k+1/2}+\frac{a_2\delta}{h_2^{\beta}}\sum_{j=0}^{m}q_jT_{n,m-j+1}^{k+1/2}.\qquad(18)$$

Therefore, from (18) there was found the set of solutions $\left\{U_{n,m}^{k+1}\colon k=\overline{0,K-1}; n=\overline{1,N}; m=\overline{1,M}\right\}$, and from (17) was found the set of solutions $\left\{T_{n,m}^{k+1}\colon k=\overline{0,K-1}; n=\overline{1,N}; m=\overline{1,M}\right\}$.

To determine the stability conditions of the obtained difference equations (8)–(9) of the associated heat-and-mass transfer, we used the method of conditionally defined known functions of the system, according to which the following relation was found:

$$\Delta t^{\alpha}\left(\frac{C_1}{h_1^{\beta}}+\frac{C_2}{h_2^{\beta}}\right)\leqslant\frac{\left(\alpha+1\right)C_3}{\left(2+\beta\right)\Gamma(2-\alpha)},\qquad(19)$$

where $C_1=\lambda_1,a_1$; $C_2=\lambda_2,a_2$; $C_3=\left(c\rho-\varepsilon\rho_0 r\right),\left(1+\delta\right)^{-1}$. Assuming that the fractal parameters α, β take integer values, an analysis and comparison are made as a result of which it was found that the found stability condition (19) coincides with the stability condition for classical equations of thermal conductivity.

3. Mathematical models of deformation taking into account the effect of "memory", and their analytical and numerical implementations

Using the methods of the mechanics of inherited media [20] and fractional integro-differential apparatus, a one-dimensional mathematical Kelvin's model of capillaryporous materials was constructed, taking into account the fractal structure of the medium:

$$E_1\tau^{\alpha}D_t^{\alpha}\sigma(t)+(E_1+E_2)\sigma(t)=E_1E_2\big(\varepsilon(t)+\tau^{\beta}D_t^{\beta}\varepsilon(t)\big),\qquad(20)$$

where t is time, τ is relaxation time, E_1 is the Voigt element elastic modulus for the Kelvin model, E_2 is the elastic modulus for the Kelvin model, $\sigma(t)$ is stress, $\varepsilon(t)$ is deformation, D_t^{α}, D_t^{β} are fractional derivatives in the sense of Riemann-Liouville by time t with order α,β respectively $0<\alpha,\beta<1$.

Using the Laplace transform method and the properties of fractional integro-differential operators, analytical relations in the integral form were found to determine the deformations ε and stresses σ for the Kelvin model:

$$\sigma_K(t)=c_Kt^{\alpha-1}tA(t)+\frac{E_2}{\tau^{\alpha}}\int_0^t(t-z)^{\alpha-1}A(t-z)\left(\varepsilon(z)+\tau^{\beta}D_z^{\beta}\varepsilon(z)\right)dz,\qquad(21)$$

$$\varepsilon_K\left(t\right)=\tilde{c}_Kt^{\beta-1}E_{\beta,\beta}\left(-\frac{t^{\beta}}{\tau^{\beta}}\right)+\frac{1}{E_2\tau^{\beta}}\int_0^t(t-z)^{\beta-1}E_{\beta,\beta}\left(-\frac{(t-z)^{\beta}}{\tau^{\beta}}\right)\left[\tau^{\alpha}D_z^{\alpha}\sigma(z)+\frac{(E_1+E_2)}{E_1}\sigma(z)\right]dz.\qquad(22)$$

The construction of two-dimensional mathematical models of deformationrelaxation processes, taking into account the fractal structure of the material, is based on the use of the properties of fractional integrodifferentiation apparatus and studies of visco-elastic media under conditions of interaction with heat transfer processes [17, 18]. In general terms, the mathematical model of two-dimensional deformation of capillary-porous materials, taking into account the effects of "memory" and self-organization for rheological models, can be described by equilibrium equations of fractional order γ $(0<\gamma\leqslant1)$ in spatial coordinates x_1 and x_2:

$$C_{11}\left(\frac{\partial^\gamma \varepsilon_{11}}{\partial x_1^\gamma}(1-\bar{R}_{11}) - \frac{\partial^\gamma \varepsilon_{T1}}{\partial x_1^\gamma} + \tilde{R}_{11}\right) + C_{12}\frac{\partial^\gamma \varepsilon_{22}}{\partial x_1^\gamma}(1-\bar{R}_{12})$$
$$- C_{12}\frac{\partial^\gamma \varepsilon_{T2}}{\partial x_1^\gamma} + C_{12}\tilde{R}_{12} + 2C_{33}\left(\frac{\partial^\gamma \varepsilon_{12}}{\partial x_2^\gamma}(1-\bar{R}_{33}^2) - \frac{\partial^\gamma \varepsilon_{T3}}{\partial x_2^\gamma} + \tilde{R}_{33}^2\right) = 0, \quad (23)$$

$$C_{21}\left(\frac{\partial^\gamma \varepsilon_{11}}{\partial x_2^\gamma}(1-\bar{R}_{21}) - \frac{\partial^\gamma \varepsilon_{T1}}{\partial x_2^\gamma} + \tilde{R}_{21}\right) + C_{22}\frac{\partial^\gamma \varepsilon_{22}}{\partial x_2^\gamma}(1-\bar{R}_{22})$$
$$- C_{22}\frac{\partial^\gamma \varepsilon_{T2}}{\partial x_2^\gamma} + C_{22}\tilde{R}_{22} + 2C_{33}\left(\frac{\partial^\gamma \varepsilon_{12}}{\partial x_1^\gamma}(1-\bar{R}_{33}^1) - \frac{\partial^\gamma \varepsilon_{T3}}{\partial x_1^\gamma} + \tilde{R}_{33}^1\right) = 0, \quad (24)$$

where $\varepsilon^T = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$, $\varepsilon_T = (\varepsilon_{T1}, \varepsilon_{T2}, \varepsilon_{T3})^T$ are deformation vectors, the components of the second vector are conditioned by changes in temperature ($\Delta T = T - T_0$) and moisture content ($\Delta U = U - U_0$), which are obtained from the model (1)–(5):

$$\varepsilon_{T1} = \alpha_{11}\Delta T + \beta_{11}\Delta U, \quad \varepsilon_{T2} = \alpha_{22}\Delta T + \beta_{22}\Delta U, \quad \varepsilon_{T3} = 0, \quad (25)$$

α_{11}, α_{22}, β_{11}, β_{22} are coefficients of thermal expansion and moisture-dependent shrinkage; C_{ij} are components of the elasticity tensor of the orthotropic body, $\bar{R}_{ij}$, $\tilde{R}_{ij}$ are some values of integrals that include relaxation kernels:

$$\int_0^t R_{ij}(t-z, T, U)dz = \bar{R}_{ij}, \quad \int_0^t R_{ij}(t-z, T, U)\frac{\partial^\gamma \varepsilon_{T1,T2}}{\partial x_k^\gamma}dz = \tilde{R}_{ij}, \quad (26)$$

where $(k, i, j = 1, 2)$. Developed is an algorithm for the numerical method for implementing a two-dimensional mathematical model of visco-elastic deformation of a material, taking into account its fractal structure based on the use of difference approximations. A finite-difference scheme for approximating equilibrium equations (23) and (24) is obtained based on the Riemann–Liouville formula:

$$\frac{C_{11}}{\Gamma(2-\gamma)h_1^\gamma}\Big[(1-\bar{R}_{11})\big(\varepsilon^k_{11(n+1,m)} - \gamma\varepsilon^k_{11(n,m)}\big) - \big(\varepsilon^k_{T1(n+1,m)} - \gamma\varepsilon^k_{T1(n,m)}\big)\Big]$$
$$+ \frac{C_{12}}{\Gamma(2-\gamma)h_1^\gamma}\Big[(1-\bar{R}_{12})\big(\varepsilon^k_{22(n+1,m)} - \gamma\varepsilon^k_{22(n,m)}\big) - \Big(\varepsilon^k_{T2(n+1,m)} - \gamma\varepsilon^k_{T2(n,m)}\Big)\Big]$$
$$+ \frac{2C_{33}}{\Gamma(2-\gamma)h_2^\gamma}\Big[(1-\bar{R}_{33}^2)\big(\varepsilon^k_{12(n,m+1)} - \gamma\varepsilon^k_{12(n,m)}\big) - \big(\varepsilon^k_{T3(n,m+1)} - \gamma\varepsilon^k_{T3(n,m)}\big)\Big]$$
$$+ C_{11}\tilde{R}_{11} + C_{12}\tilde{R}_{12} + 2C_{33}\tilde{R}_{33}^2 = 0, \quad (27)$$

$$\frac{C_{21}}{\Gamma(2-\gamma)h_2^\gamma}\Big[(1-\bar{R}_{21})\big(\varepsilon^k_{11(n,m+1)} - \gamma\varepsilon^k_{11(n,m)}\big) - \big(\varepsilon^k_{T1(n,m+1)} - \gamma\varepsilon^k_{T1(n,m)}\big)\Big]$$
$$+ \frac{C_{22}}{\Gamma(2-\gamma)h_2^\gamma}\Big[(1-\bar{R}_{22})\big(\varepsilon^k_{22(n,m+1)} - \gamma\varepsilon^k_{22(n,m)}\big) - \big(\varepsilon^k_{T2(n,m+1)} - \gamma\varepsilon^k_{T2(n,m)}\big)\Big]$$
$$+ \frac{2C_{33}}{\Gamma(2-\gamma)h_1^\gamma}\Big[(1-\bar{R}_{33}^1)\big(\varepsilon^k_{12(n+1,m)} - \gamma\varepsilon^k_{12(n,m)}\big) - \big(\varepsilon^k_{T3(n+1,m)} - \gamma\varepsilon^k_{T3(n,m)}\big)\Big]$$
$$+ C_{22}\tilde{R}_{22} + C_{21}\tilde{R}_{21} + 2C_{33}\tilde{R}_{33}^1 = 0, \quad (28)$$

where $\big(\varepsilon^k_{11(n,m)}, \varepsilon^k_{22(n,m)}, \varepsilon^k_{12(n,m)}\big)$ are deformation components in a finite-difference form.

The method of splitting two-dimensional creep nuclei for fractional-differential rheological models has also been adapted, which allows to determine creep nuclei and the function of bulk creep velocity according to experimental data for one-dimensional models, to identify fractional-differential parameters of models and to investigate effects of "memory" and self-organization of material [17].

4. The results of the identification of fractal parameters and results of numerical implementation of mathematical models

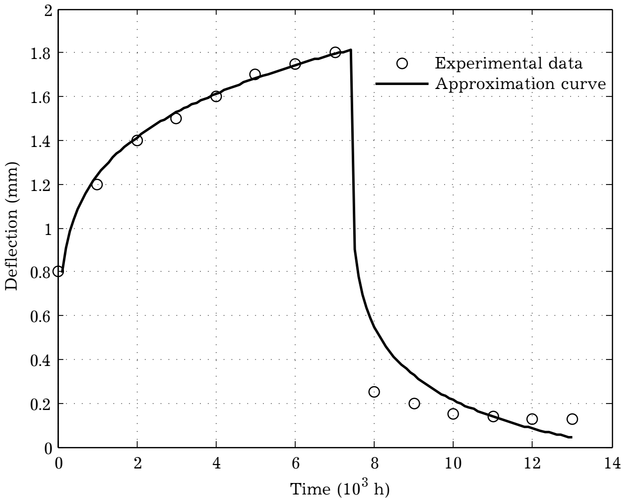

Fig. 1. The results of the identification of fractal parameters of the creep function according to experimental data.

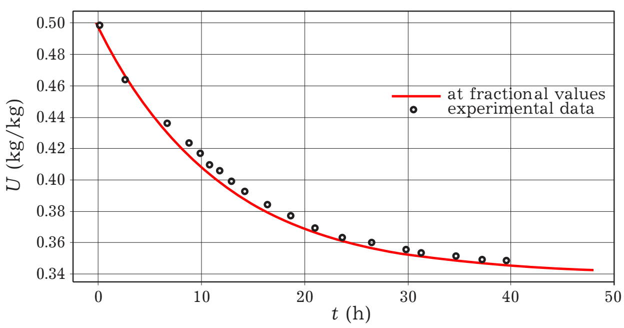

Fig. 2. The results of numerical implementation of mathematical model of heat-and-moisture transfer and experimental data.

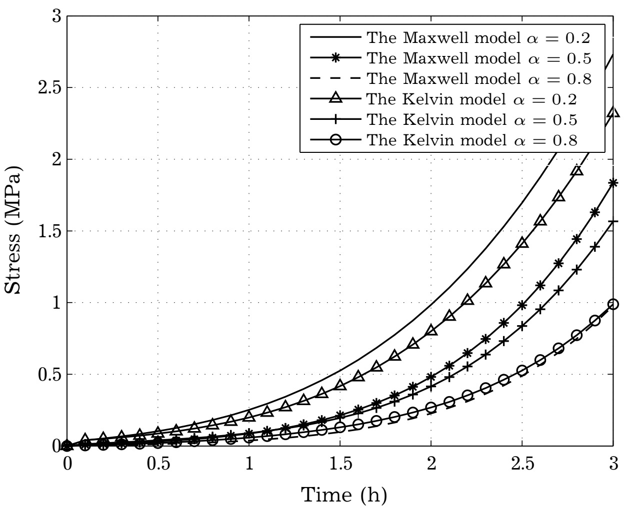

Fig. 3. Stress dynamics of fractional differential rheological models.

We present the results of the developed algorithm for identifying non-integer fractional differential parameters according to the experimental data [24] using the iterative method. The application of the iterative method has two stages. At the first stage, assuming that the parameters α, β of the deformation function are integer, based on the method of least squares, the search for the initial value of stress, relaxation time, modulus of elasticity. The obtained identification results at the first stage are taken into account in the next stage, where the values of fractional-differential parameters are obtained by minimizing the expressions describing the law of creep for rheological models. The coordinate descent method was used to clarify the identified parameters. A statistical criterion based on the application of the correlation coefficient was used to assess the divergence of the results. Thus, the correlation coefficient r is equal to the value of 0.992, which indicates a good agreement between the approximant and the experimental data shown in Fig. 1. It is shown that moisture content is released faster from materials with the effects of "memory" and self-organization with a lower base density. A decrease in the fractional differential parameters of the heat-and-mass transfer model leads to the acceleration of the process of moisture removal. So, we can observe the influence of the fractal structure of the material on the process of moisture removal. The simulation results with experimental data and are shown in Fig. 2.

Investigated is the variation of stress for the rheological models of fractional-differential type: Maxwell, Kelvin (Fig. 3). The elastic modulus of the material $E = 10200$ МПа, the fractional differential parameters of the models were taken, $\beta = 0.9$, α is variable. These numerical values are taken into account from previous studies — identification of fractal parameters of models. For the Maxwell and Kelvin models, the stress curves almost coincide, which is explained by other researchers as follows: three-element schemes containing two eponymous elements are mechanically equivalent to two-element schemes.

5. Conclusions

A mathematical model of heat-and-mass transfer based on the use of fractional integro-differential apparatus is constructed in the work, which allows analyzing the dynamics of change in temperature and moisture content of a material, taking into account the effects of "memory" and self-organization. Finite-difference schemes are developed to approximate the mathematical model of non-isothermal moisture transfer of capillary-porous materials with fractal structure, which makes it possible to carry out the algorithm of the numerical method. The stability conditions for difference schemes are established, which, by comparison, are consistent with the results of other studies. The fractal Kelvin model in fractional differential form is obtained. Based on the non-integer integro-differential apparatus and the Laplace transform method, analytical relationships in an integral form are found to determine the deformations and stresses of the rheological model, which makes it possible to determine the dynamics of the stress-strain state of the material taking into account eridarity (process memory) and self-organization, to obtain thermodynamic functions, relaxation and creep kernels of fractional differential models. Two-dimensional mathematical models of deformation processes are constructed, which allows accounting for the fractal structure of a material, depending on the initial values of temperature and moisture content, thermomechanical characteristics of the anisotropy, the base density of materials. The developed algorithm for the numerical implementation of two-dimensional mathematical models of visco-elastic deformation allows us to calculate the components of the stress-strain state of the material, taking into account the effects of "memory" and self-organization. That is why the paper presents a finite-difference scheme for the approximation of equilibrium equations based on the Riemann–Liouville formula. Presented are the results of the developed algorithm for the identification of fractional-differential parameters of the models. New patterns of heat-and-mass transfer and deformation processes are defined taking into account the fractal structure of the material, which makes it possible to give due consideration for the effects of "memory" and self-organization of the material depending on its base density, thermomechanical characteristics, technological parameters, and anisotropy directions. The results obtained were compared with the experimental data and the results of numerical studies which did not take into account the fractal structure of the material nor took into account the time. Thus, the consistency of new results with existing ones is shown.

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Дослідження теплоперенесення та напружено-деформаційного стану матеріалу з урахуванням фрактальної структури

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У роботі на основі апарату дробового інтегро-диференціювання побудовано математичні моделі тепловологоперенесення, деформаційно-релаксаційних процесів в умовах середовища з ефектами "пам'яті" та самоорганізації. Чисельна реалізація математичних моделей теплоперенесення та вологоперенесення грунтується на адаптації методу предиктор-коректор. Саме тому у роботі отримано математичні моделі у скінченно-різницевому вигляді. Для явної різницевої схеми на основі методу умовного задання відомих функцій та методу інтеграла Фур'є визначено умови стійкості. Отримано інтегральне представлення деформації та напруження дробово-диференціальної реологічної моделі за допомогою методу перетворення Лапласа. Враховуючи чисельні та аналітичні методи реалізацій побудованих моделей у роботі наведено основні результати, зокрема, ідентифікацію фрактальних параметрів для функції повзучості за експериментальними даними.

Ключові слова: *фрактальна структура, самоорганізація, тепломасоперенесення, дробово-диференціальний апарат, модель Кельвіна.*