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Existence of periodic solution for a higher-order p-Laplacian differential equation with multiple deviating arguments

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By applying Mawhin's continuation theorem, theory of Fourier series, Bernoulli numbers theory and some new inequalities, we study the higher-order p-Laplacian differential equation with multiple deviating arguments of the form

$$(\varphi_p(x^{(m)}(t)))^{(m)} = f(x(t))x'(t) + g(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_k(t))) + e(t).$$

Some new results on the existence of periodic solutions for the previous equation are obtained.

Keywords: *periodic solution, higher order, p-Laplacian equation, deviating argument, Mawhin's continuation.*

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1. Introduction

The periodic solution problem for p-Laplacian differential equation has extensively studied by many researchers, we refer the reader to see papers [1–3] and the references cited therein.

Recently, the higher-order p-Laplacian differential equations have received more and more attention, which are derived from many fields, such as fluid mechanics and nonlinear elastic mechanics. However, as far as we know, work on the existence of periodic solutions for higher-order p-Laplacian differential equations has been partially discussed [4, 5]. For instance, Li [5] has studied the existence and uniqueness of periodic solutions for a kind of higher-order p-Laplacian differential equation of the following form:

$$(\varphi_p(x^{(m)}(t)))^{(m)} + \beta(t))x'(t) + g(t, x(t)) = e(t).$$

In this paper, inspired by the results presented in [1, 4, 5], we study the existence of periodic solution for the following higher-order p-Laplacian differential equation with multiple deviating arguments of the form:

$$(\varphi_p(x^{(m)}(t)))^{(m)} = f(x(t))x'(t) + g(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_k(t))) + e(t). \quad (1)$$

Where $p > 1$ is a fixed real number. The conjugate exponent of p is denoted by q, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Let $\varphi_p\colon \mathbb{R} \to \mathbb{R}$ be the mapping defined by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$, and $\varphi_p(0) = 0$, $f, e, \in C(\mathbb{R}, \mathbb{R})$ are continuous T-periodic functions defined on $\mathbb{R}$ and $T > 0$, $g \in C(\mathbb{R}^{k+2}, \mathbb{R})$ and $g(t+T, u_0, u_1, \ldots, u_k) = g(t, u_0, u_1, \ldots, u_k), \forall (t, u_0, u_1, \ldots, u_k) \in \mathbb{R}^{k+2}$, $\tau_i \in C^1(\mathbb{R}, \mathbb{R})$ $(i = 1, 2, \ldots, k)$ with $\tau_i(t+T) = \tau_i(t)$. Therefore, in this paper, based on the Mawhin's continuation theorem and some analysis skills, without the assumption of $\int_0^T e(t)dt = 0$, some new sufficient conditions for the existence of T-periodic solution of p-Laplacian equation (1) will be established.

2. Preliminaries

Before stating the results, some necessary Lemmas are introduced.

Lemma 1 (Ref. [4]). *Let $T > 0$ be constant, $x \in C^m(\mathbb{R}, \mathbb{R})$, $m \geqslant 2$ and $x(t+T) = x(t)$, $|x^{(i)}|_0 = \max_{t\in[0,T]} |x^{(i)}(t)|$, then there are $M_i(m) > 0$ independent of x such that*

$$|x^{(i)}|_0 \leqslant M_i(m) \int_0^T |x^{(m)}(t)|dt, \quad i = 1, 2, \dots, m-1 \tag{2}$$

where, if m is an even integer

$$M_i(m) = \begin{cases} M_{2s-1}(m) = T^{m-2s}\sqrt{\dfrac{-B_{2m-4s}}{12(2m-4s)!}}, & s = 1, 2, \dots, \dfrac{m}{2} - 1; \\ M_{2s}(m) = \dfrac{(-1)^{\frac{m-2s}{2}+1} T^{m-2s-1} B_{m-2s}}{(m-2s)!}, & s = 1, 2, \dots, \dfrac{m}{2} - 1; \\ M_{m-1}(m) = \dfrac{1}{2}, & \end{cases} \tag{3}$$

if m is an odd integer

$$M_i(m) = \begin{cases} M_{2s+1}(m) = \dfrac{(-1)^{\frac{m-2s-1}{2}+1} T^{m-2s-2} B_{m-2s-1}}{(m-2s-1)!}, & s = 1, 2, \dots, \dfrac{m+1}{2} - 2; \\ M_{2s}(m) = T^{m-2s-1}\sqrt{\dfrac{-B_{2m-4s-2}}{12(2m-4s-2)!}}, & s = 1, 2, \dots, \dfrac{m+1}{2} - 2; \\ M_{m-1}(m) = \dfrac{1}{2} & \end{cases} \tag{4}$$

and B_{m-2s}, B_{2m-4s}, B_{m-2s-1}, $B_{2m-4s-2}$ are Bernoulli numbers, which can be calcuLed using the following recursion formula: $B_0 = 1$, $B_p = \frac{-\sum_{i=0}^{p-1} C_{p+1}^i B_i}{p+1}$, where C_{p+1}^i is the combination number.

Lemma 2. *Let $r > 0$, $T > 0$ be two constants, $s \in C(\mathbb{R}, \mathbb{R})$ such that $s(t+T) = s(t)$, $\tau_i \in C^1(\mathbb{R}, \mathbb{R})$ with $\tau_i(t+T) = \tau_i(t)$ and $|\tau_i'|_0 < 1$. Then*

$$\int_0^T |s(t - \tau_i(t)|^r dt \leqslant \delta_i \int_0^T |s(t)|^r dt,$$

where $\delta_i = \frac{1}{1-|\tau_i'|_0}$, $|\tau_i'|_0 = \max_{t\in[0,T]} |\tau_i'(t)|$.

Proof. It is easy to see that

$$\int_0^T |s(t - \tau_i(t))|^r dt = \int_0^T |s(t - \tau_i(t))|^r d(t - \tau_i(t)) + \int_0^T \tau_i'(t)|s(t - \tau_i(t))|^r dt,$$

i.e.

$$(1 - |\tau_i'|_0) \int_0^T |s(t - \tau_i(t))|^r dt \leqslant \int_0^T |s(t)|^r dt$$

and thus

$$\int_0^T |s(t - \tau_i(t)|^r dt \leqslant \frac{1}{1 - |\tau_i'|_0} \int_0^T |s(t)|^r dt.$$

This completes the proof. ■

Lemma 3 (Borsuk [6]). *$\Omega \subset \mathbb{R}^n$ is an open bounded set, and symmetric with respect to $0 \in \Omega$. If $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $f(x) \neq \mu f(-x)$, $\forall x \in \partial\Omega$, $\forall \mu \in [0,1]$, then $\deg(f, \Omega, 0)$ is an odd number.*

Now, we recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach spaces and $L\colon D(L) \subset X \to Y$ be a Fredholm operator with index zero. Here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim KerL = \dim(Y/ImL) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 and such that $X = \operatorname{Ker} L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$ and let $P\colon X \to \operatorname{Ker} L$ and $Q\colon Y \to Y_1$ be natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_p := L|_{D(L)\cap X_1}$ is invertible. Denote the inverse of L_p by K. Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \varnothing$, a map $N\colon \overline{\Omega} \to Y$ is said to be L-compact on $\overline{\Omega}$. If $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N\colon \overline{\Omega} \to Y$ is compact.

Lemma 4 (Mawhin [7]). *Suppose that X and Y are two Banach spaces, and $L\colon D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthemore, $\Omega \subset X$ is an open bounded set, and $N\colon \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all of the following conditions hold:*

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in]0,1[$;
(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$; *and*
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of T-periodic solution for (1), we rewrite (1) in the following system

$$\begin{cases} x_1^{(m)}(t) = \varphi_q(x_2)(t) = |x_2(t)|^{q-2}x_2(t), \\ x_2^{(m)}(t) = f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t-\tau_1(t)), \ldots, x_1(t-\tau_k(t))) + e(t). \end{cases} \tag{5}$$

Where $q > 1$ is constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^T$ is a T-periodic solution to equation set (5), then $x_1(t)$ must be a T-periodic solution to equation (1). Thus, in order to prove that (1) has a T-periodic solution, it suffices to show that equation set (5) has a T-periodic solution.

Now, we set $C_T = \{x \in C(\mathbb{R}, \mathbb{R})\colon x(t+T) = x(t)\}$ with the norm $|x|_0 = \max\limits_{t\in[0,T]} |x(t)|$, $C_T^1 = \{x \in C^1(\mathbb{R}, \mathbb{R})\colon x(t+T) = x(t)\}$ with the norm $\|x\| = \max\{|x|_0, |x'|_0\}$ $X = \{x = (x_1(t), x_2(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2)\colon x(t+T) = x(t)\}$ with the norm $\|x\|_X = \max\{\|x_1\|, \|x_2\|\}$, $Y = \{x = (x_1(t), x_2(t))^\top \in C(\mathbb{R}, \mathbb{R}^2)\colon x(t+T) = x(t)\}$ with the norm $\|x\|_Y = \max\{|x_1|_0, |x_2|_0\}$. Obviously, X and Y are two Banach spaces. Meanwhile, let

$$L\colon D(L) \subset X \to Y, \quad Lx = x^{(m)} = \begin{pmatrix} x_1^{(m)} \\ x_2^{(m)} \end{pmatrix}. \tag{6}$$

$$N\colon X \to Y,$$

$$[Nx](t) = \begin{pmatrix} \varphi_q(x_2)(t) \\ f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t-\tau_1(t)), \ldots, x_1(t-\tau_k(t))) + e(t) \end{pmatrix}. \tag{7}$$

where $D(L) = \{x = (x_1(t), x_2(t))^\top \in C^m(\mathbb{R}, \mathbb{R}^2)\colon x(t+T) = x(t)\}$ It is easy to see that equation set (5) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we see that $\operatorname{Ker} L = \mathbb{R}^2$, $\operatorname{Im} L = \{y\colon y \in Y, \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero.

Let projections $P\colon X \to \operatorname{Ker} L$ and $Q\colon Y \to \operatorname{Im} Q$ be defined by

$$Px = x(0), \quad Qy = \frac{1}{T}\int_0^T y(s)ds,$$

and let K represent the inverse of $L|_{\operatorname{Ker} P\cap D(L)}$. Clearly, $\operatorname{Ker} L = \operatorname{Im} Q = \mathbb{R}^2$ and

$$[Ky](t) = \sum_{i=1}^{m-1} \frac{1}{i!} x^{(i)}(0)t^i + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} y(s)ds, \tag{8}$$

where $x^{(i)}(0)$ $(i = 1, 2, \dots, m-1)$ are defined by the equation $AX = D$,

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m-3} & c_{m-4} & c_{m-5} & \cdots & 1 & 0 \\ c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_1 & 1 \end{pmatrix},$$

$$X = \left(x^{(m-1)}(0), x^{(m-2)}(0), \cdots, x''(0), x'(0)\right)^\top,$$

$$D = \left(d_1, d_2, \cdots, d_{m-2}, d_{m-1}\right)^\top,$$

$$d_i = -\frac{1}{i!T}\int_0^T (T-s)^i y(s)ds \quad i = 1, 2, \cdots, m-1 \quad \text{and} \quad c_j = \frac{T^j}{(j+1)!} \quad j = 1, 2, \cdots, m-2.$$

From (7) and (8), it is not difficult to find that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X. For the sake of convenience, we list the following assumptions which will be used by us in studing the existence of T-periodic solution to the equation (1)

$[H_1]$ There is a constant $d > 0$ such that:
(1) $g(t, u_0, u_1, \dots, u_k) > |e|_0, \forall (t, u_0, u_1, \dots, u_k) \in [0,T] \times \mathbb{R}^{k+1}$ with $u_i > d$ $(i = 0, 1, \dots, k)$.
(2) $g(t, u_0, u_1, \dots, u_k) < -|e|_0, \forall (t, u_0, u_1, \dots, u_k) \in [0,T] \times \mathbb{R}^{k+1}$ with $u_i < -d$ $(i = 0, 1, \dots, k)$.

$[H_2]$ $|g(t, u_0, u_1, \dots, u_k)| \leqslant \sum_{i=0}^k \alpha_i |u_i|^{p-1} + \beta$, where α_i $(i = 0, \dots, k), \beta$ are non-negative constants.

3. Main results

Lemma 5. *Suppose that $[H_1]$ holds, if $x \in D(L)$ is an arbitrary solution of the equation $Lx = \lambda Nx, \lambda \in]0,1[$, where L and N are defined by (6) and (7), respectively, then there must be a point $t^* \in [0,T]$ such that*

$$|x_1(t^*)| \leqslant d. \tag{9}$$

Proof. Suppose $x \in D(L)$ is an arbitrary solution of the equation $Lx = \lambda Nx$, for some $\lambda \in]0,1[$ then

$$\begin{cases} x_1^{(m)}(t) = \lambda\varphi_q(x_2)(t) = \lambda|x_2(t)|^{q-2}x_2(t), \\ x_2^{(m)}(t) = \lambda f(x_1(t))x_1'(t) + \lambda g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_k(t))) + \lambda e(t). \end{cases} \tag{10}$$

From the first equation of (10), we have $x_2(t) = \varphi_p\left(\frac{1}{\lambda}x_1^{(m)}\right)(t)$, and then by substituting it into the second equation of (10), we have

$$(\varphi_p(x_1^{(m)}(t)))^{(m)} = \lambda^p f(x_1(t))x_1'(t) + \lambda^p g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_k(t))) + \lambda^p e(t). \tag{11}$$

Integrating both sides of equation (11) on the interval $[0,T]$, we have

$$\int_0^T g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_k(t))) + \int_0^T e(t) = 0.$$

By the integral mean value theorem, there is a constant $t_0 \in [0,T]$ such that

$$g(t, x_1(t_0), x_1(t_0-\tau_1(t_0)), \ldots, x_1(t_0-\tau_k(t_0))) = -\frac{1}{T}\int_0^T e(t)dt. \tag{12}$$

If $|x_1(t_0)| \leqslant d$, then taking $t^* = t_0$ such that $|x_1(t^*)| \leqslant d$. If$|x_1(t_0)| > d$. It follows from assumption $[H_1]$ that there is some $i \in \{1, 2, \ldots, k\}$ such that $|x_1(t_0 - \tau_i(t_0))| \leqslant d$. Since $x_1(t)$ is continuous for $t \in \mathbb{R}$ and $x_1(t+T) = x_1(t)$, so there must be an integer r and a point $t^* \in [0,T]$ such that $t_0 - \tau_i(t_0) = rT + t^*$. So $|x_1(t^*)| = |x_1(t_0 - \tau_i(t_0))| \leqslant d$. ■

Theorem 6. *Suppose $|\tau_i'|_0 < 1$, $(i = 0, 1 \cdots, k)$ and assumption $[H_1], [H_2]$ hold. Then equation (1) has at one least one T-periodic solution, if $(\alpha_0 + \sum_{i=1}^{k} \alpha_i\delta_i)^{\frac{1}{p}} T^2 M_1(m) < 1$, where $M_1(m)$ and δ_i are defined in Lemma 1, Lemma 2.*

Proof. Let $\Omega_1 = \{x \in X \colon Lx = \lambda Nx, \lambda \in]0,1[\}$ if $x(.) = (x_1(.), x_2(.))^\top \in \Omega_1$, then from (6) and (7), we have

$$\begin{cases} x_1^{(m)}(t) = \lambda\varphi_q(x_2)(t) = \lambda|x_2(t)|^{q-2}x_2(t), \\ x_2^{(m)}(t) = \lambda f(x_1(t))x_1'(t) + \lambda g(t, x_1(t), x_1(t-\tau_1(t)), \ldots, x_1(t-\tau_k(t))) + \lambda e(t). \end{cases} \tag{13}$$

From Lemma 5, we have

$$|x_1(t)| = \left| x_1(t^*) + \int_{t^*}^{t} x_1'(s)ds \right| \leqslant d + \int_{t^*}^{t} |x_1'(s)|ds, \quad t \in [t^*, t^*+T],$$

and

$$|x_1(t)| = |x_1(t-T)| = \left| x(t^*) - \int_{t-T}^{t^*} x_1'(s)ds \right| \leqslant d + \int_{t^*-T}^{t^*} |x_1'(s)|ds, \quad t \in [t^*, t^*+T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x_1|_0 = \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[t^*,t^*+T]} |x_1(t)| &\leqslant \max_{t\in[t^*,t^*+T]} \left\{ d + \frac{1}{2}\left(\int_{t^*}^{t} |x_1'(s)|ds + \int_{t-T}^{t^*} |x_1'(s)|ds \right) \right\} \\ &\leqslant d + \frac{1}{2}\int_0^T |x_1'(s)|ds. \end{aligned} \tag{14}$$

On the hand, multiplying both sides of (11) by $x_1(t)$ and integrating it from 0 to T, we obtain

$$\begin{aligned} \int_0^T (\varphi_p(x_1^{(m)}(t)))^{(m)} x_1(t)dt &= \lambda^p \int_0^T f(x_1(t))x_1'(t)x_1(t)dt \\ &+ \lambda^p \int_0^T g(t, x_1(t), x_1(t-\tau_1(t)), \ldots, x_1(t-\tau_k(t)))x_1(t)dt + \lambda^p \int_0^T e(t)x_1(t)dt. \end{aligned} \tag{15}$$

Case 1. If m is even, we obtain

$$\int_0^T (\varphi_p(x_1^{(m)}(t)))^{(m)} x_1(t)dt = (-1)^m \int_0^T |x_1^{(m)}(t)|^p dt = \int_0^T |x_1^{(m)}(t)|^p dt.$$

Hence

$$\begin{aligned} \int_0^T |x_1^{(m)}(t)|^p dt &= \lambda^p \int_0^T f(x_1(t))x_1'(t)x_1(t)dt \\ &+ \lambda^p \int_0^T g(t, x_1(t), x_1(t-\tau_1(t)), \ldots, x_1(t-\tau_k(t)))x_1(t)dt + \lambda^p \int_0^T e(t)x_1(t)dt. \end{aligned} \tag{16}$$

In view of assumption $[H_2]$, Lemma 2 and (16), we have

$$\int_0^T |x_1^{(m)}(t)|^p dt \leqslant \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right) T|x_1|_0^p + (\beta + |e|_0)T|x_1|_0$$

i.e.

$$\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}} \leqslant T^{\frac{1}{p}} \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right)^{\frac{1}{p}} |x_1|_0 + T^{\frac{1}{p}}(\beta + |e|_0)^{\frac{1}{p}}|x_1|_0^{\frac{1}{p}},$$

which together with (14), yields

$$\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}} \leqslant \frac{T^{\frac{1}{p}+1}}{2} \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right)^{\frac{1}{p}} |x_1'|_0 + \left(\frac{T^2}{2}\right)^{\frac{1}{p}} (\beta + |e|_0)^{\frac{1}{p}}|x_1'|_0^{\frac{1}{p}} + T^{\frac{1}{p}} d \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right)^{\frac{1}{p}} + T^{\frac{1}{p}} d^{\frac{1}{p}}(\beta + |e|_0)^{\frac{1}{p}}. \tag{17}$$

From Lemma 1, there exists $M_1(m) > 0$ independent of λ and x such that

$$|x_1'|_0 \leqslant M_1(m) \int_0^T |x_1^{(m)}(t)|dt,$$

which together with (17) yields

$$\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}} \leqslant \frac{T^2}{2} M_1(m) \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right)^{\frac{1}{p}} \left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}} + \left(\frac{T^{\frac{2}{p}+\frac{1}{pq}}}{2^{\frac{1}{p}}}\right) M_1(m)^{\frac{1}{p}} (\beta + |e|_0)^{\frac{1}{p}} \left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p^2}} + T^{\frac{1}{p}} d \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right)^{\frac{1}{p}} + T^{\frac{1}{p}} d^{\frac{1}{p}}(\beta + |e|_0)^{\frac{1}{p}}. \tag{18}$$

In view of $p > 1$ and $\frac{T^2}{2} M_1(m)(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i)^{\frac{1}{p}} < 1$, from (18) we see that there is a constant M_0 independent of λ such that

$$\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}} \leqslant M_0. \tag{19}$$

Thus, it follows from Lemma 1 and (19) that we have

$$|x_1'|_0 \leqslant M_1(m) \int_0^T |x_1^{(m)}(t)|dt \leqslant M_1(m) T^{\frac{1}{q}} M_0 := M_{11}. \tag{20}$$

By means of (14) and (20), we have

$$|x_1|_0 \leqslant d + TM_{11} := M_{12}. \tag{21}$$

Let $M_f = \max\limits_{|u|\leqslant M_{12}} |f(u)|$, $M_g = \max\limits_{t\in[0,T],|u_0|\leqslant M_{12},\ldots,|u_k|\leqslant M_{12}} |g(t,u_0,\ldots,u_k)|$ and from the second equation of (13), we have

$$\begin{aligned}\int_0^T |x_2^{(m)}(t)|dt &\leqslant \int_0^T |f(x_1(t))x_1'(t)|dt + \int_0^T |g(t,x_1(t),x_1(t-\tau_1(t)),\ldots,x_1(t-\tau_k(t)))|dt + \int_0^T |e(t)| \\ &\leqslant M_f \int_0^T |x_1'(t)|dt + T(M_g + |e|_0) \\ &\leqslant M_f T|x_1'|_0 + T(M_g + |e|_0) \\ &\leqslant M_f T M_{11} + T(M_g + |e|_0) := \overline{M_0}. \end{aligned} \tag{22}$$

Again, from Lemma 1, we have

$$|x_2'|_0 \leqslant M_1(m)\int_0^T |x_2^{(m)}(t)|dt \leqslant M_1(m)M_{21} := M_{21}.$$

Integrating the first equation of (13), we have $\int_0^T |x_2(t)|^{q-2}x_2(t)dt = 0$, which implies that there is a constant $\eta \in [0,T]$ such that $x_2(\eta) = 0$, thus

$$|x_2|_0 \leqslant \int_0^T |x_2'(t)|dt \leqslant TM_{21} := M_{22}. \tag{23}$$

Let $\Omega_2 = \{x | x \in \operatorname{Ker} L, QNx = 0\}$ if $x \in \Omega_2$ then $x \in \mathbb{R}^2$ is a constant vector with

$$\begin{cases} |x_2|^{q-2}x_2 = 0, \\ \dfrac{1}{T}\displaystyle\int_0^T [f(x_1(t))x_1'(t) + g(t,x_1(t),x_1(t-\tau_1(t)),...,x_1(t-\tau_k(t))) + e(t)]dt = 0. \end{cases} \tag{24}$$

According to the first formula of (24), we have $x_2 = 0$, which together with the second equation of (24) yields

$$\frac{1}{T}\int_0^T [g(t,x_1,x_1,\ldots,x_1) + e(t)]dt = 0.$$

In view of $[H_1]$, we see that $|x_1| \leqslant d$. Now, let $M_1 = \max\{M_{11}, M_{12}\}$, $M_2 = \max\{M_{21}, M_{22}\}$, then $\|x_1\| \leqslant M_1, \|x_2\| \leqslant M_2$. Taking $\Omega = \{x | x = (x_1,x_2)^\top \in X, \|x_1\| < M_1 + d, \|x_2\| < M_2 + d\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. So from (21) and (23), it is easy to see that conditions (1) and (2) of Lemma 4 are satisfied.

Next, we verify the condition (3) of Lemma 4. To do this, we define the isomorphism

$$J\colon \operatorname{Im} Q \to \operatorname{Ker} L, \quad J(x_1,x_2)^T = (x_1,x_2)^\top,$$

then

$$JQN(x) = \begin{pmatrix} \varphi_q(x_2) \\ \frac{1}{T}\int_0^T [g(t,x_1,x_1,\ldots,x_1) + e(t)]dt \end{pmatrix}, \quad x \in \overline{\operatorname{Ker} L \cap \Omega}.$$

By Lemma 3, we need to prove that

$$JQN(x) \neq \mu(JQN(-x)), \quad \forall x \in \partial\Omega \cap \operatorname{Ker} L, \quad \mu \in [0,1],$$

Case 1. If $x = (x_1,x_2)^\top \in \partial\Omega \cap \operatorname{Ker} L \backslash \{(M_1+d,0)^\top, (-M_1-d,0)^\top\}$, then $x_2 \neq 0$ which, gives us $\varphi_q(x_2) \neq 0$

$$\varphi_q(x_2)\varphi_q(-x_2) < 0,$$

obviously, $\forall \mu \in [0,1]$ $JQN(x) \neq \mu(JQN(-x))$.

Case 2. If $x = (M_1 + d, 0)^\top$ or $x = (-M_1 - d, 0)^\top$ then

$$JQN(x) = \begin{pmatrix} 0 \\ \frac{1}{T}\int_0^T [g(t, x_1, x_1, \ldots, x_1) + e(t)]dt \end{pmatrix},$$

which, together with $[H_1]$, yields $\forall \mu \in [0,1]$, $JQN(x) \neq \mu(JQN(-x))$.

Thus, the condition (3) of Lemma 4 is also satisfied. Therefore, by applying Lemma 4, we conclude that the equation $Lx = Nx$ has at least one T-periodic solution on $\overline{\Omega}$, so (1).

The case m is odd can be treated similarly. This completes the proof of Theorem 6. ■

4. Example

In this section, we provide an example to illustrate effectiveness of Theorem 6. Let us consider the following equation

$$(\varphi_3(x^{(8)}(t)))^{(8)} = f(x(t))x'(t) + g\left(t, x(t), x\left(t - \tfrac{\cos 20\pi t}{90}\right), x\left(t - \tfrac{\sin 20\pi t}{100}\right)\right) + \cos(20\pi t), \tag{25}$$

where

$$p = 3, \quad T = \frac{1}{10}, \quad \tau_1(t) = \frac{\cos 20\pi t}{90}, \quad \tau_2(t) = \frac{\sin 20\pi t}{100}, \quad e(t) = \cos 20\pi t,$$

$$g(t, u, v, w) = \operatorname{sgn}(u)u^2(2 + \sin 20\pi t) + \frac{3}{225}\left(\operatorname{sgn}(v)v^2 + \operatorname{sgn}(w)w^2\right)|\cos 20\pi t|.$$

Therefore we can choose $d = 1$, $\alpha_0 = 3$, $\alpha_1 = \alpha_2 = 0.014$, $M_1(8) = (2\pi)^6\sqrt{\frac{691}{2730\times 12\times 12!}}$.

We can easily check that condition $[H_1], [H_2]$ of Theorem 6 holds. We can compute

$$\left(\alpha_0 + \sum_{i=1}^{k} \alpha_i\delta_i\right)^{\frac{1}{p}} T^2 M_1(m) < 1,$$

by Theorem 6, (25) has at least one $\frac{1}{10}$-periodic solution.

[1] Anane A., Chakrone O., Moutaouekkil L. Periodic solutions for p-laplacian neutral functional differential equations with multiple deviating arguments. Electronic Journal of Differential Equations. **2012** (148), 1–12 (2012).

[2] Anane A., Chakrone O., Moutaouekkil L. Liénard type p-laplacian neutral rayleigh equation with a deviating argument. Electronic Journal of Differential Equations. **2010** (177), 1–8 (2010).

[3] Anane A., Chakrone O., Moutaouekkil L. Existence of periodic solution for p-Laplacian neutral Rayleigh equation with sign-variable coefficient of non linear term. International Journal Of Mathematical Sciences. **7** (2), 487–494 (2013).

[4] Xiaojing L., Shiping L. Periodic solutions for a kind of high-order p-Laplacian differential equation with sign-changing coefficient ahead of the non-linear term. Nonlinear Analysis. **70** (2), 1011–1022 (2009).

[5] Xiaojing L. Existence and uniqueness of periodic solutions for a kind of high-order p-Laplacian Duffing differential equation with sign-changing coefficient ahead of linear term. Nonlinear Analysis. **71** (7–8), 2764–2770 (2009).

[6] Zhong C., Fan X., Chen W. Introduction to Nonlinear Functional Analysis. Lanzhou University Press, Lan Zhou (2004).

[7] Gaines R. E., Mawhin J. L. Coincidence Degree and Nonlinear Differential Equations. Springer Verlag, Berlin (1977).

Існування періодичного розв'язку для p-лапласівського диференціального рівняння вищого порядку з багатьма аргументами, що відхиляються

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Застосовуючи теорему продовження Мовхіна, теорію рядів Фур'є, теорію чисел Бернуллі та деякі нові нерівності, досліджується p-лапласівське диференціальне рівняння вищого порядку з аргументами, що відхиляються, виду

$$(\varphi_p(x^{(m)}(t)))^{(m)} = f(x(t))x'(t) + g(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_k(t))) + e(t).$$

Отримано деякі нові результати щодо існування періодичних розв'язків такого рівняння.

Ключові слова: *періодичний розв'язок, вищий порядок, p-рівняння Лапласа, аргумент, що відхиляється, продовження Мовхіна.*