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ON THE ADEQUACY OF THE FREQUENCY-SYMBOLIC METHOD FOR LINEAR PARAMETRIC CIRCUITS ANALYSIS

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Abstract: a frequency-symbolic method (FS-method) of the analysis of steady-state mode of linear parametric circuits is intended for forming their transfer functions in the frequency domain. Transfer functions are approximated by Fourier polynomials and contain a complex variable, time variable and parameters of circuit elements in the form of symbols. The coefficients of such Fourier polynomials by the FS-method are unknown in the symbolic systems of linear algebraic equations (SSLAE), and are defined as their solutions in symbolic form.

In the paper we present a method of forming an approximation expression which ensures the adequacy of calculations. Examples and results of computer experiments are given. The system of functions MAOPCs based on a frequency-symbolic method is used for the optimal design of electronic devices of noise-immune hidden radio engineering systems using code signals.

Key words: circuit analysis computing, linear periodically time-variable circuits, frequency-symbolic method, frequency-symbolic models, approximations of transfer functions by Fourier polynomials.

1. Introduction

A number of works [1,2,3] reveal the content of the frequency- symbolic method of analysis of linear parametric circuits, in which the parameters of the elements change over time. Moreover, this change is considered periodic. It is also believed that the circuit has a steady- state. The analysis of such circuits involves formation of their transfer functions $W(s,t)$ with a number of parameters specified by the symbols, *s, t* – a complex variable and time, respectively. In the transfer functions of this type, it is convenient to substitute specific values of the element parameters that is a significant advantage in solving multivariate problems of circuit analysis, such as statistical analysis, optimization, etc.

The transfer functions formed by the frequency-symbol method are approximated by Fourier polynomials, so the problem of forming transfer functions is translated into determining the coefficients of such polynomials. The adequacy of such transfer functions is discussed in the work proposed.

2. Problem Statement. Formation of transfer functions by the frequency-symbolic method.

Suppose we have a linear parametric circuit in which the parameter of one element changes over time (by period T), and the parameters of other elements are constant. Let such a circuit be in a steady-state mode. It is known that the transfer function of such a circuit can be determined from a linear differential equation with complex and time-varying coefficients [4, 5, 2]:

$$\frac{1}{n!}\frac{d^n A(s,t)}{ds^n}\frac{d^n W(s,t)}{dt^n}+\ldots \\ +\frac{dA(s,t)}{ds}\frac{dW(s,t)}{dt}+A(s,t)W(s,t)=B(s,t) \quad (1)$$

where S is the complex variable, t represents the time, $W(s,t)=\dfrac{Y(s,t)}{X(s)}$ stands for the conjugate parametric transfer function of a linear parametric circuit in the frequency domain S; $A(s,t)=a_n(t)s^n+\ldots+a_1(t)s+a_0(t)$, $B(s,t)=b_m(t)s^m+\ldots+b_1(t)s+b_0(t)$ are the known symbolic expressions, which are determined from the corresponding basic differential equation of the parametric circuit

$$a_n(t)y^{(n)}+a_{n-1}(t)y^{(n-1)}+\ldots+a_0(t)y= \\ =b_m(t)x^{(m)}+b_{m-1}(t)x^{(m-1)}+\ldots+b_0(t)x \quad (2)$$

describing the relationship of the signal $x(t)$ applied to the circuit with its original response *y(t)*;

$a_i(t)$, $b_j(t)$ are the known defined for a given circuit valid time functions *t*; *Y*(*s*,*t*),*X*(*s*) denote the images of the output *y(t)* and input *x(t)* variables in the frequency domain *S*, respectively.

The specified symbolic definition of parametric transfer function $W(s,t)$ involves forming the function $W(s,t)$ of such kind in which variables *s, t*, and some or all circuit element parameters defining the coefficients $a_i(t)$, $b_j(t)$ from (2) are given not by specific numerical values, but symbols.

It is obvious that the parametric transfer function $W(s,t)$ of such a symbolic form is, in essence, a macromodel of a given parametric circuit and can be the basis for the following studies of the circuit by way of of

– studying the properties of this function $W(s,t)$ through the assignment of various numerical values to the parameters specified by the symbols;

– assessing the stability of the circuit;

– determining by this function other functions that characterize the circuit, in particular, those derived from $W(s,t)$ with respect to the parameters of the circuit elements, such as conductivity y, capacitance c or depth of modulation m required to determine the sensitivity of the circuit S_y^W, S_c^W, S_m^W to the change in these parameters, etc.

It is often difficult or impossible to solve such problems without symbolic circuit functions. Equation (1) (L.A. Zade's equation), as his publication [4] was probably the first to derive this equation and to present the methods of its solution.

As practice has shown, the known methods for solving equation (1) turned out not to be effective enough. Therefore, the frequency- symbolic method we have developed for solving (1) is designed to close this gap.

3. Physical content of the frequency-symbolic method

Accept some prerequisites as a basis for the following material.

Prerequisite 1. Consider the content of the proposed frequency-symbolic (FS-method) definition of parametric transfer functions under the assumption that when the parameter of the parametric element changes periodically, the parametric time -varying transfer function is also periodic. This fact is noted by a number of specialists in the theory of circuits and signals [6, 7]. If the parametric element changes with the period T, then the parametric transfer function changes with the period T [4].

Prerequisite 2. It is known [8] that in general, there is no exact analytical solution to $W(s,t)$ either by employing equation (1) or by using any other methods. In this regard, we will define this solution in an approximate form, but in such a way that the methodological error of the result could always be reduced if necessary.

Prerequisite 3. To solve equation (1) we apply the approach of projection (Galerkin) methods, which involves approximation of the solution on the basis of a certain chosen system of orthogonal functions [8,9].

Given the accepted prerequisites, to solve equation (1) we use the approximation of the desired periodic function $W(s,t)$ with respect to the time variable t in the form of a trigonometric polynomial, which is widely used in the theory of circuits and signals, takes into account the accepted three prerequisites [6] and has the following form:

$$\hat{W}(s,t) = \\ = W_0(s) + \sum_{i=1}^{k}[W_{ci}(s)\cos(i\Omega t) + W_{si}(s)\sin(i\Omega t)], \quad (3)$$

where $\Omega = \dfrac{2\pi}{T}$.

The problem of determining the transfer function $W(s,t)$ due to the selected approximation is transferred to the problem of determining the corresponding time-independent functions $W_0(s)$, $W_{ci}(s)$, $W_{si}(s)$.

Note that such an approximation of parametric transfer functions, although for other purposes, has been proposed before, for example, in [6].

Along with all traditional advantages of the trigonometric polynomial, it has another, fundamental and decisive in our case, property – the fact that the coefficients $W_0(s)$, $W_{ci}(s)$, $W_{si}(s)$ of polynomial (3) are functions of a complex variable s only and do not depend on time t. This property makes it possible to determine from (3) in symbolic form the required derivatives $\dfrac{d^i\hat{W}(s,t)}{dt^i}$ for $i = 0,1,\dots,n$, substitute them into expression (1) and, thus, convert expression (1) from a differential to an algebraic form from $(2k+1)$ by the unknown:

$$W_0(s), W_{c1}(s), W_{s1}(s), W_{c2}(s), \\ W_{s2}(s)\dots, W_{ck}(s), W_{sk}(s). \quad (4)$$

Perform such substitution. In addition, move $B(s,t)$ from the right side of equation (1) to the left, and present the parameter of the parametric element of the circuit, as it changes with the period $T = \dfrac{2\pi}{\Omega}$, by a trigonometric Fourier polynomial with m harmonic components:

$$c(t) = c_0(1 + \sum_{i=1}^{m}[m_{ci}\cos(i\Omega t) + m_{si}\sin(i\Omega t)]),$$

where c_0 is the average value of the parameter $c(t)$, m_{ci}, m_{si} are the depth of modulation of the cosine and sine parts of the i th harmonic component of the parameter $c(t)$, respectively.

The resulting symbolic linear algebraic expression is denoted by:

$$\delta(W_0, W_{c1}, W_{s1}, W_{c2}, W_{s2}, \dots, W_{ck}, W_{sk}, t) = 0. \quad (5)$$

An important property of the functional δ from (5) is that it is periodic over time *t*, and changes with the same period $T=\frac{2\pi}{\Omega}$ as the parametric element of the circuit *c*(*t*). This fact follows from the following. Since the parameter of only one element of the circuit is variable, and it together with its time derivatives *t* is represented by trigonometric functions $\cos(i\Omega t)$ and $\sin(i\Omega t)$, the sums of the products of the members present in symbolic expression (5) (division operations are absent) will contain the products of different quantities (one, two or more) of such harmonic functions with frequencies from the series $1\Omega,\ 2\Omega,\ldots,m\Omega$. Therefore, these products will form new total and difference harmonics, but, importantly, only from the series $1\Omega,\ 2\Omega,\ldots,m\Omega$, and higher. And this does not change the period of change in the resulting functional δ.

Since the functional δ in time *t*is periodic with the period of change $T=\frac{2\pi}{\Omega}$, it, in turn, can be represented by a Fourier series and decomposed into separate harmonics from the series $1\Omega,\ 2\Omega,\ldots,k\Omega,\ldots$ and a constant component. It is obvious that the solution to equation (5) is considered to be the values of the coefficients from (5), which turn it into an identity. It is also obvious that these solutions must ensure the equality of the constant component to zero and all harmonic components of the functional δ, and hence, their cosine and sine parts. Since the number of unknown coefficients is $(2k+1)$, we can choose arbitrary $(2k+1)$ of such components from all harmonic components δ, equate them to zero and determine $(2k+1)$ of the unknown coefficients (4) from the $(2k+1)$ equations obtained. Which harmonic components to choose is a separate question. If, for example, we choose the equation, starting with the constant component and the first *k*-harmonic components of the functional δ, we obtain a system of equations:

$$\begin{cases} \frac{1}{T}\int_0^T \delta(s,t)dt = 0, \\ \frac{2}{T}\int_0^T \delta(s,t)\cos\Omega t dt = 0, \\ \frac{2}{T}\int_0^T \delta(s,t)\sin\Omega t dt = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{2}{T}\int_0^T \delta(s,t)\cos k\Omega t dt = 0, \\ \frac{2}{T}\int_0^T \delta(s,t)\sin k\Omega t dt = 0, \end{cases} \tag{6}$$

After performing integration actions in the system, we obtain $(2k+1)$ linear algebraic equations with respect to unknowns from (4), which form a system of linear algebraic equations of $(2k+1)$ order with a missing variable t, for example, of the following type:

$$F(s)\cdot W(s) = D(s)\,, \tag{7}$$

where $F(s)$, $D(s)$ are the $(2k+1)$-dimensional matrix and vector of free members are determined by the parameters of constant elements of the circuit, the variable s and the parameters c_0, m_{ci}, m_{si}, Ω of the parametric element *c*(*t*) of this circuit; $W(s)$ is the $(2k+1)$-dimensional vector of unknowns (4) of the required approximation of the conjugate [5] parametric transfer function $W(s,t)$.

Although the solution of SSLAE (7) in symbolic form presents certain difficulties, the means of such solutions for simple cases are present in MATLAB. Methods for symbolic solution of SSLAE (7) for more complex cases are considered in [11].

EXAMPLE 1. Consider one of the simplest examples, which still demonstrates the content of the presented FS-method for determining the parametric transfer function $W(s,t)$ of a linear parametric circuit. Let a separate parametric capacitance $c(t)=c_0(1+m\cos(\Omega t))$ described by a differential equation $i(t)=c(t)\cdot u'(t)+c'(t)\cdot u(t)$ be such a circuit, with *i*(*t*) being the given current flowing through the capacitor *c*(*t*) and *u*(*t*) representing the voltage on the capacitor *c*(*t*), respectively. Define the parametric transfer function $W(s,t)=U(s,t)/I(s)$ of such a capacitance. According to expression (1) we construct, for the given differential equation, the polynomials $A(s,t)$ and $B(s,t)$: $A(s,t)=c(t)s+c'(t)$, $B(s,t)=1$ and obtain L.A. Zade equation (1):

$$c(t)\cdot\frac{dW(s,t)}{dt}+(c(t)s+c'(t))\cdot W(s,t)=1\,. \tag{8}$$

This example is chosen also because equation (8) has an analytical solution $W(s,t)=1/(s\cdot c(t))$. But it is all the more interesting to compare the analytical solution with the solution obtained by the FS- method. Thus, according to the FS-method, we choose, for example, the simplest approximation for $W(s,t)$, containing one harmonic component:

$$\hat{W}(s,t)=W_0(s)+W_{c1}(s)\cdot\cos(\Omega t)+W_{s1}(s)\cdot\sin(\Omega t)\,. \tag{9}$$

From (9) determine

$$\begin{aligned}&\frac{d\hat{W}(s,t)}{dt}= \\ &=-\Omega\cdot W_{c1}(s)\cdot\sin(\Omega t)+\Omega\cdot W_{s1}(s)\cdot\cos(\Omega t)\end{aligned}\,. \tag{10}$$

Substitute expressions (9), (10) into (8) and obtain functional (5) in the form:

$$\begin{aligned}&\delta(W_0,W_{c1},W_{s1},t)=c_0\left(1+m\cdot\cos(\Omega t)\right)\cdot\\&\cdot\left(W_{s1}\Omega\cdot\cos(\Omega t)-W_{c1}\Omega\cdot\sin(\Omega t)\right)+\\&+(c_0\cdot(1+m\cdot\cos(\Omega t))\cdot s-c_0\Omega m\cdot\sin(\Omega t))\cdot\\&\cdot\left(W_0+W_{c1}\cdot\cos(\Omega t)+W_{s1}\sin(\Omega t)\right)-1=0.\end{aligned}\tag{11}$$

Since functional (11) is equal to zero, its constant component and the component of the first harmonic in the expansion (11) to the Fourier series for $k=1$ are also equal to zero. Therefore

$$\begin{cases}\dfrac{1}{T}\cdot\int_0^T\delta dt=\dfrac{1}{2}\cdot c_0\cdot s\cdot m\cdot W_{c1}+c_0\cdot s\cdot W_0-1=0,\\ \dfrac{2}{T}\cdot\int_0^T\delta\cdot\cos(\Omega t)dt=\\ \quad=c_0\cdot\left(s\cdot m\cdot W_0+s\cdot W_{c1}+\Omega\cdot W_{s1}\right)=0,\\ \dfrac{2}{T}\cdot\int_0^T\delta\cdot\sin(\Omega t)dt=\\ \quad=c_0\cdot\left(-\Omega\cdot m\cdot W_0-\Omega\cdot W_{c1}+s\cdot W_{s1}\right)=0.\end{cases}\tag{12}$$

Three $(2k+1=3)$ equations (12) define a SSLAE with three unknowns W_0, W_{c1}, W_{s1}, which in matrix form has the form:

$$\begin{bmatrix}c_0 s & \frac{1}{2}c_0 sm & 0\\ c_0 sm & c_0 s & c_0\Omega\\ -c_0\Omega m & -c_0\Omega & c_0 s\end{bmatrix}\cdot\begin{bmatrix}W_0\\W_{c1}\\W_{s1}\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}.\tag{13}$$

The symbolic solution of SSLAE (13) is as follows:

$$\begin{aligned}&W_0=-\frac{2}{c_0\cdot m^2\cdot s-2\cdot c_0\cdot s},\\&W_{c1}=\frac{2\cdot m}{c_0\cdot m^2\cdot s-2\cdot c_0\cdot s},\ W_{s1}=0.\end{aligned}\tag{14}$$

If the approximation of the transfer function $W(s,t)$ takes into account 2 harmonic components ($k=2$), then expression (9) will look like:

$$\begin{aligned}\hat{W}(s,t)&=W_0(s)+W_{c1}(s)\cos(\Omega t)+\\&+W_{s1}(s)\sin(\Omega t)+W_{c2}(s)\cos(2\Omega t)+\\&+W_{s2}(s)\sin(2\Omega t),\end{aligned}\tag{9a}$$

its time derivative is as follows:

$$\begin{aligned}&\frac{d\hat{W}(s,t)}{dt}=-\Omega\cdot W_{c1}(s)\cdot\sin(\Omega t)+\\&+\Omega\cdot W_{s1}(s)\cdot\cos(\Omega t)-2\Omega\cdot W_{c2}(s)\cdot\\&\cdot\sin(2\Omega t)+2\Omega\cdot W_{s2}(s)\cdot\cos(2\Omega t).\end{aligned}\tag{10a}$$

Substituting expressions (9a), (10a) in (8), we obtain functional (5) in the following form:

$$\begin{aligned}&\delta(W_0,W_{c1},W_{s1},W_{c2},W_{s2},t)=c_0\left(1+m\cdot\cos(\Omega t)\right)\cdot\\&\cdot(W_{s1}\Omega\cdot\cos(\Omega\cdot t)-W_{c1}\Omega\cdot\sin(\Omega t)-\\&-2\cdot W_{c2}\Omega\cdot\sin(2\Omega t)+2\cdot W_{s2}\Omega\cdot\cos(2\Omega t))+\\&+(c_0\cdot\left(1+m\cdot\cos(\Omega\cdot t)\right)\cdot s-c_0\Omega m\cdot\sin(\Omega t))\cdot\\&\cdot\left(W_0+W_{c1}\cdot\cos(\Omega t)+W_{s1}\cdot\sin(\Omega t)+\right.\\&\left.+W_{c2}\cdot\cos(2\Omega t)+W_{s2}\cdot\sin(2\Omega t)\right)-1=0.\end{aligned}\tag{11a}$$

According to expression (11a) a SLAR of $2k+1=5$ order is formed:

$$\begin{cases}\dfrac{1}{T}\cdot\int_0^T\delta dt=0,\\ \dfrac{2}{T}\cdot\int_0^T\delta\cdot\cos(\Omega t)dt=0,\\ \dfrac{2}{T}\cdot\int_0^T\delta\cdot\sin(\Omega t)dt=0,\\ \dfrac{2}{T}\cdot\int_0^T\delta\cdot\cos(2\Omega t)dt=0,\\ \dfrac{2}{T}\cdot\int_0^T\delta\cdot\sin(2\Omega t)dt=0,\end{cases}$$

or after the symbolic definition of integrals:

$$\begin{cases}c_0 sW_0+\frac{1}{2}c_0 msW_{c1}=1\\ c_0 msW_0+c_0 sW_{c1}+c_0\Omega W_{s1}+\frac{1}{2}c_0 msW_{c2}+\\ \quad+\frac{1}{2}c_0 m\Omega W_{s2}=0\\ -c_0 m\Omega W_0-c_0\Omega W_{c1}+c_0 sW_{s1}-\frac{1}{2}c_0 m\Omega W_{c2}+\\ \quad+\frac{1}{2}c_0 msW_{s2}=0\\ \frac{1}{2}c_0 msW_{c1}+c_0\Omega mW_{s1}+c_0 sW_{c2}+\\ \quad+2c_0\Omega W_{s2}=0\\ -c_0 m\Omega W_{c1}+\frac{1}{2}c_0 smW_{s1}-2c_0\Omega W_{c2}+\\ \quad+c_0 sW_{s2}=0,\end{cases}\tag{13a}$$

where the coefficients $W_0, W_{c1}, W_{s1}, W_{c2}, W_{s2}$ from (9a) are unknown.

The solution to SSLAE (13a) are the expressions:

$$W_0=-\frac{m^4-4}{4\cdot c_0\cdot s-3\cdot c_0\cdot m^2\cdot s},$$
$$W_{c1}=-\frac{4\cdot m}{4\cdot c_0\cdot s-3\cdot c_0\cdot m^2\cdot s}, W_{s1}=0, \qquad (15)$$
$$W_{c2}=\frac{2\cdot m^2}{4\cdot c_0\cdot s-3\cdot c_0\cdot m^2\cdot s}, W_{s2}=0.$$

Compare the obtained solutions (14) and (15) with the analytical solution. To do this, set of numerical values $c_0=1$ F, $\Omega=1$ rad/s, $m=0{,}1$ to the parameters of the parametric capacitor. Voltage U is determined by the expression $U=\mathrm{Re}\left[\hat{W}(s,t)\cdot e^{st}\right]$ for the FS-determined transfer function $W(s,t)$ and by the expression $U=\mathrm{Re}\left[\frac{1}{s\cdot c(t)}\cdot e^{st}\right]$ for the analytical solution, $s = j\cdot 1$. The calculation results for k harmonic components in the approximation of the transfer function $\hat{W}(s,t)$ and the analytical solution for $t=100,101,\ldots,105$ s are given in Table 1. Here and in the following Tables, the results of the calculations are presented taking into account the four digits after comma.

Table 1

Voltage calculations on parametric capacitor

t, c	*U* by FS-method, B			*U* by analytical expression, B
	k=1	*k*=2	*k*=3	
1	2	3	4	5
100	-0,4650	-0,4662	-0,4662	-0,4662
101	0,4138	0,4150	0,4150	0,4150
102	0,9897	0,9847	0,9848	0,9848
103	0,6751	0,6759	0,6759	0,6759
104	-0,3538	-0,3552	-0,3553	-0,3553
105	-0,9989	-0,9947	-0,9945	-0,9945

Table 1a shows the relative errors for each k of Table 1 compared to the solution obtained by an analytical expression.

As follows from Table 1 and Table 1a, the voltage values are closer to the analytical solution when more harmonic components are taken into account in the transfer function $W(s,t)$.

Table 1a

Relative errors of voltage calculations on parametric capacitor

t, c	Relative error of FS-method, %		
	k=1	*k*=2	*k*=3
1	2	3	4
100	2,5740E-03	0,0000E+00	0,0000E+00
101	2,8916E-03	0,0000E+00	0,0000E+00
102	4,9756E-03	1,0154E-04	0,0000E+00
103	1,1836E-03	0,0000E+00	0,0000E+00
104	4,2218E-03	2,8145E-04	0,0000E+00
105	4,4243E-03	2,0111E-04	0,0000E+00

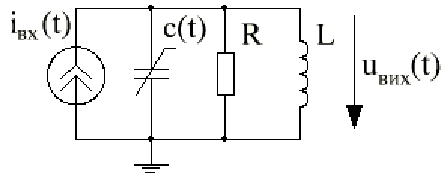


Fig. 1. Single-circuit parametric amplifier,
$i_{вх}(t)=I_m\cdot\cos(\omega\cdot t+\varphi)$; I_m=1 A; ω=1 c^{-1} ;
Y=1/R=1Sm; L=1 H; $c(t)=c_0\cdot(1+m\cdot\cos(\Omega t))$; c_0=1 F; $\Omega=2\omega$.

EXAMPLE 2. Consider the features of determining the transfer function of a single-circuit parametric amplifier shown in Fig.1 by using the FS-method.

The differential equation describing a given circuit with respect to the input current $i_{вх}(t)$ and output voltage $u_{вих}(t)$ is as follows:

$$\begin{aligned} L\cdot i'_{вх} &= L\cdot c(t)\cdot u''_{вих}+ \\ &+\left(2\cdot c'(t)\cdot L+L\cdot Y\right)\cdot u'_{вих}+. \\ &+\left(c''(t)\cdot L+1\right)\cdot u_{вих} \end{aligned} \qquad (16)$$

L.A. Zade equation (1) derived from expression (16) has the form:

$$\begin{aligned} &L\cdot c(t)\cdot W''(s,t)+ \\ &+\left(2\cdot L\cdot c(t)\cdot s+2\cdot c'(t)\cdot L+L\cdot Y\right)\cdot W'(s,t)+ \\ &+\left(L\cdot c(t)\cdot s^2+\left(2\cdot c'(t)\cdot L+L\cdot Y\right)\cdot s+\right. \\ &\left.+\left(c''(t)\cdot L+1\right)+\left(c''(t)\cdot L+1\right)\right)\cdot W(s,t)=L\cdot s. \end{aligned} \qquad (17)$$

By analogy with example 1 in the approximation of the parametric transfer function $W(s,t)$ by one harmonic component (9), we obtain the following SSLAE, whose unknowns are the coefficients of approximation W_0, W_{c1}, W_{s1}:

$$\begin{bmatrix} \left(1+sLY+C_0Ls^2\right) & \frac{1}{2}C_0Ls^2m & 0 \\ \left(C_0Ls^2m-C_0L\Omega^2m\right) & \left(1+C_0Ls^2-C_0L\Omega^2+LYs\right) & \left(LY\Omega+2L\Omega C_0s\right) \\ -2C_0m\Omega Ls & -\left(L\Omega Y+2L\Omega C_0s\right) & \left(1+sLY+C_0Ls^2-C_0L\Omega^2\right) \end{bmatrix} \cdot \begin{bmatrix} W_0 \\ W_{c1} \\ W_{s1} \end{bmatrix} = \begin{bmatrix} Ls \\ 0 \\ 0 \end{bmatrix}. \quad (18)$$

The complete symbolic solution (18) obtained by MATLAB is rather cumbersome, so it is not given here. The same solution in fractional-rational form (the symbol specifies only a complex variable s) at m=0.1 is as follows:

$$W0 = \frac{200s\left(s^4+2s^3+11s^2+10s+13\right)}{199s^6+599s^5+2791s^4+4596s^3+6788s^2+4600s+2600},$$

$$Wc1 = -\frac{20s\left(s^4+s^3+9s^2+4s+12\right)}{199s^6+599s^5+2791s^4+4596s^3+6788s^2+4600s+2600},$$

$$Ws1 = -\frac{40s\left(s^2+2s+4\right)}{199s^6+599s^5+2791s^4+4596s^3+6788s^2+4600s+2600}.$$

The symbolic solution in fractional-rational form (the symbol specifies only a complex variable s) when approximating the parametric transfer function by two harmonic components, obtained in the MATLAB program, has the following form:

*W*0= *s** (549420160* *s*^2+159201* *s*^8+455654400* *s*+638400* *s*^7+499389696+ 115871728* *s*^4+7965640* *s*^6+196723200* *s*^3+21710400* *s*^5)/(30241688* *s*^7+ 333736816* *s*^5+145060808* *s*^6+859839984* *s*^4+158403* *s*^10+1200700672* *s*^3+795203* *s*^9+1502158336* *s*^2+8726523* *s*^8+955044096* *s*+499389696),

*W*c1= -40* *s** (1087552* *s*^2+548544* *s*+18359* *s*^6+243480* *s*^4+1199* *s*^7+ 1152960+284256* *s*^3+36376* *s*^5+399* *s*^8)/(30241688* *s*^7+333736816* *s*^5+ 145060808* *s*^6+859839984* *s*^4+158403* *s*^10+1200700672* *s*^3+795203* *s*^9+ 1502158336* *s*^2+8726523* *s*^8+955044096* *s*+499389696),

*W*s1= 160* (201* *s*^6+802* *s*^5+8624* *s*^4+22416* *s*^3+89944* *s*^2+123632* *s*+ 193056)* *s*/(30241688* *s*^7+333736816* *s*^5+145060808* *s*^6+859839984* *s*^4+ 158403* *s*^10+1200700672* *s*^3+795203* *s*^9+1502158336* *s*^2+8726523* *s*^8+ 955044096* *s*+499389696),

*W*c2= 2* (399* *s*^8+800* *s*^7+17160* *s*^6+24800* *s*^5+223872* *s*^4+198400* *s*^3+ 968640* *s*^2+332800* *s*+943104)* *s*/(30241688* *s*^7+333736816* *s*^5+145060808* *s*^6+859839984* *s*^4+158403* *s*^10+1200700672* *s*^3+795203* *s*^9+1502158336* *s*^2+8726523* *s*^8+955044096* *s*+499389696),

*W*s2=-4800* (*s*^6+3* *s*^5+31* *s*^4+54* *s*^3+244* *s*^2+240* *s*+448)* *s*/(30241688* *s*^7+333736816* *s*^5+145060808* *s*^6+859839984* *s*^4+158403* *s*^10+ 1200700672* *s*^3+795203* *s*^9+1502158336* *s*^2+8726523* *s*^8+955044096* *s*+ 499389696).

When completing example 2, compare the calculations of the output voltage of the circuit (Fig.1) performed by MicroCap7.0 (column 2 of Table 2) and by the program that implements the developed FS- method with the number of harmonic components in the approximation of the parametric transfer function of 1, 2, 3 and 4 (columns 3, 4, 5, 6 of Table 2), respectively. The results shown in Table 2 were obtained at the modulation coefficient of the parametric capacitance $m = 0.2$ and the phase difference between the harmonic change in the capacitance and the input signal (current) $\phi = -45°$ at k harmonic components in the approximation $W(s,t)$. Column 7 shows the relative errors of calculations by the FS-method at k=4 compared to the results of calculations using MicroCap7 (column 2).

From Table 2 it follows that the values in columns 2, 3, 4, and 5 differ slightly from each other, but the values of columns 2 and 6 coincide completely. These results, as well as the results of example 1, also prove the adequacy of the FS-method and the program MAOPCs for a symbolic analysis of parametric circuits developed by the team including the author of this paper [10].

Table 2

Time, c	Result by Micro-Cap 7.0, B	Result by FS-method, B				Relative error of FS-method, %
		k=1	*k*=2	*k*=3	*k*=4	
1	2	3	4	5	6	7
62,832	0,6924	0,6922	0,6925	0,6924	0,6924	0,0000
63,146	0,8918	0,8986	0,8911	0,8919	0,8918	0,0000
63,460	1,0714	1,0837	1,0720	1,0713	1,0714	0,0000
63,774	1,1754	1,1707	1,1767	1,1756	1,1754	0,0000
64,088	1,1092	1,0860	1,1069	1,1091	1,1092	0,0000
64,403	0,8244	0,8188	0,8242	0,8244	0,8244	0,0000
64,717	0,4207	0,4411	0,4230	0,4208	0,4207	0,0000
65,031	0,0449	0,0588	0,0440	0,0447	0,0449	0,0000
65,345	-0,2494	-0,2527	-0,2502	-0,2493	-0,2494	0,0000
65,659	-0,4839	-0,4881	-0,4835	-0,4839	-0,4839	0,0000
65,973	-0,6920	-0,6918	-0,6921	-0,6920	-0,6920	0,0000
66,288	-0,8921	-0,8989	-0,8914	-0,8921	-0,8921	0,0000
66,602	-1,0716	-1,0839	-1,0722	-1,0715	-1,0716	0,0000
66,916	-1,1754	-1,1708	-1,1768	-1,1756	-1,1754	0,0000
67,230	-1,1090	-1,0858	-1,1068	-1,1089	-1,1090	0,0000
67,544	-0,8251	-0,8195	-0,8249	-0,8251	-0,8251	0,0000
67,858	-0,4215	-0,4418	-0,4238	-0,4216	-0,4215	0,0000
68,173	-0,0445	-0,0583	-0,0436	-0,0443	-0,0445	0,0000
68,487	0,2497	0,2530	0,2505	0,2496	0,2497	0,0000
68,801	0,4842	0,4883	0,4838	0,4842	0,4842	0,0000

4. Choosing the number of harmonic components in the approximation of the function $W(s,t)$.

It was noted above that SSLAE (7) can be formed from arbitrary harmonic components of functional (5). It is obvious that the presence of certain harmonic components in approximating expression (3) can significantly affect its adequacy. The difficulty of the question is that the expression being approximated is missing in analytical form. Only differential equation (1) is known, which this approximation expression must satisfy. Therefore, without solving the problem of forming an approximation polynomial with mathematical thoroughness, we present an engineering method for solving this problem.

Previously, remind:

1. If the parametric element of the circuit changes with the period $T = \dfrac{2\pi}{\Omega}$, then the conjugate parametric transfer function $W(s,t)$ also changes with the period *T*.

2. Regardless of the number of harmonic components that specify the parametric element (one or more), the function $W(s,t)$ contains an unlimited number of harmonic components with frequencies $i\Omega$, $i = 0,1,2,\dots$.

As a rule [9], the highest accuracy for $\hat{W}(s,t)$ is obtained when harmonic components are included in the approximation in a row, starting with the basic frequency of change in the parametric element Ω: constant component, first harmonic component Ω, second 2Ω, etc. This is physically clear: if the parameter *c*(*t*) of a parametric element is taken to be constant, then when approximating $\hat{W}(s,t)$, the constant term of a trigonometric polynomial is also sufficient. If the parametric element begins to change harmoniously with the period T, but with insignificant depths of modulation ($m_{c1} \ll 1$, $m_{s1} \ll 1$), then often in the approximation of $\hat{W}(s,t)$, it is enough to take into account only the first or the first two harmonic components. With a significant change in the parametric element - in the approximation of $\hat{W}(s,t)$ we have to take into account the first, second and higher harmonic components. We expect that in the general case, an increase in the approximation of the number of harmonic components to k should lead to a decrease in the methodological error of the frequency- symbolic method, however, this is due to an increase in the order of SSLAE, which is obtained.

On the other hand, we can notice one of the positive features of the method, which is that the frequency-symbolic method does not calculate the members of the series, but determines the unknown coefficients of approximation of the desired parametric transfer function, chosen as a trigonometric polynomial. Although these are close concepts, but not identical. In particular, the process of calculating a series consists, for example, in the alternate determination of each of its subsequent members with the already defined previous ones, which remain unchanged in the future. In this case, no doubt, you should control the convergence of the formed series. By the frequency-symbolic method, the unknown coefficients of the approximating function by the trigonometric Fourier polynomial are determined from the SSLAE independently of each other, and an increase in k, for example, by one leads to changing all members of the series obtained at the previous value k, not just the last (last two).). In addition, the required function is periodic, differentiated the required number of times, and the trigonometric polynomial is used not as a series, but as a system of orthogonal functions. Therefore, in our case, it is incorrect to speak about the convergence of the series, but it is enough to ensure only the adequacy of approximation, which is determined by the harmonic components selected for the approximating expression, and their number.

Illustrate the latter in Example 1 (parametric capacitance). L.A. Zade equation (8) for parametric capacitance, as mentioned, has an analytical solution $W(s,t) = 1/(s \cdot c(t))$. Therefore, it is easy to show how the number of selected members in the approximation of $\hat{W}(s,t)$ affects the accuracy of the result. In particular, Fig. 2 shows the dependences of the relative error η of determining the individual coefficients of approximation of the function $\hat{W}(s,t)$ depending on the accepted number of k harmonic components in it. The relative error was determined between the corresponding coefficients of the trigonometric polynomial, determined by the frequency-symbolic method, and the expansion of the analytical solution in a Fourier series (in both cases the value k was chosen to be the same and equal in turn 1, 2,…, 8). From the above dependences it follows that:

– the relative error of determining each approximation coefficient decreases significantly with increasing the number of harmonic components in it;

– in the approximation, with the coefficient belonging to the higher harmonic component, its accuracy decreases.

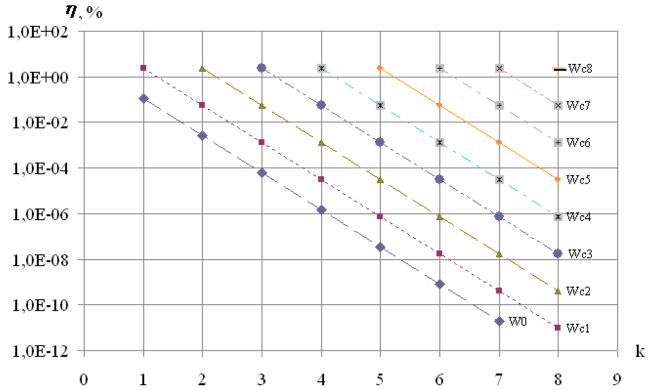


Fig. 2. Dependences of relative errors η of separate approximation coefficients for $\hat{W}(s,t)$ at different quantity of k harmonic components in it.

Figure 3 shows the relative errors of calculating the function $W(s,t)$ by the frequency-symbolic method and the expansion of the analytical solution in a Fourier series at two time points t_1 =1,0472 s and t_2 =4,1888 s depending on the number of harmonic components taken into account in the function $\hat{W}(s,t)$. The error in both cases is determined in relation to this analytical solution. From the above dependences it follows that:

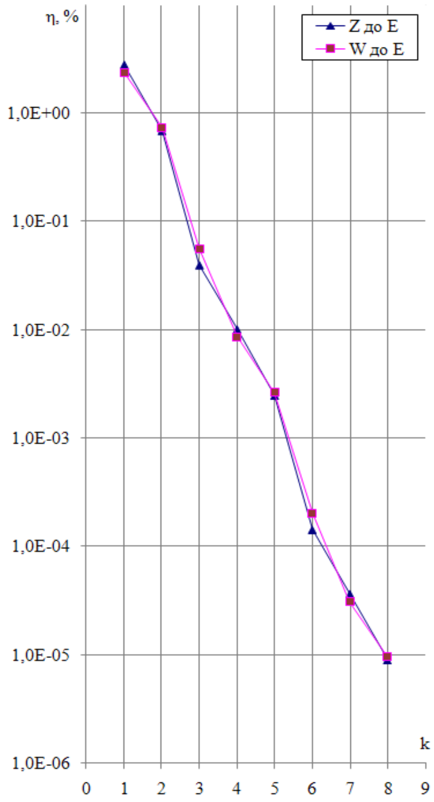
– the result of the expansion of analytical solution in a Fourier series is not always more accurate than the approximation determined by the frequency-symbolic method;

– the values of the solution approximation determined by the frequency- symbolic method at some time points are more accurate than the values of the series obtained from the analytical solution;

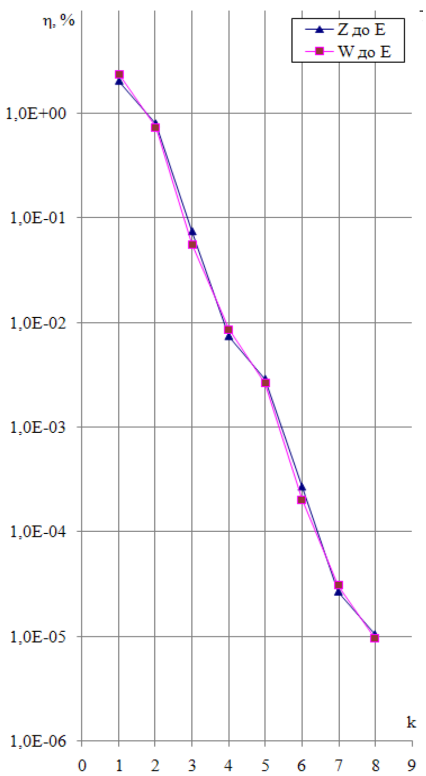
– increasing the number of harmonic components in both cases always leads to a decrease in error.

As the results of computational experiments shown in Fig. 2 and Fig. 3, the main tool for reducing the error in determining the transfer function by the frequency-symbolic method is to increase the number of harmonic components in its approximation. But the latter leads to the cumbersomeness of symbolic expressions and, as a consequence, to emergency stops of calculations in the MATLAB program. Therefore, in practice, an approximation expression should contain the least number of harmonic components, but those that ensure the adequacy of the result.

On the other hand, no matter how many harmonic components are taken into account in the approximation of $\hat{W}(s,t)$, there will never be a certainty that the approximation expression includes all “significant” harmonic components. However, for the practical application of the frequency-symbolic method, we can use the method of accuracy control presented in the Conclusion.



a)



b)

Fig. 3. Relative errors of the calculation of the function W(s,t) by the FS-method (W) and by the expansion of the analytical solution (Z) in a Fourier series with respect to the analytical solution (E) for: a) t = 1,0472 *s; b) t* = 4.1888*s.*

5. Conclusion

Based on the material presented above, in conclusion, we can present the following method of controlling the accuracy of the frequency-symbolic method for the analysis of linear parametric circuits.

The accuracy of the obtained transfer function with the constant component and the k first harmonic components taken into account in the approximation expression is sufficient if the increase in their number in the approximation n does not lead to a significant change in the results obtained on its basis, insufficient - if vice versa. In the latter case, it is necessary to increase k and repeat the calculation.

For example, if we have two variants of the output signal of the circuit defined by the product of the input signal $e^{j\omega t}$ by the approximation of the transfer function $\hat{W}(s,t)$ by the expression [4,5]:

$$y(t) = \mathrm{Re}[\hat{W}(s,t)\cdot e^{j\omega t}] \qquad (20)$$

when the number of harmonics k and k+n, respectively, and these both output signals are within a given deviation λ, we assume that the transfer function $\hat{W}(s,t)$ is defined with sufficient accuracy. If not - then, as mentioned above, repeat the calculation of the transfer function $\hat{W}(s,t)$ with a larger value of k.

Obviously, this method does not guarantee compliance with the requirements for the accuracy of the results, but the greater n, the more chances that with a given number of harmonic components k, the proposed method of accuracy control will provide an adequate result.

In our opinion, the final determination of the number of required harmonic components k in the approximation of the transfer function can be put on a specialist-researcher dealing with the linear parametric circuit, who understands the electrical processes occurring in the circuit and can predict the choice of “significant” harmonic components in the parametric transfer functions.

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ПРО АДЕКВАТНІСТЬ ЧАСТОТНОГО СИМВОЛЬНОГО МЕТОДУ АНАЛІЗУ ЛІНІЙНИХ ПАРАМЕТРИЧНИХ КІЛ

Юрій Шаповалов

Частотний символьний метод (ЧС-метод) аналізу усталеного режиму лінійних параметричних кіл призначений для формування їх передавальних функцій у частотній області. Передавальні функції апроксимуються поліномами Фур’є та містять комплексну змінну, змінну час та параметри елементів кола у вигляді символів. Коефіцієнти таких поліномів Фур’є за ЧС-методом виступають невідомими у символьних системах лінійних алгебраїчних рівнянь (ССЛАР), і визначаються як їх розв’язки у символьному вигляді.

Подано спосіб формування апроксимаційного виразу, який забезпечує адекватність обчислень.

Наведено приклади та результати комп’ютерних експериментів. Основана на частотному символьному методі система функцій MAOPCs використовується при оптимальному проектуванні електронних пристроїв завадостійких скритних радіотехнічних систем з використанням кодових сигналів.

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