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DERIVATIVES OF PARAMETER MATRICES

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Abstract. In the paper there is proposed a method of differentiation with respect to an argument of direct and inverse matrices of static and differential parameters through the derivative of static parameters; the method claims to be the simplest one. The method proposes, after finding the simplest derivative, to find three more complicated derivatives on the basis of formal matrix operations. The results may be used in the theory of parametric sensitivity of first and higher degrees. As an example, the direct and inverse matrices of static and differential inductances of a saturated three-phase induction motor are considered, provided that single-phase and three-phase power supply is applied.

Key words: derivative of matrix with respect to an argument, static and differential parameter, saturated induction motor, matrix of static and differential inductances.

1. Introduction

In the problems of analysis of nonlinear physical systems often there is a necessity to differentiate the parameter matrices, static or differential, with respect to an argument. In most cases one can circumvent such differentiation if so-called auxiliary variational equations [1-4] are applied. But there are some problems in which this differentiation can not be avoided, for example, at the construction of variational equations of higher order in the theory of parametric sensitivity etc. The way of simplifying such a complicated problem is shown below [5].

2. Theoretical foundations

Let us consider the vector function $\vec{y}$ with the argument of $\vec{x}$

$$\vec{y} = \vec{y}(\vec{x}) \tag{1}$$

The relation between the vectors can be written in four possible ways:

$$\begin{aligned} &\vec{y} = \mathbf{S}(\vec{x})\vec{x}; \quad \vec{x} = \mathbf{S}(\vec{x})^{-1}\vec{y}; \\ &d\vec{y} = \mathbf{D}(\vec{x})d\vec{x}; \quad d\vec{x} = \mathbf{D}(\vec{x})^{-1}d\vec{y}, \end{aligned} \tag{2}$$

where $\mathbf{S}(\vec{x}), \mathbf{D}(\vec{x})$ are matrices of static and differential parameters, accordingly.

In general case these matrices are difficult enough for differentiation, especially the reverse ones. In practice, only the matrix of static parameters (the simplest among them) can be easily differentiated. Therefore, the derivative $d\mathbf{S}(\vec{x})/d\vec{x}$ is considered to be given. In such case a question arises how to express other derivatives through the given derivative.

The derivative of the inverse matrix of static para– meters $d\mathbf{S}(\vec{x})^{-1}/d\vec{x}$ can be found through the operations of multiplication according to the formula (3)

$$\frac{d\mathbf{S}(\vec{x})^{-1}}{d\vec{x}} = -\mathbf{S}(\vec{x})^{-1}\frac{d\mathbf{S}(\vec{x})}{d\vec{x}}\mathbf{S}(\vec{x})^{-1}. \tag{3}$$

The matrix of differential parameters $\mathbf{D}(\vec{x})$ can also be obtained formally from the matrix of static parameters as a result of operations of multiplication and addition according to a formula given in [2, 4]

$$\mathbf{D}(\vec{x}) = \frac{d\mathbf{S}(\vec{x})}{d\vec{x}}\vec{x} + \mathbf{S}(\vec{x}). \tag{4}$$

Having differentiated (4) with respect to $\vec{x}$, we obtain the sought derivative of the matrix of differential parameters

$$\frac{d\mathbf{D}(\vec{x})}{d\vec{x}} = \frac{d^2\mathbf{S}(\vec{x})}{d\vec{x}^2}x + 2\frac{d\mathbf{S}(\vec{x})}{d\vec{x}}, \tag{5}$$

which is the aim of this operation.

For finding the corresponding derivative of the inverse matrix of differential parameters, let us use the same expression (3)

$$\frac{d\mathbf{D}(\vec{x})^{-1}}{d\vec{x}} = -\mathbf{D}(\vec{x})^{-1}\frac{d\mathbf{D}(\vec{x})}{d\vec{x}}\mathbf{D}(\vec{x})^{-1}. \tag{6}$$

Substituting (5) in (6), we obtain

$$\frac{d\mathbf{D}(\vec{x})^{-1}}{d\vec{x}} = -\mathbf{D}(\vec{x})^{-1}\left(\frac{d^2\mathbf{S}(\vec{x})}{d\vec{x}^2}\vec{x} + 2\frac{d\mathbf{S}(\vec{x})}{d\vec{x}}\right)\mathbf{D}(\vec{x})^{-1}. \tag{7}$$

Thus, using the simplest derivative $dS(x)/dx$ as well the expressions (3), (5), (7) we can find other much more entangled derivatives by formal matrix operations.

3. An example of application

The fact that formulae (3), (5) (7) give us a practical mathematical instrument, we can illustrate on the example of a rather complicated multipole element, that is, a saturated induction motor. The matrix of static inductances used in the reduced equations of its state and received as a result of ortogonal linear coordinate transformations has a comparatively simple form [1,3]^

$$\mathbf{S}(\vec{i}) = \mathbf{L}_\sigma + l_\tau(i_m)\mathbf{A}_2, \tag{8}$$

where

$$\mathbf{L}_\sigma = \begin{array}{|c|c|c|c|}\hline l_S & & & \\ \hline & l_S & & \\ \hline & & l_R & \\ \hline & & & l_R \\ \hline\end{array}; \quad \mathbf{A}_2 = \begin{array}{|c|c|c|c|}\hline 1 & & 1 & \\ \hline & 1 & & 1 \\ \hline 1 & & 1 & \\ \hline & 1 & & 1 \\ \hline\end{array}, \tag{9}$$

where l_S, l_R are constant leakage inductances of the stator and rotor windings; $l_\tau = l_\tau(i_m)$ is basic static inductance, found from the magnetization characteristic $\psi_m(i_m)$ of the magnetic conductor

$$l_\tau(i_m) = \frac{\psi_m(i_m)}{i_m}, \tag{10}$$

where i_m is a magnetization current

$$i_m = \sqrt{\left(i_x^2 + i_y^2\right)}; \; i_x = i_{Sx} + i_{Rx}; \; i_y = i_{Sy} + i_{Ry}, \tag{11}$$

and

$$\vec{i} = (i_{Sx}, i_{Sy}, i_{Rx}, i_{Ry})_t \tag{12}$$

is the column of the transformed stator and rotor currents serving as argument of the vector function $\vec{\Psi} = \vec{\Psi}(\vec{i})$.

Regarding other matrices $\mathbf{S}(\vec{x})^{-1}, \mathbf{D}(\vec{x}), \mathbf{D}(\vec{x})^{-1}$, we can say that they are much more complicated, so their direct differentiation is highly problematic. To illustrate the task comlexity, we give examples of those matrices below.

The inverse matrix of static parameters looks like this [3]

$$\mathbf{S}(\vec{i})^{-1} = T \begin{array}{|c|c|c|c|}\hline \alpha_S(\alpha_R+\tau) & & -\alpha_S\alpha_R & \\ \hline & \alpha_S(\alpha_R+\tau) & & -\alpha_S\alpha_R \\ \hline -\alpha_S\alpha_R & & \alpha_R(\alpha_S+\tau) & \\ \hline & -\alpha_S\alpha_R & & \alpha_R(\alpha_S+\tau) \\ \hline\end{array}, \tag{13}$$

where

$$a_S = \frac{1}{l_S}; \quad a_R = \frac{1}{l_R}; \qquad T = \frac{1}{t + a_S + a_R}; \; t(i_m) = \frac{1}{l_t(i_m)}. \tag{14}$$

The matrix of differential parameters can be received by the substitution of (8) in (4). Then we obtain

$$\mathbf{D}(\vec{i}) = L_\sigma \mathbf{1} + \frac{1}{i_m^2} \cdot \begin{array}{|c|c|c|c|}\hline l_\rho i_x^2 + l_\tau i_y^2 & (l_\rho - l_\tau)i_x i_y & l_\rho i_x^2 + l_\tau i_y^2 & (l_\rho - l_\tau)i_x i_y \\ \hline (l_\rho - l_\tau)i_x i_y & l_\tau i_x^2 + l_\rho i_y^2 & (l_\rho - l_\tau)i_x i_y & l_\tau i_x^2 + l_\rho i_y^2 \\ \hline l_\rho i_x^2 + l_\tau i_y^2 & (l_\rho - l_\tau)i_x i_y & l_\rho i_x^2 + l_\tau i_y^2 & (l_\rho - l_\tau)i_x i_y \\ \hline (l_\rho - l_\tau)i_x i_y & l_\tau i_x^2 + l_\rho i_y^2 & (l_\rho - l_\tau)i_x i_y & l_\tau i_x^2 + l_\rho i_y^2 \\ \hline\end{array}, \tag{15}$$

where $l_\rho(i_m)$ is found in the same way as it has been done for $l_\tau(i_m)$ in (10)

$$l_\rho(i_m) = \frac{d\psi_m(i_m)}{di_m}. \tag{16}$$

The inverse matrix of differential parameters is the most complicated. On the basis of theory of electric circuits it is even impossible to obtain it analytically. However, it is possible to find it with help of the theory of electromagnetic circuits developed in [1-4]

$$\mathbf{D}(\vec{i})^{-1} = \begin{array}{|c|c|c}\hline a_S(1-a_S(T+bi_x^2)) & -a_S^2 b i_x i_y & \ldots \\ \hline -a_S^2 b i_x i_y & a_S(1-a_S(T+bi_y^2)) & \ldots \\ \hline -a_S a_R(T+bi_x^2) & -a_S a_R b i_x i_y & \ldots \\ \hline -a_S a_R b i_x i_y & -a_S a_R(T+bi_y^2) & \ldots \\ \hline\end{array}$$
$$\begin{array}{c|c|c|}\hline \ldots & -a_S a_R(T+bi_x^2) & -a_S a_R b i_x i_y \\ \hline \ldots & -a_S a_R b i_x i_y & -a_S a_R(T+bi_y^2) \\ \hline \ldots & a_R(1-a_R(T+bi_x^2)) & -a_R^2 b i_x i_y \\ \hline \ldots & -a_R^2 b i_x i_y & a_R(1-a_R(T+bi_y^2)) \\ \hline\end{array}, \tag{17}$$

where

$$R = \frac{1}{\rho + \alpha_S + \alpha_R}; \quad \rho = \frac{1}{l_\rho}; \quad b = \frac{1}{i_m^2}(R - T). \tag{18}$$

Now let us give our attention to the derivative $\partial\mathbf{S}(\vec{i})/\partial\vec{i}$. and differentiate it (8) with respect to $\vec{i}$

$$\frac{\partial\mathbf{S}(\vec{i})}{\partial\vec{i}} = \frac{\partial l_\tau}{\partial i_m}\left(\frac{\partial i_m}{\partial i_x}\frac{\partial i_x}{\partial\vec{i}} + \frac{\partial i_m}{\partial i_y}\frac{\partial i_y}{\partial\vec{i}}\right) \to$$
$$\to \mathbf{A}_2 = \frac{1}{i_m}\frac{\partial l_\tau}{\partial i_m}\mathbf{A}_3(\vec{i}), \tag{19}$$

where

$$\mathbf{A}_3(\vec{i}) = \vec{N} \to \mathbf{A}_2; \quad \vec{N} = \begin{array}{|c|c|c|c|}\hline i_x & i_y & i_x & i_y \\ \hline\end{array}. \tag{20}$$

Here and further the arrow $\rightarrow$ indicates that the left one-row matrix replaces every nonzero element of the right matrix, enlarging its dimension by a unit, since the result of differentiation of a 2-dimensional matrix with respect to a vector is a 3-dimensional matrix. Every matrix element should be multiplied by the coefficient before the matrix.

The second derivative we receive as a result of differentiating with respect to $\vec{i}$ of the first derivative (19)

$$\frac{\partial^2 \mathbf{S}(\vec{i})}{\partial \vec{i}^{\,2}} = \frac{1}{i_m}\left(\frac{\partial^2 l_\tau}{\partial i_m^2} + \frac{1}{i_m}\frac{\partial l_\tau}{\partial i_m}\right)\mathbf{A}_4(\vec{i}) + \frac{1}{i_m}\frac{\partial l_\tau}{\partial i_m}\mathbf{A}_4. \quad (21)$$

where

$$\mathbf{A}_4(i) = \vec{N} \rightarrow \mathbf{A}_3; \quad \mathbf{A}_4 = \partial \mathbf{A}_3 / \partial \vec{i} = const. \quad (22)$$

Naturally, the direct differentiation of $\mathbf{S}(\vec{i})^{-1}, \mathbf{D}(\vec{i}), \mathbf{D}(\vec{i})^{-1}$ can not be performed.

One may have the impression that, the higher the order of parameter matrix is, the more difficult the calculation of its elements is. It is not true, as we will show on the example of the parameter matrices of a saturated singly fed induction motor (three-phase motor in the mode of single phase power supply) [3, 6].

The inverse matrix of differential parameters of such a machine looks like

$$\mathbf{D}(\vec{i})^{-1} = q \begin{array}{|c|c|c|} \hline 1 & \mathrm{A}_{SA} & \mathrm{A}_{SB} \\ \hline \mathrm{A}_{AS} & \mathrm{A}_{A} & \mathrm{A}_{AB} \\ \hline \mathrm{A}_{BS} & \mathrm{A}_{BA} & \mathrm{A}_{B} \\ \hline \end{array}. \quad (23)$$

Here A-coefficients are calculated by the following formulae:

$$\begin{aligned} &\mathrm{A}_{SA} = -c_2 c_5 - c_3 c_8; \qquad \mathrm{A}_{SB} = -c_2 c_6 - c_3 c_9; \\ &\mathrm{A}_{AS} = -c_5 c_4 - c_6 c_7; \qquad \mathrm{A}_{BS} = -c_8 c_4 - c_9 c_7; \\ &\mathrm{A}_{A} = c_5 / q + \mathrm{A}_{AS}\mathrm{A}_{SA}; \quad \mathrm{A}_{AB} = c_6 / q + \mathrm{A}_{AS}\mathrm{A}_{SB}; \quad (24) \\ &\mathrm{A}_{BA} = c_8 / q + \mathrm{A}_{BS}\mathrm{A}_{SA}; \quad \mathrm{A}_{B} = c_9 / q + \mathrm{A}_{BS}\mathrm{A}_{SB}; \\ &\qquad q = 1/(c_1 + c_4 \mathrm{A}_{SA} + c_7 \mathrm{A}_{SB}), \end{aligned}$$

where

$$\begin{gathered} \Delta = g_1 g_2; \\ c_1 = c_2 - c_3 + 2/\alpha_s; c_2 = b(2d_1 - d_3 - d_2) + 1/\tau; \\ c_3 = b(d_1 + d_3 - 2d_2) - 1/\tau; c_4 = \Delta(c_6 + c_9) - 1/\alpha_R; \\ c_5 = b(2d_2 + d_3)/\Delta + 1/g_1; c_6 = -b(d_1 + 2d_3)/\Delta; \\ c_7 = -\Delta(c_5 + c_8) + 1/\alpha_R; \tilde{n}_8 = -b(d_2 + 2d_3)/\Delta; \\ c_9 = b(2d_2 + d_3)/\Delta + 1/g_1; \;\; d_1 = i_A^2,\, d_2 = i_B^2,\, d_3 = i_A i_B; \\ b = \frac{2}{3}\left(\frac{1}{\rho} - \frac{1}{\tau}\right)\frac{1}{i_m^2}; g_1 = \frac{1}{\rho} + \frac{1}{\alpha_R}; \;\; g_2 = \frac{1}{\tau} + \frac{1}{\alpha_R}, \qquad (25) \end{gathered}$$

and t, r are reverse static and differential inductances. They can be found with the help of the magnetization curve of the singly fed machine as follows

$$\tau = \left[\frac{\psi_m(i_m)}{i_m}\right]^{-1}; \; \rho = \left[\frac{d\psi_m(i_m)}{di_m}\right]^{-1}, \qquad (26)$$

where i_m is the module of a spatial vector of magnetization currents

$$i_m = 2\sqrt{(i_A^2 + i_A i_B + i_B^2)/3}, \qquad (27)$$

The magnetization currents by spatial phase axes are found in such a way

$$i_A = i_S + i_{RA}; \; i_B = -i_S + i_{RB}. \qquad (28)$$

Input data for the construction of this matrix are α_S, α_R, the magnetization curve $\psi_m(i_m)$ and input signals U_m, ω_0, $M(\omega, t)$.

The inverse matrix of static parameters is simpler than the corresponding inverse matrix of differential parameters (23), but yet complex enough:

$$\mathbf{S}(\vec{i})^{-1} = \frac{T}{2} \begin{array}{|c|c|c|} \hline \alpha_S(\alpha_R + \tau) & -\alpha_S\alpha_R & \ldots \\ \hline -\alpha_S\alpha_R & \dfrac{\alpha_R}{\alpha_R + \tau}\left(\alpha_S\alpha_R + \dfrac{2\tau}{T}\right) & \ldots \\ \hline \alpha_S\alpha_R & -\dfrac{\alpha_S\alpha_R^2}{\alpha_R + \tau} & \ldots \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline \ldots & \alpha_S\alpha_R \\ \hline \ldots & -\dfrac{\alpha_S\alpha_R^2}{\alpha_R + \tau} \\ \hline \ldots & \dfrac{\alpha_R}{\alpha_R + \alpha_m}\left(\alpha_S\alpha_R + \dfrac{2\tau}{T}\right) \\ \hline \end{array}, \qquad (29)$$

where

$$T = 1/(\alpha_S + \alpha_R + \tau) \qquad (30)$$

As we can see, matrices of lower order (23) and (29) are more difficult for differentiating with respect to the argument, than corresponding matrices of higher order (13) and (17).

We have given the concrete examples from the field of electromechanics in order to show how much difficult the mathematical problem solved in this article is.

The proposed formulae of differentiation (3)–(7) are not simple, but their complexity is related only to the formal mathematical matrix operations in case of highdimensional spatial matrices, because $\frac{\partial \mathbf{S}(\vec{x})}{\partial \vec{x}}$, $\frac{\partial \mathbf{S}^{-1}(\vec{x})}{\partial \vec{x}}$, $\frac{\partial \mathbf{D}(\vec{x})}{\partial \vec{x}}$ $\frac{\partial \mathbf{D}^{-1}(\vec{x})}{\partial \vec{x}}$ are 3-dimensional matrices and $\frac{\partial^2 \mathbf{S}(\vec{x})}{\partial \vec{x}^2}$ is a 4- dimensional matrix.

The computational algorithm is based on the rules of differentiation with respect to a 1-dimensional matrix (a vector) $\vec{x}$. Let φ be a scalar, $\vec{q}$ be a 1- dimensional matrix, $\mathbf{A}$ be a 2- dimensional matrix, $\mathbf{B}$ be a 3-dimensional matrix, and $\mathbf{C}$ be a 4- dimensional matrix:

$$y_i = \frac{d\varphi}{dx_i};\; a_{ij} = \frac{dq_i}{dx_j};\; b_{ijk} = \frac{da_{ij}}{dx_k};\; c_{ijkz} = \frac{db_{ijk}}{dx_z}. \quad (31)$$

Every differentiation of a matrix with respect to a vector enlarges its dimensionality by a unit, every multiplying of matrix by a vector, on the contrary, reduces its dimensionality by a unit

$$(\mathbf{A}\vec{x})_i = \sum_j a_{ij} x_j;$$

$$(\mathbf{B}\vec{x})_{jk} = \sum_k b_{ijk} x_k;\quad (\mathbf{C}\vec{x})_{ijk} = \sum_z c_{ijkz} x_z. \quad (32)$$

Let us remark that operations with multi–dimensional matrices are simple enough. However, their visual presentation is rather cumbersome. This statement refers to 3-dimensional matrices and matrices of higher dimensions.

4. Conclusion

If the analysis of nonlinear systems requires the differentiation of direct and inverce matrices of static and differential parameters цшер respect to the argument, then the practical option of solving the problem consists, in general, in finding the simplest derivative (i.e. the matrix of static parameters). The remaining derivatives can be computed through formal matrix operations such as multiplication and addition.

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ПОХІДНІ МАТРИЦЬ ПАРАМЕТРІВ

Василь Чабан

Запропоновано метод диференціювання за аргументом прямих та обернених матриць статичних і диференціальних параметрів. Його здійснюють через похідну матриці статичних параметрів як найпростішу і таку, що вважається заданою в нашому випадку. Метод аналізу полягає в тому, щоб через відому найпростішу похідну було можливо на підставі формальних матричних операцій знайти інші три складніші похідні. Одержані результати можна використати в теорії параметричної чутливости першого й вищих порядків. Як приклад розглянуто прямі й обернені матриці індуктивностей насиченого трифазного індукційного мотора за однофазного й трифазного живлення.

Vasyl Tchaban – DSc, full professor at Lviv Polytechnic National University (Ukraine). He obtained his Doctor Habilitation degree in Electrical Engineering at Moscow Energetic University (Russia) in 1987. His research interests lie in the areas of mathematical modelling of electro-mechanical processes and electromagnetic field theory, physics, mathemtics as well as short surrealistic story writting. He is the author of about 500 scentific publications and, besides, 600 short surrealistic stories, including 33 books (8 monographs, 13 didactic, 2 humanistic and 10 of the arts).