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Exact difference scheme for system nonlinear ODEs of second order on semi-infinite intervals

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We constructed and substantiated the exact three-point differential scheme for the numerical solution of boundary value problems on a semi-infinite interval for systems of second order nonlinear ordinary differential equations with non-selfadjoint operator. The existence and uniqueness of the solution of the exact three-point difference scheme and the convergence of the method of successive approximations for its findings are proved under the conditions of existence and uniqueness of the solution of the boundary value problem.

Keywords: *nonlinear ordinary differential equation, boundary value problem, exact three-point difference scheme*

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1. Introduction

The bases of theory of exact difference schemes and truncated schemes of any order of accuracy for nonlinear ordinary differential equations with boundary conditions of the general form are described in [1,4]. In the papers [4,5], exact and truncated three-point difference schemes on finite irregular grid with nonlinear asymptotic boundary condition on right end of the grid for numerical solution of boundary value problems on a semi-axis are constructed in the form

$$\frac{d^2u}{dx^2} - m^2u = -f(x,u), \quad x \in (0,\infty), \quad u(0) = \mu_1, \quad \lim_{x\to\infty} u(x) = 0,$$
$$m \neq 0, \quad u, \mu_1, f(x,u) \in \mathbb{R}^1.$$

In this paper, in contrast to [4–6], the exact three-point difference scheme (ETDS) of not divergent type is constructed and justified on finite irregular grid for the system of nonlinear ordinary second order differential equations

$$\frac{d^2u}{dx^2} + m^2\frac{du}{dx} = -f(x,u), \quad x \in (0,\infty), \quad m \neq 0, \tag{1}$$

with the boundary conditions

$$u(0) = \mu_1, \quad \lim_{x\to\infty} u(x) = 0, \tag{2}$$

where $u, \mu_1, f(x,u) \in \mathbb{R}^n$.

2. Existence and uniqueness of the solution

In [3, p. 80] sufficient conditions for existence of a solution of the problem (1), (2) are given. The following assertion gives the sufficient conditions of existence and uniqueness of the solution of the problem (1), (2), which follows from Banach's Fixed Point Theorem.

We introduce the function $u^{(0)}(x) = \mu_1 e^{-m^2 x}$ and set

$$\Omega(D, \beta) = \left\{ u(x) : u(x) \in C^1(D), \ \left\| u - u^{(0)} \right\|_{1,\infty,D} \leqslant \beta, \ D \subseteq [0, \infty) \right\},$$

$$\|u\|_{1,\infty,D} = \max \left\{ \|u\|_{0,\infty,D}, \ \left\| \frac{du}{dx} \right\|_{0,\infty,D} \right\}, \quad \|u\|_{0,\infty,D} = \max_{x \in D} \|u(x)\|,$$

where (u, v) is scalar product of vectors $u, v \in \mathbb{R}^n$, $\|u\| = (u, u)^{1/2}$ is norm of the vector $u \in \mathbb{R}^n$.

Theorem 1. *Let the following conditions are satisfied*

$$\begin{gathered} f_u(x) \equiv f(x, u) \in Q^0[0, \infty) \quad \forall u \in \mathbb{R}^n, \\ \forall r > 0 \ \exists K(x) \in L_1[0, \infty) : \|f(x, u)\| \leqslant K(x) \quad \forall x \in [0, \infty), u \in \Omega([0, \infty), r), \end{gathered} \tag{3}$$

$$\lim_{x \to \infty} e^{-m^2 x} \int_0^x e^{m^2 \xi} K(\xi) d\xi = 0, \quad \max \left\{ \frac{1}{m^2}, 1 \right\} \int_0^\infty K(\xi) d\xi \leqslant r, \tag{4}$$

$$\forall r > 0 \ \exists L(x) \in L_1 : \|f(x, u) - f(x, v)\| \leqslant L(x) \|u - v\| \quad \forall x \in [0, \infty), \ u, v \in \Omega([0, \infty), r), \tag{5}$$

$$q = \max \left\{ \frac{1}{m^2}, 1 \right\} \int_0^\infty L(\xi) d\xi < 1, \tag{6}$$

then the boundary value problem (1), (2) *on the set* $\Omega([0, \infty), r)$ *has a unique solution* $u(x)$*, which can be found by fixed point iteration*

$$\begin{gathered} \frac{d^2 u^{(k)}}{dx^2} + m^2 \frac{du^{(k)}}{dx} = -f(x, u^{(k-1)}), \quad x \in (0, \infty), \\ u^{(k)}(0) = \mu_1, \quad \lim_{x \to \infty} u^{(k)}(x) = 0, \quad k = 1, 2, ... \end{gathered} \tag{7}$$

with error estimation

$$\left\| u^{(k)} - u \right\|_{1,\infty,[0,\infty)} \leqslant \frac{q^k}{1 - q} r. \tag{8}$$

Here $Q^0[0, \infty)$ is the class of piecewise continuous functions with a finite number of discontinuity points of first kind.

Proof. We write the problem (1), (2) in the equivalent integral form

$$u(x) = \Re(x, u(\cdot)) = \int_0^\infty G(x, \xi) f(\xi, u(\xi)) d\xi + u^{(0)}(x), \quad x \geqslant 0, \tag{9}$$

where

$$G(x, \xi) = \begin{cases} \frac{1}{m^2} \left(1 - e^{-m^2 x} \right), & 0 \leqslant x \leqslant \xi, \\ \frac{1}{m^2} e^{-m^2 x} \left(e^{m^2 \xi} - 1 \right), & x \geqslant \xi \end{cases}$$

is the Green's function of problem (1), (2). Since

$$\|u(x)\| \leqslant \frac{e^{-m^2x}}{m^2}\int\limits_0^x e^{m^2\xi}K(\xi)d\xi - \frac{e^{-m^2x}}{m^2}\int\limits_0^\infty K(\xi)d\xi + \frac{1}{m^2}\int\limits_x^\infty K(\xi)d\xi + \|\mu_1\|\, e^{-m^2x},$$

then due to the assumption of the theorem, function (9) satisfies the boundary condition as $x\to\infty$.

Let us show that the operator (9) transforms the set $\Omega([0,\infty),r)$ into itself. Taking into account (3), from the equality (9) we get

$$\left\|\mathfrak{R}(x,v(\cdot)) - u^{(0)}\right\|_{1,\infty,[0,\infty)} \leqslant \int\limits_0^\infty \|G(x,\xi)\|_{1,\infty,[0,\infty)} K(\xi)d\xi \quad \forall v\in\Omega([0,\infty),r).$$

From inequalities

$$\int\limits_0^\infty |G(x,\xi)|\, K(\xi)d\xi = \frac{e^{-m^2x}}{m^2}\int\limits_0^x (e^{m^2\xi}-1)K(\xi)d\xi + \frac{1-e^{-m^2x}}{m^2}\int\limits_x^\infty K(\xi)d\xi \leqslant$$
$$\leqslant \frac{1-e^{-m^2x}}{m^2}\int\limits_0^\infty K(\xi)d\xi \leqslant \frac{1}{m^2}\int\limits_0^\infty K(\xi)d\xi,$$

$$\int\limits_0^\infty \left|\frac{\partial G(x,\xi)}{\partial x}\right| K(\xi)d\xi = e^{-m^2x}\int\limits_0^x (e^{m^2\xi}-1)K(\xi)d\xi + e^{-m^2x}\int\limits_x^\infty K(\xi)d\xi \leqslant \int\limits_0^\infty K(\xi)d\xi$$

follows that

$$\left\|\mathfrak{R}(x,v(\cdot)) - u^{(0)}\right\|_{1,\infty,[0,\infty)} \leqslant r \quad \forall v\in\Omega([0,\infty),r),$$

i.e. the operator (9) transforms the set $\Omega([0,\infty),r)$ into itself.

Moreover, under the condition (6), the operator $\mathrm{Re}(x,u(\cdot))$ is contractive on the set $\Omega([0,\infty),r)$, thus, we have

$$\|\mathfrak{R}(x,u(\cdot)) - \mathfrak{R}(x,v(\cdot))\|_{1,\infty,[0,\infty)} \leqslant \|u-v\|_{1,\infty,[0,\infty)} \int\limits_0^\infty \|G(x,\xi)\|_{1,\infty,[0,\infty)} L(\xi)d\xi \leqslant$$
$$\leqslant q\,\|u-v\|_{1,\infty,[0,\infty)} \quad \forall u,v\in\Omega([0,\infty),r).$$

Therefore, the operator $\mathfrak{R}(x,u(\cdot))$ satisfies all conditions of Banach's Fixed Point Theorem, and this implies that the equation (9) has a unique solution that can be obtained by fixed point iterations (7) with the error estimation (8) (see, for example, [7]). ■

Remark 1. Since

$$\left\|\frac{du}{dx}\right\| \leqslant e^{-m^2x}\int\limits_0^x e^{m^2\xi}K(\xi)d\xi + e^{-m^2x}\int\limits_0^\infty K(\xi)d\xi + \|\mu_1\|\, m^2 e^{-m^2x},$$

then in accordance with the condition (4)

$$\lim_{x\to\infty}\frac{du}{dx} = 0.$$

3. The existence of the exact three-point scheme

We introduce irregular grid on the interval $[0, \infty)$

$$\hat{\hat{\omega}}_N = \left\{ x_j \in [0, \infty),\ j = 0, 1, ..., N,\ x_0 = 0,\ h_j = x_j - x_{j-1} > 0,\ \sum_{j=1}^{N} h_j = x_N \right\}$$

so that the discontinuity points of the function $f(x, u)$ coincide with the grid nodes. The set of all discontinuity points is denoted by ρ and we will assume that N is such that $\rho \subseteq \hat{\omega}_N$. According to [2,5], we introduce the following conditions on the step h_j of grid $\hat{\omega}_N$:

$$c_1 \leqslant \frac{h_{\max}}{h_{\min}} \leqslant c_2, \tag{10}$$

where c_1, c_2 are real constants. For achievement of the maximum order of convergence of difference schemes, one needs

$$\frac{1}{h_{\max}} \leqslant x_N \leqslant \frac{1}{h_{\min}}. \tag{11}$$

From the inequalities $h_{\min} N \leqslant x_N = h_1 + h_2 + ... + h_N \leqslant h_{\max} N$ and (11), the relation follows

$$h_{\min} \leqslant \frac{1}{x_N} \leqslant \frac{1}{N h_{\min}}, \quad \frac{1}{N h_{\max}} \leqslant \frac{1}{x_N} \leqslant h_{\max}.$$

On the basis of (10) we obtain

$$\frac{h_{\max}}{c_2} \leqslant h_{\min} \leqslant \frac{1}{\sqrt{N}}, \quad c_2 h_{\min} \geqslant h_{\max} \geqslant \frac{1}{\sqrt{N}}.$$

This yields inequalities

$$h_{\max} \leqslant \frac{c_2}{\sqrt{N}}, \quad h_{\min} \geqslant \frac{1}{c_2\sqrt{N}}, \quad \frac{\sqrt{N}}{c_2} \leqslant h_{\min} N \leqslant x_N \leqslant h_{\max} N \leqslant c_2\sqrt{N}. \tag{12}$$

Hence, we will have that $h_{\max} \to 0$, $x_N \to \infty$ for $N \to \infty$.

Now we consider boundary value problems

$$\begin{gathered} \frac{d^2 Y_\alpha^j(x,u)}{dx^2} + m^2 \frac{dY_\alpha^j(x,u)}{dx} = -f(x, Y_\alpha^j(x,u)), \quad x_{j-2+\alpha} < x < x_{j-1+\alpha}, \\ Y_\alpha^j(x_{j-2+\alpha}, u) = u(x_{j-2+\alpha}), \quad Y_\alpha^j(x_{j-1+\alpha}, u) = u(x_{j-1+\alpha}), \\ j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1, 2, \end{gathered} \tag{13}$$

$$\begin{gathered} \frac{d^2 Y_2^N(x,u)}{dx^2} + m^2 \frac{dY_2^N(x,u)}{dx} = -f(x, Y_2^N(x,u)), \quad x > x_N, \\ Y_2^N(x_N, u) = u(x_N), \quad \lim_{x\to\infty} Y_2^N(x,u) = 0. \end{gathered} \tag{14}$$

Lemma 2. *Let the conditions* (3)–(6) *are satisfied, then problems* (13), (14) *will have a unique solution* $Y_\alpha^j(x,u), j=2-\alpha, 3-\alpha, \ldots, N+1-\alpha, \alpha=1,2$, $Y_2^N(x,u)$, *and the solution of the problem* (1), (2) *can be represented in the form*

$$\begin{gathered} u(x)=Y_\alpha^j(x,u), \quad x\in[x_{j-2+\alpha},x_{j-1+\alpha}], \\ j=2-\alpha,3-\alpha,...,N+1-\alpha, \quad \alpha=1,2, \\ u(x)=Y_2^N(x,u), \quad x\in[x_N,\infty). \end{gathered} \tag{15}$$

Proof. The boundary value problems (13), (14) can be written in the equivalent form

$$\begin{gathered} Y_\alpha^j(x,u)=\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} G^{j-1+\alpha}(x,\xi)f(\xi,Y_\alpha^j(\xi,u))d\xi+\hat{u}(x), \\ x\in[x_{j-2+\alpha},x_{j-1+\alpha}], \quad j=2-\alpha,3-\alpha,...,N+1-\alpha, \quad \alpha=1,2, \end{gathered} \tag{16}$$

$$\hat{u}(x)=\frac{e^{-m^2x_{j-2+\alpha}}-e^{-m^2x}}{e^{-m^2x_{j-2+\alpha}}-e^{-m^2x_{j-1+\alpha}}}u_{j-1+\alpha}+\frac{e^{-m^2x}-e^{-m^2x_{j-1+\alpha}}}{e^{-m^2x_{j-2+\alpha}}-e^{-m^2x_{j-1+\alpha}}}u_{j-2+\alpha}, \tag{17}$$

$$Y_2^N(x,u)=\int\limits_{x_N}^{\infty} G^\infty(x,\xi)f(\xi,Y_2^N(\xi,u))d\xi+u_N e^{-m^2(x-x_N)}, \quad x\in[x_N,\infty), \tag{18}$$

where

$$G^{j-1+\alpha}(x,\xi)=\begin{cases} \dfrac{\left(e^{-m^2x_{j-2+\alpha}}-e^{-m^2x}\right)\left(1-e^{-m^2(x_{j-1+\alpha}-\xi)}\right)}{m^2\left(e^{-m^2x_{j-2+\alpha}}-e^{-m^2x_{j-1+\alpha}}\right)}, & x_{j-2+\alpha}\leqslant x\leqslant\xi, \\ \dfrac{\left(e^{-m^2x}-e^{-m^2x_{j-1+\alpha}}\right)\left(e^{m^2(\xi-x_{j-2+\alpha})}-1\right)}{m^2\left(e^{-m^2x_{j-2+\alpha}}-e^{-m^2x_{j-1+\alpha}}\right)}, & \xi\leqslant x\leqslant x_{j-1+\alpha}, \end{cases} \tag{19}$$

$$G^\infty(x,\xi)=\begin{cases} \dfrac{1}{m^2}\left(1-e^{-m^2(x-x_N)}\right), & x_N\leqslant x\leqslant\xi, \\ \dfrac{1}{m^2}e^{-m^2x}\left(e^{m^2\xi}-e^{m^2x_N}\right), & x\geqslant\xi. \end{cases} \tag{20}$$

For $\alpha=1$ we get

$$\begin{aligned} \hat{u}(x)=&\frac{e^{-m^2x_{j-1}}-e^{-m^2x}}{e^{-m^2x_{j-1}}-e^{-m^2x_j}}\left[\int\limits_0^\infty G(x_j,\xi)f(\xi,u)d\xi+\mu_1e^{-m^2x_j}\right]+ \\ &+\frac{e^{-m^2x}-e^{-m^2x_j}}{e^{-m^2x_{j-1}}-e^{-m^2x_j}}\left[\int\limits_0^\infty G(x_{j-1},\xi)f(\xi,u)d\xi+\mu_1e^{-m^2x_{j-1}}\right], \quad x\in[x_{j-1},x_j], \end{aligned}$$

$$u_N(x)e^{-m^2(x-x_N)}=e^{-m^2(x-x_N)}\left[\int\limits_0^\infty G(x_N,\xi)f(\xi,u)d\xi+\mu_1e^{-m^2x_N}\right], \quad x\in[x_N,\infty).$$

Since

$$\left[\frac{e^{-m^2x_{j-1}}-e^{-m^2x}}{e^{-m^2x_{j-1}}-e^{-m^2x_j}}e^{-m^2x_j}+\frac{e^{-m^2x}-e^{-m^2x_j}}{e^{-m^2x_{j-1}}-e^{-m^2x_j}}e^{-m^2x_{j-1}}\right]\mu_1=e^{-m^2x}\mu_1=u^{(0)}(x),$$

that

$$\hat{u}(x) = \frac{e^{-m^2 x_{j-1}} - e^{-m^2 x}}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} \int\limits_0^\infty G(x_j, \xi) f(\xi, u) d\xi +$$

$$+ \frac{e^{-m^2 x} - e^{-m^2 x_j}}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} \int\limits_0^\infty G(x_{j-1}, \xi) f(\xi, u) d\xi + u^{(0)}(x), \quad x \in [x_{j-1}, x_j],$$

$$u_N(x) e^{-m^2(x - x_N)} = e^{-m^2(x - x_N)} \int\limits_0^\infty G(x_N, \xi) f(\xi, u) d\xi + +u^{(0)}(x), \quad x \in [x_N, \infty).$$

Then

$$Y_1^j(x, u) = \frac{e^{-m^2 x_{j-1}} - e^{-m^2 x}}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} \int\limits_0^\infty G(x_j, \xi) f(\xi, u) d\xi + \frac{e^{-m^2 x} - e^{-m^2 x_j}}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} \int\limits_0^\infty G(x_{j-1}, \xi) f(\xi, u) d\xi +$$

$$+ \int\limits_{x_{j-1}}^{x_j} G^j(x, \xi) f(\xi, Y_1^j(\xi, u)) d\xi + u^{(0)}(x), \quad x \in [x_{j-1}, x_j],$$

$$Y_2^N(x, u) = e^{-m^2(x - x_N)} \int\limits_0^\infty G(x_N, \xi) f(\xi, u) d\xi + \int\limits_{x_N}^\infty G^\infty(x, \xi) f(\xi, Y_2^N(\xi, u)) d\xi + u^{(0)}(x), \quad x \in [x_N, \infty).$$

Based on equality $Y_2^j(x, u) = Y_1^{j+1}(x, u)$ we have

$$Y_2^j(x, u) = \frac{e^{-m^2 x_j} - e^{-m^2 x}}{e^{-m^2 x_j} - e^{-m^2 x_{j+1}}} \int\limits_0^\infty G(x_{j+1}, \xi) f(\xi, u) d\xi + \frac{e^{-m^2 x} - e^{-m^2 x_{j+1}}}{e^{-m^2 x_j} - e^{-m^2 x_{j+1}}} \int\limits_0^\infty G(x_j, \xi) f(\xi, u) d\xi +$$

$$+ \int\limits_{x_j}^{x_{j+1}} G^{j+1}(x, \xi) f(\xi, Y_2^j(\xi, u)) d\xi + u^{(0)}(x), \quad x \in [x_j, x_{j+1}].$$

Thus, the question of the existence and uniqueness of the solution of the problem (16), (18) is equivalent to the analogous problem for the equations

$$\begin{gathered} U_\alpha^j(x) = \mathfrak{F}_\alpha^j(x, u, U_\alpha^j) = \frac{e^{-m^2 x_{j-2+\alpha}} - e^{-m^2 x}}{e^{-m^2 x_{j-2+\alpha}} - e^{-m^2 x_{j-1+\alpha}}} \int\limits_0^\infty G(x_{j-1+\alpha}, \xi) f(\xi, u) d\xi + \\ + \frac{e^{-m^2 x} - e^{-m^2 x_{j-1+\alpha}}}{e^{-m^2 x_{j-2+\alpha}} - e^{-m^2 x_{j-1+\alpha}}} \int\limits_0^\infty G(x_{j-2+\alpha}, \xi) f(\xi, u) d\xi + \\ + \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} G^{j-1+\alpha}(x, \xi) f(\xi, U_\alpha^j(\xi, u)) d\xi + u^{(0)}(x), \\ x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \quad j = 2 - \alpha, 3 - \alpha, ..., N + 1 - \alpha, \quad \alpha = 1, 2, \end{gathered} \tag{21}$$

$$\begin{gathered}U_2^N(x)=\mathfrak{F}_2^N(x,u,U_2^N)=e^{-m^2(x-x_N)}\int\limits_0^\infty G(x_N,\xi)f(\xi,u)d\xi+\\+\int\limits_{x_N}^\infty G^\infty(x,\xi)f(\xi,U_2^N(\xi,u))d\xi+u^{(0)}(x),\quad x\in[x_N,\infty).\end{gathered}\tag{22}$$

We show that the operators $\mathfrak{F}_\alpha^j(x,u,U_\alpha^j)$, $\mathfrak{F}_2^N(x,u,U_2^N)$ transform sets $\Omega\left([x_{j-2+\alpha},x_{j-1+\alpha}],r\right)$, $\Omega\left([x_N,\infty),r\right)$ accordingly into themselves. Let $U_\alpha^j(x)\in\Omega([x_{j-2+\alpha},x_{j-1+\alpha}],r)$, $U_2^N(x)\in\Omega\left([x_N,\infty),r\right)$. Then

$$\begin{gathered}\left\|\mathfrak{F}_\alpha^j(x,u,U_\alpha^j)-u^{(0)}(x)\right\|\leqslant\frac{e^{-m^2x_{j-2+\alpha}}-e^{-m^2x}}{e^{-m^2x_{j-2+\alpha}}-e^{-m^2x_{j-1+\alpha}}}\int\limits_0^\infty G(x_{j-1+\alpha},\xi)K(\xi)d\xi+\\+\frac{e^{-m^2x}-e^{-m^2x_{j-1+\alpha}}}{e^{-m^2x_{j-2+\alpha}}-e^{-m^2x_{j-1+\alpha}}}\int\limits_0^\infty G(x_{j-2+\alpha},\xi)K(\xi)d\xi+\\+\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}G^{j-1+\alpha}(x,\xi)K(\xi)d\xi=\frac{e^{-m^2x}}{m^2}\int\limits_0^x(e^{m^2\xi}-1)K(\xi)d\xi+\\+\frac{1-e^{-m^2x}}{m^2}\int\limits_x^\infty K(\xi)d\xi\leqslant\frac{1}{m^2}\int\limits_0^\infty K(\xi)d\xi,\end{gathered}$$

$$\left\|\frac{\partial\mathfrak{F}_\alpha^j(x,u,U_\alpha^j)}{\partial x}-\frac{du^{(0)}(x)}{dx}\right\|\leqslant e^{-m^2x}\int\limits_0^x(e^{m^2\xi}-1)K(\xi)d\xi+e^{-m^2x}\int\limits_x^\infty K(\xi)d\xi\leqslant\int\limits_0^\infty K(\xi)d\xi,$$

$$\begin{gathered}\left\|\mathfrak{F}_2^N(x,u,U_2^N)-u^{(0)}(x)\right\|\leqslant e^{-m^2(x-x_N)}\int\limits_0^\infty G(x_N,\xi)K(\xi)d\xi+\int\limits_{x_N}^\infty G^\infty(x,\xi)K(\xi)d\xi=\\=\frac{e^{-m^2x}}{m^2}\int\limits_0^x(e^{m^2\xi}-1)K(\xi)d\xi+\frac{1-e^{-m^2x}}{m^2}\int\limits_x^\infty K(\xi)d\xi\leqslant\frac{1}{m^2}\int\limits_0^\infty K(\xi)d\xi,\end{gathered}$$

$$\left\|\frac{\partial\mathfrak{F}_2^N(x,u,U_2^N)}{\partial x}-\frac{du^{(0)}(x)}{dx}\right\|\leqslant e^{-m^2x}\int\limits_0^x(e^{m^2\xi}-1)K(\xi)d\xi+e^{-m^2x}\int\limits_x^\infty K(\xi)d\xi\leqslant\int\limits_0^\infty K(\xi)d\xi.$$

From here

$$\begin{gathered}\left\|\mathfrak{F}_\alpha^j(x,u,U_\alpha^j)-u^{(0)}\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}\leqslant r,\\\left\|\mathfrak{F}_2^N(x,u,U_2^N)-u^{(0)}\right\|_{1,\infty,[x_N,\infty)}\leqslant r.\end{gathered}$$

Moreover,

$$\begin{gathered}\left\|\mathfrak{F}_\alpha^j(x,u,U_\alpha^j)-\mathfrak{F}_\alpha^j(x,u,\tilde U_\alpha^j)\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}\leqslant\\\leqslant\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left\|G^{j-1+\alpha}(x,\xi)\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}L(\xi)d\xi\left\|U_\alpha^j-\tilde U_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}\leqslant\\\leqslant q\left\|U_\alpha^j-\tilde U_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]},\end{gathered}$$

$$\left\|\mathfrak{F}_2^N(x,u,U_2^N)-\mathfrak{F}_2^N(x,u,\tilde{U}_2^N)\right\|_{1,\infty,[x_N,\infty)} \leqslant \int\limits_{x_N}^{\infty} \|G^{\infty}(x,\xi)\|_{1,\infty,[x_N,\infty)} L(\xi)d\xi \left\|U_2^N-\tilde{U}_2^N\right\|_{1,\infty,[x_N,\infty)} \leqslant$$

$$\leqslant q\left\|U_2^N-\tilde{U}_2^N\right\|_{1,\infty,[x_N,\infty)},$$

where

$$q=\max\left\{\frac{1}{m^2},1\right\}\int\limits_0^{\infty} L(\xi)d\xi<1.$$

Then, for the operators (21) on the sets $\Omega\left([x_{j-2+\alpha},x_{j-1+\alpha}],r\right)$ and for operator (22) on $\Omega\left([x_N,\infty),r\right)$ all the conditions of Banach's Fixed Point Theorem are fulfilled. Therefore, the problems (13) and (14) have a unique solution.

■

The form of ETDS for the problem (1), (2) and conditions of its existence give the assertion.

Theorem 3. *Let the conditions of the Theorem 1 are satisfied. Then for the problem (1), (2) there is ETDS in the form*

$$\frac{1}{\hbar_j}\left(b_j u_{x,j}-a_j u_{\bar{x},j}\right)=-\hat{T}^{x_j}(f(\xi,u(\xi))),\quad j=1,2,...,N-1, \tag{23}$$

$$u_0=\mu_1,\quad -a_N u_{\bar{x},N}=m^2 u_N-\hat{T}^{x_N}(f(\xi,u(\xi))), \tag{24}$$

where

$$u_{\bar{x},j}=\frac{u_j-u_{j-1}}{h_j},\quad u_{x,j}=\frac{u_{j+1}-u_j}{h_{j+1}},\quad \hbar_j=\frac{h_j+h_{j+1}}{2},$$

$$a_j=\frac{m^2h_j}{e^{m^2h_j}-1},\quad b_j=\frac{m^2h_{j+1}}{1-e^{-m^2h_{j+1}}},\quad j=1,2,...,N,$$

$$\begin{aligned}\hat{T}^{x_j}(f(\xi,u(\xi)))=&\frac{1}{\hbar_j\left(e^{m^2\hbar_j}-1\right)}\int\limits_{x_{j-1}}^{x_j}\left(e^{m^2(\xi-x_{j-1})}-1\right)f(\xi,u(\xi))d\xi+\\&+\frac{1}{\hbar_j\left(1-e^{-m^2h_{j+1}}\right)}\int\limits_{x_j}^{x_{j+1}}\left(1-e^{-m^2(x_{j+1}-\xi)}\right)f(\xi,u(\xi))d\xi,\quad j=1,2,...,N-1,\end{aligned} \tag{25}$$

$$\hat{T}^{x_N}(f(\xi,u(\xi)))=\int\limits_{x_N}^{\infty}f(\xi,u(\xi))d\xi+\frac{1}{e^{m^2h_N}-1}\int\limits_{x_{N-1}}^{x_N}\left(e^{m^2(\xi-x_{N-1})}-1\right)f(\xi,u(\xi))d\xi,$$

the function $u(\xi)$ on the right side of (23), (24) are defined by the formula (15) and depends only on $u_0,u_1,\dots,u_N$.

Proof. ETDS for the problem

$$\frac{d^2\tilde{u}}{dx^2}+m^2\frac{d\tilde{u}}{dx}=-f(x,\tilde{u}),\quad x\in(0,x_{N+1}),$$

$$\tilde{u}(0)=\mu_1,\quad \tilde{u}(x_{N+1})=0$$

has a form

$$\frac{1}{\hbar_j}\left(b_j\tilde{u}_{x,j}-a_j\tilde{u}_{\bar{x},j}\right)=-\hat{T}^{x}(f(\xi,\tilde{u}(\xi))),\quad j=1,2,...,N, \tag{26}$$

$$\tilde{u}(0)=\mu_1,\quad \tilde{u}(x_{N+1})=0. \tag{27}$$

Multiplying both sides of the equation (26) for $j=N$ by $\hbar_N=\frac{1}{2}(h_{N+1}+h_N)$, $h_{N+1}=x_{N+1}-x_N$, we have

$$\frac{m^2(\tilde u_{N+1}-\tilde u_N)}{\left(1-e^{-m^2h_{N+1}}\right)}-\frac{m^2h_N\tilde u_{\bar x,N}}{e^{m^2h_N}-1}=$$
$$=-\frac{1}{e^{m^2h_N}-1}\int\limits_{x_{N-1}}^{x_N}\left(e^{m^2(\xi-x_{N-1})}-1\right)f(\xi,\tilde u(\xi))d\xi-$$
$$-\frac{1}{1-e^{-m^2h_{N+1}}}\int\limits_{x_N}^{x_{N+1}}\left(1-e^{-m^2(x_{N+1}-\xi)}\right)f(\xi,\tilde u(\xi))d\xi.$$

Passing to the limit as $x_{N+1}\to\infty$, we obtain difference schemes (23), (24). ■

We introduce a set of grid functions

$$\Omega(\hat{\hat\omega}_N,\beta)=\left\{v(x),\ x\in\hat{\hat\omega}_N:\left\|v-u^{(0)}\right\|^*_{1,\infty,\hat{\hat\omega}_N}\leqslant\beta\right\},$$

where

$$\|v\|_{0,\infty,\hat\omega_N}=\max_{0\leqslant j\leqslant N}|v_j|,\quad \|v\|_{0,\infty,\hat\omega_N^+}=\max_{1\leqslant j\leqslant N}|v_j|,$$

$$\|v\|^*_{1,\infty,\hat{\hat\omega}_N}=\max\left\{\|v\|_{0,\infty,\hat\omega_N^+},\left\|\frac{dv}{dx}\right\|_{0,\infty,\hat{\hat\omega}_N}\right\},\quad \left.\frac{dv}{dx}\right|_{x=x_j}=\left.\frac{dY_\alpha^j(x,v)}{dx}\right|_{x=x_j}.$$

The existence of the solutions of nonlinear ETDS (23), (24) is proved in the Theorem 3, and uniqueness is established in the next lemma.

Lemma 4. *Let the conditions of the Theorem 1 are satisfied. Then there exists a number $h_0>0$ such that for $|h|=\max\limits_{1\leqslant j\leqslant N}h_j\leqslant h_0$, ETDS (23),(24) have a unique solution $\forall(u_j)_{j=0}^N\in\Omega(\hat{\hat\omega}_N,r)$ that can be obtained by the method of successive approximations*

$$\frac{1}{\hbar_j}\left(b_ju^{(k)}_{x,j}-a_ju^{(k)}_{\bar x,j}\right)=-\hat T^{x_j}(f(\xi,u^{(k-1)}(\xi))),\quad j=1,2,...,N-1,$$
$$u_0^{(k)}=\mu_1,\quad -a_Nu^{(k)}_{\bar x,N}=m^2u_N^{(k)}-\hat{\hat T}^{x_N}(f(\xi,u^{(k-1)}(\xi))),\tag{28}$$
$$u^{(k)}(x)=Y_\alpha^j(x,u^{(k)}),\quad x\in[x_{j-2+\alpha},x_{j-1+\alpha}],$$
$$j=2-\alpha,3-\alpha,...,N+1-\alpha,\quad \alpha=1,2,$$
$$u^{(k)}(x)=Y_2^N(x,u^{(k)}),\quad x\in[x_N,\infty),\quad k=1,2,...,\quad u^{(0)}(x)=\mu_1e^{-m^2x}$$

with the error estimate

$$\left\|u^{(k)}-u\right\|^*_{1,\infty,\hat{\hat\omega}}=\max\left\{\left\|u^{(k)}-u\right\|_{0,\infty,\hat\omega_N^+},\left\|\frac{du^{(k)}}{dx}-\frac{du}{dx}\right\|_{0,\infty,\hat{\hat\omega}}\right\}\leqslant\frac{q_1^k}{1-q_1}r,\tag{29}$$

where $q_1=q+M|h|<1$, M is the constant, which does not dependent on $|h|$.

Proof. Due to the difference scheme (23), (24) is exact, i.e. its solution is a projection of the exact solution boundary value problem on the grid, then this solution for $\forall x\in\hat{\hat\omega}_N$ can be written in the form

$$
\begin{aligned}
u(x) \;=\; \mathfrak{R}_h(x,u) &= \int\limits_0^{\infty} G(x,\eta) f(\eta,u(\eta))d\eta + u^{(0)}(x) = \\
&= \sum_{i=1}^{N} \int\limits_{x_{i-1}}^{x_i} G(x,\eta) f(\eta,u(\eta))d\eta + \int\limits_{x_N}^{\infty} G(x,\eta) f(\eta,u(\eta))d\eta + u^{(0)}(x),
\end{aligned}
\tag{30}
$$

where

$$
u(\eta) = Y_1^i(\eta,u), \quad \eta \in [x_{i-1},x_i], \quad i=1,2,...,N,
$$
$$
u(\eta) = Y_2^N(\eta,u), \quad \eta \in [x_N,\infty).
$$

We will investigate the properties of the operator $\mathfrak{R}_h(x,u)$. The operator (30) transforms the set $\Omega\left(\hat{\bar{\omega}}_N, r\right)$ into itself. Let $(v_j)_{j=0}^N \in \Omega\left(\hat{\bar{\omega}}_N, r\right)$, then

$$
v(x) = Y_1^i(x,v) \in \Omega([x_{j-1},x_j],r), \quad v(x) = Y_2^N(x,v) \in \Omega([x_N,\infty),r),
$$

$$
\left\| \mathfrak{R}_h(x,v) - u^{(0)} \right\|^*_{1,\infty,\hat{\bar{\omega}}_N} \leqslant \int\limits_0^{\infty} \|G(x,\eta)\|^*_{1,\infty,\hat{\bar{\omega}}_N} K(\eta) d\eta \leqslant r \quad \forall (v_j)_{j=0}^N \in \Omega(\hat{\bar{\omega}}_N, r).
$$

Besides,

$$
\begin{gathered}
\|\mathfrak{R}_h(x,u) - \mathfrak{R}_h(x,v)\|^*_{1,\infty,\hat{\bar{\omega}}_N} \leqslant q \, \|u-v\|_{1,\infty,[0,\infty)} \\
\forall (u_j)_{j=0}^N, (v_j)_{j=0}^N \in \Omega(\hat{\bar{\omega}}_N, r).
\end{gathered}
\tag{31}
$$

We will show that

$$
\|u-v\|_{1,\infty,[0,\infty)} \leqslant (1+M|h|) \, \|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N}. \tag{32}
$$

For this, we consider boundary problems

$$
\begin{gathered}
\frac{d^2u}{dx^2} + m^2\frac{du}{dx} = -f(x,u), \quad x \in (x_{j-1},x_j), \\
u(x_{j-1}) = u_{j-1}, \quad u(x_j) = u_j, \quad j=1,2,...,N,
\end{gathered}
$$

and

$$
\begin{gathered}
\frac{d^2u}{dx^2} + m^2\frac{du}{dx} = -f(x,u), \quad x \in (x_N,\infty), \\
u(x_N) = u_N, \quad \lim_{x\to\infty} u(x) = 0,
\end{gathered}
$$

we can write the solutions in the form

$$
u(x) = \int\limits_{x_{j-1}}^{x_j} G^j(x,\xi) f(\xi,u(\xi))d\xi + \hat{u}(x), \quad x_{j-1} \leqslant x \leqslant x_j, \quad j=1,2,...,N,
$$

$$
u(x) = \int\limits_{x_N}^{\infty} G^{\infty}(x,\xi) f(\xi,u(\xi))d\xi + u_N e^{-m^2(x-x_N)}, \quad x \geqslant x_N,
$$

where $\hat{u}(x), G^j(x,\xi)$ are given by the formulas (17), (19) for $\alpha=1$ and $G^{\infty}(x,\xi)$ by the formula (20).

Based on the Lipschitz condition (5)

$$\|u-v\|_{1,\infty,[x_{j-1},x_j]} \leqslant \int\limits_{x_{j-1}}^{x_j} \left\|G^j(x,\xi)\right\|_{1,\infty,[x_{j-1},x_j]} L(\xi)d\xi \, \|u-v\|_{1,\infty,[x_{j-1},x_j]} + \\ + \|\hat{u}-\hat{v}\|_{1,\infty,[x_{j-1},x_j]}, \quad j=1,2,...,N,$$

$$\|u-v\|_{1,\infty,[x_N,\infty)} \leqslant \int\limits_{x_N}^{\infty} \|G^{\infty}(x,\xi)\|_{1,\infty[x_N,\infty)} L(\xi)d\xi \, \|u-v\|_{1,\infty,[x_N,\infty)} + \\ + \left\|(u_N-v_N)\, e^{-m^2(x-x_N)}\right\|_{1,\infty,[x_N,\infty)}.$$

Since

$$\|\hat{u}-\hat{v}\|_{0,\infty,[x_{j-1},x_j]} = \max_{x\in[x_{j-1},x_j]} \left\{ \frac{\left(e^{-m^2 x_{j-1}} - e^{-m^2 x}\right)\|u_j-v_j\|}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} + \right. \\ \left. + \frac{\left(e^{-m^2 x} - e^{-m^2 x_j}\right)\|u_{j-1}-v_{j-1}\|}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} \right\} \leqslant \|u-v\|_{0,\infty,\hat{\omega}_N^+},$$

$$\left\|\frac{d\hat{u}}{dx} - \frac{d\hat{v}}{dx}\right\|_{0,\infty,[x_{j-1},x_j]} = \max_{x\in[x_{j-1},x_j]} \frac{m^2 h_j e^{-m^2 x}\|u_{\bar{x},j}-v_{\bar{x},j}\|}{e^{-m^2 x_{j-1}} - e^{-m^2 x_j}} = \\ = \frac{m^2 h_j \|u_{\bar{x},j}-v_{\bar{x},j}\|}{1-e^{-m^2 h_j}} \leqslant (1+M_1|h|) \left\|\frac{du}{dx} - \frac{dv}{dx}\right\|_{0,\infty,\hat{\bar{\omega}}_N},$$

$$\left\|(u_N-v_N)e^{-m^2(x-x_N)}\right\|_{0,\infty,[x_N,\infty)} \leqslant \|u-v\|_{0,\infty,\hat{\omega}_N^+},$$

$$\left\|(u_N-v_N)m^2 e^{-m^2(x-x_N)}\right\|_{0,\infty,[x_N,\infty)} \leqslant \frac{e^{m^2 x_N}}{x_N} \|u-v\|_{0,\infty,\hat{\omega}_N^+} \leqslant M_2|h| \, \|u-v\|_{0,\infty,\hat{\omega}_N^+},$$

then

$$\|u-v\|_{1,\infty,[x_{j-1},x_j]} \leqslant (1+M_1|h|)\,\|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N} + |h|\,M_3\,\|u-v\|_{1,\infty,[x_{j-1},x_j]},$$

$$\|u-v\|_{1,\infty,[x_N,\infty)} \leqslant (1+M_2|h|)\,\|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N} + |h|\,M_4\,\|u-v\|_{1,\infty,[x_N,\infty)}.$$

From here we get estimates

$$\|u-v\|_{1,\infty,[x_{j-1},x_j]} \leqslant \frac{1+M_1|h|}{1-M_3|h|}\|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N} \leqslant \\ \leqslant (1+|h|\,M)\,\|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N}, \quad j=1,2,...,N,$$

$$\|u-v\|_{1,\infty,[x_N,\infty)} \leqslant \frac{1+M_2|h|}{1-M_4|h|}\|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N} \leqslant (1+|h|\,M)\,\|u-v\|^*_{1,\infty,\hat{\bar{\omega}}_N},$$

from which the inequality (32) follows.

Taking into account (32), from estimates (31) we obtain

$$\|\mathfrak{R}_h(x,u) - \mathfrak{R}_h(x,v)\|^*_{1,\infty,\hat{\omega}_N^+} \leqslant (q+M|h|)\,\|u-v\|^*_{1,\infty,\hat{\omega}_N^+} = q_1\,\|u-v\|^*_{1,\infty,\hat{\omega}_N^+}.$$

According to (6), for sufficiently small h_0 we have $q_1 < 1$, and the operator (30) is a contraction $\forall (u_j)_{j=0}^N, (v_j)_{j=0}^N \in \Omega(\hat{\bar{\omega}}_N, r)$. So, due to Banach's Fixed Point Theorem, if h_0 is sufficiently small, then ETDS (23), (24) has a unique solution, which can be obtained by successive approximations. ■

Lemma 5. *Let there exists a constant $\Delta > 0$ such that the conditions (3), (5) are satisfied on the set $\Omega\left([0,\infty), r+\Delta\right)$. Then there exists such $h_0 > 0$, that for $|h| \leqslant h_0$ and $\forall\,(v_j)_{j=0}^{N} \in \Omega\left(\hat{\bar{\omega}}_N, r\right)$ problems*

$$\begin{gathered}\frac{d^2Y_\alpha^j(x,v)}{dx^2} + m^2\frac{dY_\alpha^j(x,v)}{dx} = -f(x, Y_\alpha^j(x,v)), \quad x_{j-2+\alpha} < x < x_{j-1+\alpha},\\ Y_\alpha^j(x_{j+(-1)^\alpha}, v) = v(x_{j+(-1)^\alpha}), \qquad \frac{dY_\alpha^j(x_{j+(-1)^\alpha}, v)}{dx} = \left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}},\\ j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1,2,\end{gathered} \tag{33}$$

$$\begin{gathered}\frac{d^2Y_2^N(x,v)}{dx^2} + m^2\frac{dY_2^N(x,v)}{dx} = -f(x, Y_2^N(x,v)), \quad x > x_N,\\ Y_2^N(x_N, v) = v(x_N), \qquad \lim_{x\to\infty} Y_2^N(x,v) = 0\end{gathered} \tag{34}$$

will have a unique solution.

Proof. Problems (33), (34) are equivalent to the operator equations

$$\begin{gathered}U_\alpha^j(x) = \mathfrak{R}_\alpha^j(x, v, U_\alpha^j) = -\frac{1}{m^2}\int\limits_{x_{j+(-1)^\alpha}}^{x}\left(1 - e^{-m^2(x-\xi)}\right) f(\xi, U_\alpha^j)d\xi +\\ + v_{j+(-1)^\alpha} + \frac{1 - e^{-m^2(x - x_{j+(-1)^\alpha})}}{m^2}\left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}},\\ x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \quad j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1,2,\end{gathered}$$

$$U_2^N(x) = \mathfrak{R}_2^N(x, v, U_2^N) = \int\limits_{x_N}^{\infty} G^\infty(x,\xi) f(\xi, U_2^N(\xi, v))d\xi + v_N e^{-m^2(x-x_N)}, \quad x \in [x_N, \infty),$$

where $G^\infty(x,\xi)$ are given by the formula (20).

We investigate properties of operators $\mathfrak{R}_\alpha^j(x, v, U_\alpha^j)$, $\alpha = 1,2$, $\mathfrak{R}_2^N(x, v, U_2^N)$. We should note, that

$$\begin{gathered}u^{(0)}(x) = \mu_1 e^{-m^2 x} = u_{j+(-1)^\alpha}^{(0)} + \frac{1 - e^{-m^2(x - x_{j+(-1)^\alpha})}}{m^2}\left.\frac{du^{(0)}}{dx}\right|_{x=x_{j+(-1)^\alpha}},\\ x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \quad j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1,2,\\ u^{(0)}(x) = u_N^{(0)} e^{-m^2(x-x_N)}, \quad x > x_N.\end{gathered}$$

Let $U_\alpha^j \in \Omega\left([x_{j-2+\alpha}, x_{j-1+\alpha}], r+\Delta\right)$, $U_2^N \in \Omega\left([x_N, \infty), r+\Delta\right)$, then

$$\begin{gathered}\left\|\mathfrak{R}_\alpha^j(x, v, U_\alpha^j) - u^{(0)}\right\| \leqslant \left\|v_{j+(-1)^\alpha} - u_{j+(-1)^\alpha}^{(0)}\right\| +\\ + (-1)^{\alpha+1}\frac{1 - e^{-m^2(x - x_{j+(-1)^\alpha})}}{m^2}\left[\left\|\left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}} - \left.\frac{du^{(0)}}{dx}\right|_{x=x_{j+(-1)^\alpha}}\right\| + \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} K(\xi)d\xi\right] \leqslant\\ \leqslant r + M|h|,\end{gathered}$$

$$\left\| \frac{\partial \mathfrak{R}_\alpha^j(x,v,U_\alpha^j)}{\partial x} - \frac{du^{(0)}(x)}{dx} \right\| \leqslant (-1)^{\alpha+1} e^{-m^2(x-x_{j+(-1)^\alpha})} \times$$
$$\times \left[\left\| \left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}} - \left.\frac{du}{dx}\right|_{x=x_{j+(-1)^\alpha}} \right\| + \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} K(\xi)d\xi \right] \leqslant$$
$$\leqslant r + M|h|,$$

$$\left\| \mathfrak{R}_2^N(x,v,U_2^N) - u^{(0)} \right\| \leqslant e^{-m^2(x-x_N)} \left\| v_N - u_N^{(0)} \right\| + \frac{1-e^{-m^2(x-x_N)}}{m^2} \int\limits_{x_N}^{\infty} K(\xi)d\xi \leqslant$$
$$\leqslant r + M|h| \quad \forall v \in \Omega(\hat{\bar{\omega}}_N, r),$$

$$\left\| \frac{\partial \mathfrak{R}_2^N(x,v,U_2^j)}{\partial x} - \frac{du^{(0)}(x)}{dx} \right\| \leqslant m^2 e^{-m^2(x-x_N)} \left\| v_N - u_N^{(0)} \right\| + \int\limits_x^{\infty} K(\xi)d\xi + \int\limits_{x_N}^{x} K(\xi)d\xi \leqslant$$
$$\leqslant \int\limits_{x_N}^{\infty} K(\xi)d\xi + \frac{e^{m^2 x_N}}{x_N} \leqslant r + M|h| \quad \forall v \in \Omega(\hat{\bar{\omega}}_N, r).$$

Hence

$$\left\| \mathfrak{R}_\alpha^j(x,v,U_\alpha^j) - u^{(0)} \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \leqslant r + \Delta,$$
$$\left\| \mathfrak{R}_2^N(x,v,U_2^N) - u^{(0)} \right\|_{1,\infty,[x_N,\infty)} \leqslant r + \Delta,$$

i.e. operators $\mathfrak{R}_\alpha^j(x,v,U_\alpha^j), \alpha = 1,2,$ $\mathfrak{R}_2^N(x,v,U_2^N)$ transforms the sets $\Omega\left([x_{j-2+\alpha},x_{j-1+\alpha}], r+\Delta\right)$, $\Omega\left([x_N,\infty), r+\Delta\right)$ respectively into themselves.

Besides,

$$\left\| \mathfrak{R}_\alpha^j(x,v,U_\alpha^j) - \mathfrak{R}_\alpha^j(x,v,\tilde{U}_\alpha^j) \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \leqslant \left\| \int\limits_{x_{j+(-1)^\alpha}}^{x} \frac{1-e^{-m^2(x-\xi)}}{m^2} L(\xi)d\xi \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \times$$
$$\times \left\| U_\alpha^j - \tilde{U}_\alpha^j \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \leqslant$$
$$\leqslant \left\| \frac{1-e^{-m^2(x-x_{j+(-1)^\alpha})}}{m^2} \int\limits_{x_{j+(-1)^\alpha}}^{x} L(\xi)d\xi \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \left\| U_\alpha^j - \tilde{U}_\alpha^j \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \leqslant$$
$$\leqslant q\,|h| \left\| U_\alpha^j - \tilde{U}_\alpha^j \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]},$$

$$\left\| \mathfrak{R}_2^N(x,v,U_2^N) - \mathfrak{R}_2^N(x,v,\tilde{U}_2^N) \right\|_{1,\infty,[x_N,\infty)} \leqslant$$
$$\leqslant \left\| \frac{1-e^{-m^2(x-x_N)}}{m^2} \right\|_{1,\infty,[x_N,\infty)} \int\limits_{x_N}^{\infty} L(\xi)d\xi \left\| U_2^N - \tilde{U}_2^N \right\|_{1,\infty,[x_N,\infty)} \leqslant$$
$$\leqslant q \left\| U_2^N - \tilde{U}_2^N \right\|_{1,\infty,[x_N,\infty)}.$$

Since due to (6) $\tilde{q} = q\,|h| < 1$ at sufficiently small h_0, then operators $\mathfrak{R}_\alpha^j(x, v, U_\alpha^j)$, $\alpha = 1, 2$, $\mathfrak{R}_2^N(x, v, U_2^N)$ are a contraction for all $U_\alpha^j, \tilde{U}_\alpha^j \in \Omega\left([x_{j-2+\alpha}, x_{j-1+\alpha}], r + \Delta\right)$, $U_2^N, \tilde{U}_2^N \in \Omega\left([x_N, \infty), r + \Delta\right)$. Thus, in accordance with the satisfied Banach's Fixed Point Theorems, at sufficiently small h_0, problems (33), (34) will have a unique solution. ■

4. Conclusions

In this paper the exact three-point differential scheme for the numerical solution of boundary value problems on a semi-infinite interval for systems of second order nonlinear ordinary differential equations with non-selfadjoint operator is constructed and substantiated.

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Точна різницева схема для систем нелінійних ЗДР другого порядку на напівнескінченному інтервалі

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Для чисельного розв'язування крайових задач на півпрямій для систем нелінійних звичайних диференціальних рівнянь другого порядку з несамоспряженим оператором побудовано точну триточкову різницеву схему. За умов існування та єдиності розв'язку крайової задачі доведено існування та єдиність розв'язку точної триточкової різницевої схеми та збіжність методу послідовних наближень для його знаходження.

Ключові слова: *нелінійні звичайні диференціальні рівняння, крайова задача, точна триточкова різницева схема*

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