COMPUTATIONAL PROBLEMS OF ELECTRICAL ENGINEERING

Vol. 5, No. 2, 2015

APPLICATION OF DIAKOPTICS APPROACH TO ANALYSIS OF ELECTROMAGNETIC FIELD BY MEANS OF FINITE-DIFFERENCE METHOD

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Abstract. The paper discusses versatile constraint equations used for providing compatibility between the solutions obtained for separate sub-domains of electromagnetic field which are analyzed simultaneously. The mathematical model of the task has been developed using invariant approximation technique for finite-difference method. Different techniques for domain decomposition are considered. The discussed constraint equations and domain decomposition techniques have been applied to solving a test problem, and namely the problem of magnetic skin-effect. On the basis of obtained computational results some recommendations regarding the scope of overlapping and applicability of different constraint equations have been formulated.

Key words: domain decomposition, electromagnetic field, finite-difference method, invariant approximation technique, parallelization, parallel processing, simultaneous calculation, diakoptics.

1. Introduction

Parallelization of computations is widely used for the analysis of lumped and distributed electric circuits [1]. Its application to electromagnetic fields analysis is even more crucial because of high order of a system of equations obtained as a result of the implementation of finite-difference or finite-element method. In that case parallelization of computations leads to necessity of dividing the computational area into sub-domains (so called domain decomposition) that can be completed in many different ways [2]. Usually non-overlapping sub-domains are used [3]. The decomposition may be not only spatial one, but also performed in time-domain [4].

Our paper discusses the development of constraint equations intended for increasing the compatibility of solutions obtained for different sub-domains of field distribution. The discussion concentrates on discrete analogues of partial differential equations obtained by the application of invariant approximation technique [5] to finite-difference method.

2. Application of Finite-Difference Method to Maxwell Equations

In general, a boundary problem of electromagnetic field analysis on a domain filled with a nonlinear heterogeneous anisotropic medium is formulated as follows. Let us consider the domain Ω of the field propagation with the boundary Γ that, in general, may be divided into three parts with the Dirichlet, Neumann, and Robin boundary conditions. The electromagnetic field in any point of the domain is described by Maxwell partial differential equations supplemented by medium characteristics:

$$rot\bar{H} = \bar{J} + \partial\bar{D}/\partial t,\ div\bar{B} = 0,$$

$$rot\bar{E} = -\partial\bar{B}/\partial t, \quad div\bar{D} = r, \tag{1}$$

$$\bar{H} = \bar{H}\left[\bar{B}\right], \bar{D} = \bar{D}\left[\bar{E}\right], \bar{J} = \bar{J}\left[\bar{E}\right],$$

where $\bar{H}$, $\bar{J}$, $\bar{D}$, $\bar{B}$, $\bar{E}$ are vectors of magnetizing field, electric current density, displacement field, magnetic field, and electric field, respectively; r is electric charge density.

Let us apply to the field domain a grid consisting of $M = M_\Omega + M_\Gamma$ nodes. Using the invariant approximation technique, we can assign a nonsingular set of P nodes to any m-th node ($m = \overline{1, M}$) and calculate the algebraic analogue of Hamiltonian operator for each node. Therefore, the algebraic analogue of the system (1) can be written in the form:

$$\bar{\bar{R}}_{\nabla m} \times \bar{\bar{H}}_m = \bar{J}_m + \partial\bar{D}_m/\partial t;\ \bar{\bar{R}}_{\nabla m} \cdot \bar{\bar{B}}_m = 0;$$

$$\bar{\bar{R}}_{\nabla m} \times \bar{\bar{E}}_m = -\partial\bar{B}_m/\partial t;\ \bar{\bar{R}}_{\nabla m} \cdot \bar{\bar{D}}_m = r_m; \tag{2}$$

$$\bar{H}_m = \bar{H}\left[\bar{B}_m\right], \bar{D}_m = \bar{D}\left[\bar{E}_m\right], \bar{J}_m = \bar{J}\left[\bar{E}_m\right],$$

where $\bar{\bar{R}}_{\nabla m}$ is an algebric analogue of the Hamiltonian operator for the m-th node, $\bar{\bar{D}}_m, \bar{\bar{E}}_m, \bar{\bar{B}}_m, \bar{\bar{H}}_m$ are nodal columns of field variables for the m-th nodal set.

In $\bar{A}, \varphi$ formulation (where $\bar{A}$ is vector magnetic potential, φ is scalar electric potential):

$$\vec{R}_{\nabla m}\times\vec{H}_m=\bar{J}[\bar{E}_m]+\partial\bar{D}_m/\partial t;$$

$$\bar{H}_m=\bar{H}[\vec{R}_{\nabla m}\times\vec{A}_m]=0;$$

$$\bar{E}_m=-\partial\bar{A}_m/\partial t-\vec{R}_m\cdot\vec{\varphi};$$

$$\vec{R}_{\nabla m}\cdot\vec{D}_m=\rho_m; \qquad (3)$$

$$\bar{D}_m=\bar{D}\left[\bar{E}_m\right],$$

that can be combined into two equations:

$$\vec{R}_{\nabla m}\times\overline{\bar{H}[\vec{R}_{\nabla m}\times\vec{A}_m]}_m=$$

$$=-\bar{J}[\partial\bar{A}_m/\partial t+\vec{R}_{\nabla m}\cdot\vec{\varphi}_m]=$$

$$=-\bar{J}[\partial\bar{A}_m/\partial t+\vec{R}_{\nabla m}\cdot\vec{\varphi}_m]-\overleftrightarrow{\varepsilon}\cdot\partial^2\bar{A}_m/\partial t^2-\vec{R}_{\nabla m}\cdot d\vec{\varphi}_m/dt;$$

$$\vec{R}_{\nabla m}\cdot\overline{\bar{D}[d\bar{A}_m/dt+\vec{R}_{\nabla m}\cdot\vec{\varphi}_m]}_m=\rho_m; \qquad (4)$$

where $\overleftrightarrow{\varepsilon}$ is tensor of differential electric permittivity.

The algebraic analogue consists of equations (4) applied to internal nodes and boundary conditions applied to boundary nodes.

The application of invariant approximation technique allows us to utilize nodal sets of arbitrary form for construction of difference analogues of differential operators. The only requirement is non-singularity of Taylor matrix. Examples of internal and near-border 2D nodal sets are given in Fig. 1–3.

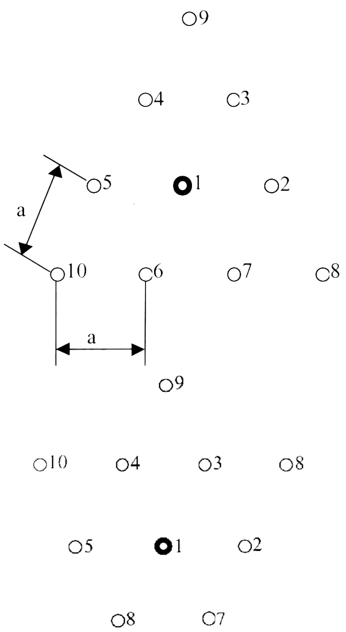

Fig. 1. Internal 2D nodal sets.

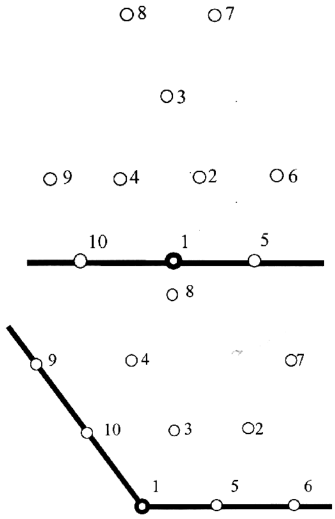

Fig. 2. Near-border 2D nodal sets in case of a plain boundary.

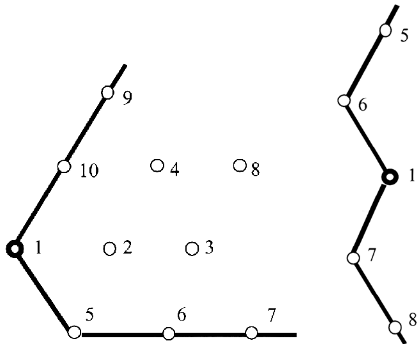

Fig. 3. Near-border 2D nodal sets in case of a piece-wise boundary.

3. Application of Diakoptics Approach

3.1. Ways of Domain Decomposition and Construction of Constraint Equations

Let us divide the domain Ω into sub-domains which means breaking up the system (3) into a set of sub-systems. Each sub-domain contains two kinds of nodal sets: sets with nodes belonging to the sub-domain (called "native") and sets with nodes belonging both to the sub-domain and to adjacent sets (called "alien"). The borders between the sub-domains may be put amid nodes, as well as include nodes as shown in Fig. 4 and Fig. 5.

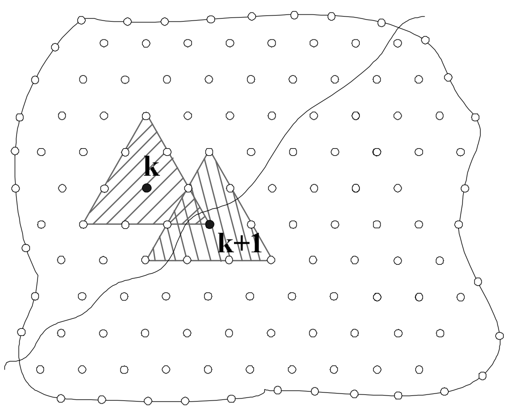

Fig. 4. Strict division of nodes into "native" and "alien" ones.

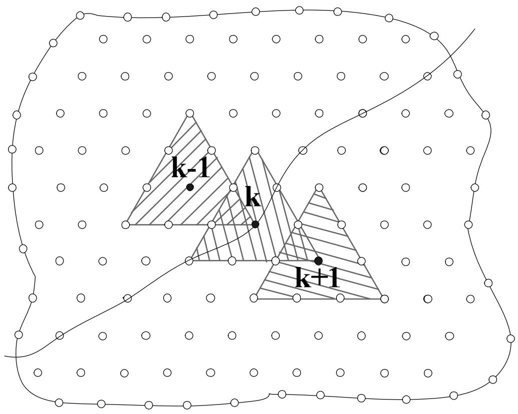

Fig. 5. Division of nodes into "native", "alien", and "common" ones.

At each integration step the values of field characteristics in "alien" nodes may be fixed at the values obtained at the previous step; constraint equations may be of two kinds: new values of field characteristics are equal to their values obtained as a result of solving "alien" sub-systems; new values of field characteristics are calculated as a combination of their values obtained as a result of solving "alien" sub-systems and old values with chosen weight coefficients. Constraint equations for "common" nodes may also be of two kinds: new values of field characteristics are calculated as the arithmetic mean of their values obtained as a result of solving different sub-systems or as the combination of the values with chosen weight coefficients. The order of a sub-system corresponds to the number of the nodes in its relevant sub-domain. Therefore, we have a dilemma. If we include into sub-systems the "common" nodes, the order of the total system increases (equations for "common" nodes are present in all adjacent sub-systems) but, presumably, the accuracy of computation increases.

3.2. Techniques for Increasing Accuracy

1) New values of field characteristics in "alien" nodes are calculated as linear combinations of their values obtained as a result of solving the "alien" sub-system and their old values with chosen weight coefficients.

2) New values of field characteristics in "common" nodes are calculated as a linear combination of the values obtained in different sub-domains with chosen weight coefficients.

3) No strict fixation of sub-domains; they may penetrate each other if it is dictated by reasons of accuracy or convergence.

4. Computational Experiment

As a test problem we have chosen a task that has an analytical solution, that is, the flow of alternating magnetic flux $\Phi = \Phi_m \cos wt$ in a cylindrical conductor of radius *a*=6 mm with magnetic permeability $m = 1000 m_0$ and specific electric conductivity $g = 10^7$ Sm/m. The problem is described by the following equations:

$$rot\bar{H} = g\,\bar{E},\ div\bar{H} = 0,$$

$$rot\bar{E} = -m\partial\bar{H} / \partial t, \qquad (5)$$

After substitution

$$rotrot\bar{H} = -mg\partial\bar{H} / \partial t, \qquad (6)$$

Taking into account the second equation

$$\nabla^2\bar{H} = mg\partial\bar{H} / \partial t. \qquad (7)$$

Magnetic field is directed along the axis *z*, electric field is directed along the axis α. That is why we can rewrite the equation (7) in complex scalar magnitudes:

$$\frac{d^2\dot{H}}{dr^2} + \frac{1}{r}\frac{d\dot{H}}{dr} - jmg\,\dot{H} = 0. \qquad (8)$$

The obtained equation may be presented in the form

$$\frac{d^2\dot{H}}{d(\sqrt{-jwmg}\,r)^2} + \frac{1}{\sqrt{-jwmg}\,r}\frac{d\dot{H}}{d(\sqrt{-jwmg}\,r)} + \dot{H} = 0.$$

We have received the Bessel equation of order zero; its solution is $\dot{H} = \dot{H}_0 \dot{J}_0[\sqrt{-jwmg}\,r]$.

Let us show how to solve this task applying finite-difference method with parallelization of computations. For example, on the *i*-th step of integration the algebraic analogue of the differential equation for the electric field in a near-border node (number 2), while applying the fourth order Taylor polynomial and constant grid step *h*, looks like:

$$(\frac{11}{12h^2}E_{1,i} - \frac{5}{3h^2}E_{2,i} + \frac{1}{2h^2}E_{3,i} + \frac{1}{3h^2}E_{4,i} - \frac{1}{12h^2}E_{5,i}) +$$

$$+\frac{1}{h}(-\frac{1}{4h}E_{1,i} - \frac{5}{6h}E_{2,i} + \frac{3}{2h}E_{3,i} - \frac{1}{2h}E_{4,i} + \frac{1}{12h}E_{5,i}) =$$

$$= (\frac{1}{4h_t}E_{2,i-4} - \frac{4}{3h_t}E_{2,i-3} + \frac{3}{h_t}E_{2,i-2} - \frac{4}{h_t}E_{2,i-1} + \frac{25}{12h_t}E_{2,i}).$$

On the 3-rd step of integration the algebraic analogue of the differential equation for electric field in *k*-th node, while applying the fourth order Taylor polynomial and constant grid step *h*, looks like:

$$(-\frac{1}{12h^2}E_{k-2,3} + \frac{4}{3h^2}E_{k-1,3} - \frac{5}{2h^2}E_{k,3} + \frac{4}{3h^2}E_{k+1,3} - \frac{1}{12h^2}E_{k+2,3})$$

$$+\frac{1}{(k-1)h}(\frac{1}{12h}E_{k-2,3} - \frac{2}{3h}E_{k-1,3} + \frac{2}{3h}E_{k+1,3} - \frac{1}{12h}E_{k+2,3} =$$

$$= (-\frac{1}{3h_t}E_{k,0} + \frac{3}{2h_t}E_{k,1} - \frac{3}{h_t}E_{k,2} + \frac{11}{6h_t}E_{k,3}).$$

Boundary conditions are obtained from Ampere's law:

$$E\big|_{r=0} = 0,\ E\big|_{r=a} = w\Phi_m / (2p\,a)\sin wt\,.$$

The computational domain was covered with the grid of 101 nodes (100 steps). The domain was divided in such sub-domains: 50/50 steps, 40/20/40 steps, 30/40/30 steps, 25/25/25/25 steps, 20/20/20/20/20 steps, 10x10 steps, as well as their modifications.

5. Conclusion

1. The technique of dividing the domain into sub-domains with strict recognition of "alien" and "native" nodes failed.

2. Internal sub-domains must be less than boundary sub-domains by at least 10%.

3. If the values of field characteristics obtained for "common" node in different sub-domains differ by more than 30%, it is necessary to apply deeper penetration of sub-domains or reduce the step of time integration.

4. If the physical condition of a problem allows us to increase the physical size of sub-domains concurrently with increasing spatial step, the speed of increasing the spatial step cannot exceed 20%.

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ЗАСТОСУВАННЯ ДІАКОПТИЧНОГО ПІДХОДУ ДО АНАЛІЗУ ЕЛЕКТРОМАГНЕТНОГО ПОЛЯ ІЗ ЗАСТОСУВАННЯМ МЕТОДУ СКІНЧЕННИХ РІЗНИЦЬ

Марія Говикович, Петро Стахів

Запропоновано різноманітні рівняння зв'язку для узгодження розв'язків, отриманих для окремих підобластей електромагнетного, розрахунок яких здійснюється паралельно. Математичну модель такої задачі отримано на підставі теорії інваріантних наближень із застосуванням методускінченних різниць. Розглянуто різні методики розбиття області розрахунку поля на підобласті паралельного розрахунку. Застосування запропонованих рівнянь зв'язку і методик розбиття показано на тестовій задачі магнетного поверхневого ефекту. На підставі отриманих числових результатів висловлено рекомендації щодо обсягу взаємного перекриття підобластей та конкретного застосування рівнянь зв'язку.

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