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# 2D integral formulae and equations for thermoelectroelastic bimaterial with thermally insulated interface

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The paper presents a complex variable approach for obtaining of the integral formulae and integral equations for plane thermoelectroelasticity of an anisotropic bimaterial with thermally insulated interface. Obtained relations do not contain domain integrals and incorporate only physical boundary functions such as temperature, heat flux, extended displacement and traction, which are the main advances of these relations.

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## 1. Introduction

Ferroelectric materials are widely used in modern technologies, especially precise devices, due to the highest values of electro-mechanical coupling among other piezoelectric materials. In turn, all ferroelectric materials are pyroelectric ones [1], thus, polarized when heated or cooled. This behavior should be definitely accounted for in the design of smart devices containing ferroelectric parts, which are not maintained at the constant temperature. The presence of different defects (e.g. cracks or inclusions) can additionally cause high stress and electric displacement intensity under the applied thermal load, especially, when the pyroelectric material is not homogeneous, or consists of homogeneous parts bonded together.

The study of anisotropic and piezoelectric bimaterial solids is widely covered in the scientific literature. Pan and Amadei [2] developed a single domain boundary element approach for fracture analysis of anisotropic bimaterials. Tian and Gabbert [3] studied cracks near the interface of piezoelectric and magnetoelectroelastic bimaterials. Ou and Chen [4] studied near-tip stress fields near interfacial cracks.

However, the problems for thermoelectroelastic bimaterials with imperfect interface are more challenging. There are fewer publications concerning these problems. Wang and Pan [5] derived the 2D Green's function for a thermoelastic anisotropic bimaterial with imperfect interface. Qin [6] derived 2D Green's functions and developed a boundary element technique for defective thermomagnetoelectroelastic bimaterial solids.

Therefore, this paper utilizes authors' novel complex variable approach [7,8] for obtaining of the Somigliana type integral formulae and integral equations for an anisotropic thermoelectroelastic bimaterial with thermally insulated interface in a strict and straightforward manner.

## 2. Governing equations of thermoelectroelasticity. Extended Stroh formalism

In a fixed rectangular system of coordinates $Ox_1x_2x_3$ under the assumption that all fields depend only on in-plane coordinates $x_1$ and $x_2$, the balance equations for stress, electric displacement and heat flux,

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and constitutive laws can be written in the following compact form [6,7]:

$$\tilde{\sigma}_{ij,j}+\tilde{f}_i=0,\quad h_{i,i}-f_h=0\quad (i=1,..,4;\ j=1,2)\,; \tag{1}$$

$$\tilde{\sigma}_{ij}=\tilde{C}_{ijkm}\tilde{u}_{k,m}-\tilde{\beta}_{ij}\theta,\quad h_i=-k_{ij}\theta_{,j}\quad (i,k=1,..,4;\ j,m=1,2) \tag{2}$$

with

$$\begin{gathered}\tilde{u}_i=u_i,\ \ \tilde{u}_4=\varphi;\ \ \tilde{f}_i=f_i,\ \ \tilde{f}_4=-q;\ \ \tilde{\sigma}_{ij}=\sigma_{ij},\ \tilde{\sigma}_{4j}=D_j;\ \ \tilde{\beta}_{ij}=\beta_{ij},\ \tilde{\beta}_{4j}=-\chi_j;\\ \tilde{C}_{ijkm}=C_{ijkm},\ \ \tilde{C}_{ij4m}=e_{mij},\ \ \tilde{C}_{4jkm}=e_{jkm},\ \ \tilde{C}_{4j4m}=-\kappa_{jm}\ \ (i,k=1,2,3;j,m=1,2)\,,\end{gathered} \tag{3}$$

where $\sigma_{ij}$ is the stress tensor; $h_i$ is the heat flux; $D_i$ is the electric displacement; $f_i$ is the body force vector; $q$ is the free charge volume density; $f_h$ is the distributed heat source density; $u_i$ is the displacement; $\varphi$ is the electric potential; $\theta$ is the temperature change with respect to the reference temperature; $C_{ijkm}$ are elastic moduli; $k_{ij}$ are heat conduction coefficients; $\beta_{ij}=C_{ijkm}\alpha_{km}+e_{mij}\lambda_m$ $(i,j,k,m=1,..,3)$ are thermal moduli (thermal stress coefficients); $\alpha_{ij}$ are thermal expansion coefficients; $e_{ijk}$ are piezoelectric constants; $\kappa_{ij}$ are dielectric constants; $\chi_i=-e_{ikm}\alpha_{km}+\kappa_{ij}\lambda_j$ are pyroelectric coefficients; and $\lambda_i$ are pyroelectric constants. Tensors $C_{ijkm}$, $k_{ij}$, $\kappa_{ij}$ and $\beta_{ij}$ are assumed to be symmetric. Here and further, the Einstein summation convention is used. A comma at subscript denotes differentiation with respect to a coordinate indexed after the comma, i.e. $u_{i,j}=\partial u_i/\partial x_j$.

According to the extended Stroh formalism [6], the general homogeneous (i.e. for $\tilde{f}_i\equiv 0$ and $f_h\equiv 0$) solution of Eqs. (1), (2) can be expressed in terms of complex analytic functions as follows

$$\begin{gathered}\theta=2\,\mathrm{Re}\left\{g'\left(z_t\right)\right\},\ \vartheta=2k_t\,\mathrm{Im}\left\{g'\left(z_t\right)\right\},\ h_1=-\vartheta_{,2},\ h_2=\vartheta_{,1},\ k_t=\sqrt{k_{11}k_{22}-k_{12}^2};\\ \tilde{\mathbf{u}}=2\,\mathrm{Re}\left\{\mathbf{A}\mathbf{f}\left(z_*\right)+\mathbf{c}g\left(z_t\right)\right\},\ \tilde{\boldsymbol{\varphi}}=2\,\mathrm{Re}\left\{\mathbf{B}\mathbf{f}\left(z_*\right)+\mathbf{d}g\left(z_t\right)\right\};\ \tilde{\sigma}_{i1}=-\tilde{\varphi}_{i,2},\ \tilde{\sigma}_{i2}=\tilde{\varphi}_{i,1};\\ z_t=x_1+p_tx_2;z_\alpha=x_1+p_\alpha x_2;\ \mathbf{f}\left(z_*\right)=\left[F_1\left(z_1\right),F_2\left(z_2\right),F_3\left(z_3\right),F_4\left(z_4\right)\right]^{\mathrm{T}},\end{gathered} \tag{4}$$

where $g\left(z_t\right)$ and $F_\alpha\left(z_\alpha\right)$ are complex analytic functions with respect to their arguments; complex matrix, vector and scalar constants included in Eq. (4) depend only on the properties of the material and are derived based on the Stroh eigenvalue problem [6].

Stroh orthogonality conditions [6] allow obtaining the following relations between the Stroh complex functions and vectors of extended displacement and stress function [7]:

$$\mathbf{f}\left(z_*\right)=\mathbf{B}^{\mathrm{T}}\tilde{\mathbf{u}}+\mathbf{A}^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}-\mathbf{B}^{\mathrm{T}}\tilde{\mathbf{u}}^t-\mathbf{A}^{\mathrm{T}}\tilde{\boldsymbol{\varphi}}^t,\ \tilde{\mathbf{u}}^t=2\,\mathrm{Re}\left\{\mathbf{c}g\left(z_t\right)\right\},\ \tilde{\boldsymbol{\varphi}}^t=2\,\mathrm{Re}\left\{\mathbf{d}g\left(z_t\right)\right\}. \tag{5}$$

Based on Eq. (4) one can derive the following relation between the function $g'\left(z_t\right)$ and temperature and heat flux function:

$$g'\left(z_t\right)=\frac{1}{2}\left(\theta+i\frac{\vartheta}{k_t}\right). \tag{6}$$

## 3. Derivation of the integral formulae for the Stroh complex functions for a bimaterial

Consider a plane problem of thermoelectroelasticity for two half-spaces $x_2>0$ and $x_2<0$ (actually only the cross-sections (half-planes) $S_1$ and $S_2$ normal to the axis $x_3$ are studied as depicted in Fig. 1), which are bonded along the plane $x_2=0$ such that the following conditions are satisfied for thermal fields (thermally insulated interface)

$$\vartheta^{(1)}\left(x_1,x_2\right)\Big|_{x_2=0}=\vartheta^{(2)}\left(x_1,x_2\right)\Big|_{x_2=0}\equiv 0\quad \forall x_1\in(-\infty;\infty) \tag{7}$$

and electric and mechanical fields (perfect electromechanical contact)

$$\tilde{\mathbf{u}}^{(1)}(x_1,x_2)\Big|_{x_2=0} = \tilde{\mathbf{u}}^{(2)}(x_1,x_2)\Big|_{x_2=0}, \quad \tilde{\boldsymbol{\varphi}}^{(1)}(x_1,x_2)\Big|_{x_2=0} = \tilde{\boldsymbol{\varphi}}^{(2)}(x_1,x_2)\Big|_{x_2=0} \quad \forall x_1 \in (-\infty;\infty). \tag{8}$$

Each of the half-spaces contains a system of cylindrical voids which are referred as corresponding holes in the 2D half-planes $S_1$ and $S_2$ that are bounded by plane contours $\Gamma_1 = \bigcup_j \Gamma_j^{(1)}$ and $\Gamma_2 = \bigcup_j \Gamma_j^{(2)}$ respectively (Fig. 1).

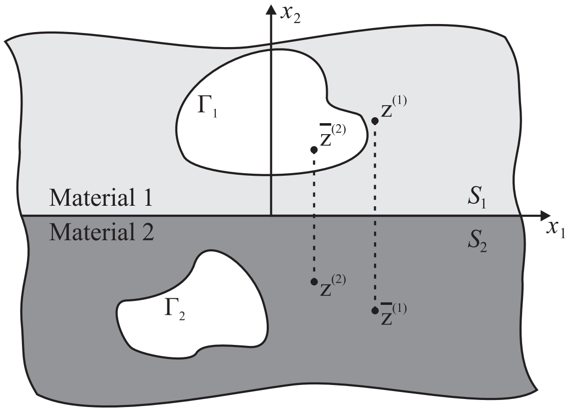


**Fig. 1.** The sketch of the problem.

For derivation of the integral formulae for the Stroh complex functions for bonded half-spaces one can use the Cauchy integral formula [9] which relates values of an arbitrary analytic function $\varphi(\tau)$ at the boundary $\partial S$ of the domain $S$ with its value $\varphi(z)$ inside this domain:

$$\frac{1}{2\pi i}\int_{\partial S} \frac{\varphi(\tau)\,d\tau}{\tau - z} = \begin{cases} \varphi(z) & \forall z \in S, \\ 0 & \forall z \notin S, \end{cases} \tag{9}$$

where $\tau, z \in \mathbb{C}$ are complex variables, which define the position of the source and field points, respectively. Herewith, if the domain $S$ is infinite, it is assumed that the function $\varphi(z)$ vanishes at $z \to \infty$.

### 3.1. Heat conduction

Due to the linearity of the problem of heat conduction one can present its solution as a superposition of the homogeneous solution given by the functions $g_{1\infty}\big(z_t^{(1)}\big)$ and $g_{2\infty}\big(z_t^{(2)}\big)$ (which should definitely satisfy the boundary conditions (7)) and the perturbed solution caused by the presence of the contours $\Gamma_1$ and $\Gamma_2$, and certain boundary conditions set on them. Denoting the Cauchy integrals of the complex temperature functions $g_i'(z_t^{(i)})$ as

$$q_t^{(i)}\big(z_t^{(j)}\big) = \int_{\Gamma_i} \frac{g_i'\big(\tau_t^{(i)}\big)d\tau_t^{(i)}}{\tau_t^{(i)} - z_t^{(j)}}, \quad \bar{q}_t^{(i)}\big(z_t^{(j)}\big) = \int_{\Gamma_i} \frac{\overline{g_i'\big(\tau_t^{(i)}\big)}d\bar{\tau}_t^{(i)}}{\bar{\tau}_t^{(i)} - z_t^{(j)}}, \tag{10}$$

and the improper integrals over the infinite path $-\infty < x_1 < \infty$ as

$$p_t\big(z_t^{(j)}\big) = \int_{-\infty}^{\infty} \frac{\theta_j(x_1)\,dx_1}{x_1 - z_t^{(j)}}, \tag{11}$$

based on Eqs. (6) and (9), satisfying the contact conditions (7) one obtains the following integral formulae for the functions $g_i'\big(z_t^{(i)}\big)$:

$$g_1'\big(z_t^{(1)}\big) = g_{1\infty}'\big(z_t^{(1)}\big) + \frac{1}{2\pi i} q_t^{(1)}\big(z_t^{(1)}\big) + \frac{1}{4\pi i} p_t\big(z_t^{(1)}\big) \quad \forall\,\mathrm{Im}\,\big(z_t^{(1)}\big) > 0; \tag{12}$$

$$g_2'\big(z_t^{(2)}\big) = g_{2\infty}'\big(z_t^{(2)}\big) + \frac{1}{2\pi i} q_t^{(2)}\big(z_t^{(2)}\big) - \frac{1}{4\pi i} p_t\big(z_t^{(2)}\big) \quad \forall\,\mathrm{Im}\,\big(z_t^{(2)}\big) > 0, \tag{13}$$

and a system of equations for determination of the improper integrals (11) through the Cauchy integrals (10):

$$\bar{q}_t^{(1)}\big(z_t^{(1)}\big) + \frac{1}{2} p_t\big(z_t^{(1)}\big) = 0 \quad \big(\mathrm{Im}\,\big(z_t^{(1)}\big) > 0\big), \tag{14}$$

$$\bar{q}_t^{(2)}\big(z_t^{(2)}\big) - \frac{1}{2} p_t\big(z_t^{(2)}\big) = 0 \quad \big(\mathrm{Im}\,\big(z_t^{(2)}\big) < 0\big). \tag{15}$$

The superscripts 1 or 2 denote corresponding half-space, which the temperature complex function belongs to. A positive orientation of the contour is selected under the condition that as we traverse the path following the positive orientation the domain occupied by the solid must always be on the left.

Solving Eqs. (14) and (15) for the unknowns (11) one can obtain that

$$p_t\big(z_t^{(1)}\big) = -2\bar{q}_t^{(1)}\big(z_t^{(1)}\big), \tag{16}$$

$$p_t\big(z_t^{(2)}\big) = 2\bar{q}_t^{(2)}\big(z_t^{(2)}\big). \tag{17}$$

Substituting Eq. (16) into Eq. (12), and Eq. (17) into Eq. (13), one obtains the following integral formulae for the temperature functions $g_i'\big(z_t^{(i)}\big)$, which do not contain improper integrals along the infinite bimaterial interface:

$$g_1'\big(z_t^{(1)}\big) = g_{1\infty}'\big(z_t^{(1)}\big) + \frac{1}{2\pi i}\left[q_t^{(1)}\big(z_t^{(1)}\big) - \bar{q}_t^{(1)}\big(z_t^{(1)}\big)\right] \quad \forall\,\mathrm{Im}\,\big(z_t^{(1)}\big) > 0; \tag{18}$$

$$g_2'\big(z_t^{(2)}\big) = g_{2\infty}'\big(z_t^{(2)}\big) + \frac{1}{2\pi i}\left[q_t^{(2)}\big(z_t^{(2)}\big) - \bar{q}_t^{(2)}\big(z_t^{(2)}\big)\right] \quad \forall\,\mathrm{Im}\,\big(z_t^{(2)}\big) < 0. \tag{19}$$

According to [8], integral formulae (10) can be reduced to the following line integrals of the first kind

$$\begin{aligned} q_t^{(i)}\big(z_t^{(j)}\big) &= -\frac{1}{2}\int\limits_{\Gamma_i} \frac{n_2(s) - p_t^{(i)} n_1(s)}{\tau_t^{(i)}(s) - z_t^{(j)}}\theta(s)ds + \frac{i}{2k_t^{(i)}}\int\limits_{\Gamma_i} \ln\left(\tau_t^{(i)}(s) - z_t^{(j)}\right) h_n(s)ds, \\ \bar{q}_t^{(i)}\big(z_t^{(j)}\big) &= -\frac{1}{2}\int\limits_{\Gamma_i} \frac{n_2(s) - \bar{p}_t^{(i)} n_1(s)}{\bar{\tau}_t^{(i)}(s) - z_t^{(j)}}\theta(s)ds - \frac{i}{2k_t^{(i)}}\int\limits_{\Gamma_i} \ln\left(\bar{\tau}_t^{(i)}(s) - z_t^{(j)}\right) h_n(s)ds. \end{aligned} \tag{20}$$

Based on Eq. (20), one can write explicit expressions for integral formulae (18) and (19) through the boundary values of the temperature $\theta$ and the normal component $h_n$ of a heat flux vector. By means of the Sokhotskii-Plemelj formula and Eqs. (18), (19), (20) one can easily obtain boundary integral equations for a temperature field in a bimaterial with thermally insulated interface.

According to Eq. (4), for derivation of the integral formulae for extended displacement and stress it is necessary to calculate the anti-derivative of the functions $g_i'\big(z_t^{(i)}\big)$ and $p_t^{(i)}\big(z_t^{(j)}\big)$:

$$g_i\big(z_t^{(i)}\big) = \int g_i'\big(z_t^{(i)}\big) dz_t^{(i)}; \tag{21}$$

$$P_t\big(z_t^{(i)}\big) = \int p_t\big(z_t^{(i)}\big) dz_t^{(i)} = -\int\limits_{-\infty}^{\infty} \ln\big(x_1 - z_t^{(i)}\big)\theta^{(i)}\big(x_1\big) dx_1. \tag{22}$$

According to Eqs. (16)–(19), one obtains

$$\begin{aligned} &P_t\big(z_t^{(1)}\big) = -2\bar{Q}_t^{(1)}\big(z_t^{(1)}\big),\\ &g_1\big(z_t^{(1)}\big) = g_{1\infty}\big(z_t^{(1)}\big) + \frac{1}{2\pi i}\Big[Q_t^{(1)}\big(z_t^{(1)}\big) - \bar{Q}_t^{(1)}\big(z_t^{(1)}\big)\Big] \quad \big(\forall\,\mathrm{Im}\,\big(z_t^{(1)}\big) > 0\big); \end{aligned} \tag{23}$$

$$\begin{aligned} &P_t\big(z_t^{(2)}\big) = 2\bar{Q}_t^{(2)}\big(z_t^{(2)}\big),\\ &g_2\big(z_t^{(2)}\big) = g_{2\infty}\big(z_t^{(2)}\big) + \frac{1}{2\pi i}\Big[Q_t^{(2)}\big(z_t^{(2)}\big) - \bar{Q}_t^{(2)}\big(z_t^{(2)}\big)\Big] \quad \big(\forall\,\mathrm{Im}\,\big(z_t^{(2)}\big) < 0\big), \end{aligned} \tag{24}$$

where according to Eqs. (20) and (21),

$$\begin{aligned} Q_t^{(i)}(z) &= \int q_t^{(i)}(z)\, dz = \\ &= \frac{1}{2}\int\limits_{\Gamma_i}\Big(n_2(s) - p_t^{(i)} n_1(s)\Big)\ln\Big(\tau_t^{(i)}(s) - z\Big)\theta(s) ds - \frac{i}{2k_t^{(i)}}\int\limits_{\Gamma_i} f^*\Big(\tau_t^{(i)}(s) - z\Big) h_n(s) ds,\\ \bar{Q}_t^{(i)}(z) &= \int \bar{q}_t^{(i)}(z)\, dz = \\ &= \frac{1}{2}\int\limits_{\Gamma_i}\Big(n_2(s) - \bar{p}_t^{(i)} n_1(s)\Big)\ln\Big(\bar{\tau}_t^{(i)}(s) - z\Big)\theta(s) ds + \frac{i}{2k_t^{(i)}}\int\limits_{\Gamma_i} f^*\Big(\bar{\tau}_t^{(i)}(s) - z\Big) h_n(s) ds, \end{aligned} \tag{25}$$

and $f^*(z) = z(\ln z - 1)$ is the anti-derivative of a logarithmic function.

### 3.2. Thermoelectroelasticity of a bimaterial

For obtaining the integral formulae of thermoelectroelasticity one should write the Cauchy integral formula (9) for the Stroh complex vector functions $\mathbf{f}^{(1)}\big(z_*^{(1)}\big)$ and $\mathbf{f}^{(2)}\big(z_*^{(2)}\big)$, which are analytic in the domains $S_1$ and $S_2$, respectively. Since the Cauchy integral formula defines the analytic function that vanishes at the infinity, the complete solution of the problem can be presented as a sum of the perturbed solution defined by the Cauchy formula and a homogeneous solution given by the functions $\mathbf{f}_\infty^{(1)}\big(z_*^{(1)}\big)$ and $\mathbf{f}_\infty^{(2)}\big(z_*^{(2)}\big)$, which satisfy interface boundary conditions (8). Consequently, one obtains

$$\mathbf{f}^{(1)}\big(z_*^{(1)}\big) = \mathbf{f}_\infty^{(1)}\big(z_*^{(1)}\big) + \frac{1}{2\pi i}\left[\int\limits_{\Gamma_1}\left\langle\frac{d\tau_*^{(1)}}{\tau_*^{(1)} - z_*^{(1)}}\right\rangle \mathbf{f}^{(1)}\big(\tau_*^{(1)}\big) + \int\limits_{-\infty}^{\infty}\left\langle\frac{dx_1}{x_1 - z_*^{(1)}}\right\rangle \mathbf{f}^{(1)}(x_1)\right]; \tag{26}$$

$$\int\limits_{\Gamma_1}\left\langle\frac{d\tau_*^{(1)}}{\tau_*^{(1)} - \bar{z}_\beta^{(1)}}\right\rangle \mathbf{f}^{(1)}\big(\tau_*^{(1)}\big) + \int\limits_{-\infty}^{\infty}\frac{dx_1}{x_1 - \bar{z}_\beta^{(1)}}\mathbf{f}^{(1)}(x_1) = \mathbf{0}; \tag{27}$$

$$\int\limits_{\Gamma_2}\left\langle\frac{d\tau_*^{(2)}}{\tau_*^{(2)} - z_\beta^{(1)}}\right\rangle \mathbf{f}^{(2)}\big(\tau_*^{(2)}\big) - \int\limits_{-\infty}^{\infty}\frac{dx_1}{x_1 - z_\beta^{(1)}}\mathbf{f}^{(2)}(x_1) = \mathbf{0} \quad \Big(\mathrm{Im}\, z_\beta^{(1)} > 0\Big); \tag{28}$$

$$\mathbf{f}^{(2)}\big(z_*^{(2)}\big) = \mathbf{f}_\infty^{(2)}\big(z_*^{(2)}\big) + \frac{1}{2\pi i}\left[\int\limits_{\Gamma_2}\left\langle\frac{d\tau_*^{(2)}}{\tau_*^{(2)} - z_*^{(2)}}\right\rangle \mathbf{f}^{(2)}\big(\tau_*^{(2)}\big) - \int\limits_{-\infty}^{\infty}\left\langle\frac{dx_1}{x_1 - z_*^{(2)}}\right\rangle \mathbf{f}^{(2)}(x_1)\right]; \tag{29}$$

$$\int_{\Gamma_1}\left\langle\frac{d\tau_*^{(1)}}{\tau_*^{(1)}-z_\beta^{(2)}}\right\rangle\mathbf{f}^{(1)}\left(\tau_*^{(1)}\right)+\int_{-\infty}^{\infty}\frac{dx_1}{x_1-z_\beta^{(2)}}\mathbf{f}^{(1)}\left(x_1\right)=\mathbf{0}; \tag{30}$$

$$\int_{\Gamma_2}\left\langle\frac{d\tau_*^{(2)}}{\tau_*^{(2)}-\bar{z}_\beta^{(2)}}\right\rangle\mathbf{f}^{(2)}\left(\tau_*^{(2)}\right)-\int_{-\infty}^{\infty}\frac{dx_1}{x_1-\bar{z}_\beta^{(2)}}\mathbf{f}^{(2)}(x_1)=\mathbf{0}\quad\left(\operatorname{Im} z_\beta^{(2)}<0\right), \tag{31}$$

where $\left\langle F\left(z_*\right)\right\rangle=\operatorname{diag}\left[F_1(z_1),F_2\left(z_2\right),F_3\left(z_3\right),F_4\left(z_4\right)\right]$; and $z_\alpha^{(j)}=x_1+p_\alpha^{(j)}x_2$ $(\alpha=1,\ldots,4)$.

Accounting for Eqs. (5) and (8), the integrals over the infinite bimaterial interface included in Eqs. (26)–(31) can be rewritten as,

$$\int_{-\infty}^{\infty}\frac{\mathbf{f}^{(j)}\left(x_1\right)dx_1}{x_1-z_\beta^{(i)}}=\mathbf{A}_j^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(i)}\right)+\mathbf{B}_j^{\mathrm{T}}p\left(z_\beta^{(i)}\right)-2\int_{-\infty}^{\infty}\frac{\left(\mathbf{A}_j^{\mathrm{T}}\operatorname{Re}\left[\mathbf{d}_jg_j(x_1)\right]+\mathbf{B}_j^{\mathrm{T}}\operatorname{Re}\left[\mathbf{c}_jg_j(x_1)\right]\right)dx_1}{x_1-z_\beta^{(i)}}, \tag{32}$$

where the following notations are used

$$\mathbf{m}\left(z_\beta^{(j)}\right)=\int_{-\infty}^{\infty}\frac{\tilde{\boldsymbol{\varphi}}\left(x_1\right)dx_1}{x_1-z_\beta^{(j)}},\quad\mathbf{p}\left(z_\beta^{(j)}\right)=\int_{-\infty}^{\infty}\frac{\tilde{\mathbf{u}}\left(x_1\right)dx_1}{x_1-z_\beta^{(j)}}. \tag{33}$$

After integration of the last term in the right-hand side of Eq. (32) and accounting for Eqs. (6), and (22), Eq. (32) is rewritten as,

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\mathbf{f}^{(j)}\left(x_1\right)dx_1}{x_1-z_\beta^{(i)}}&=\mathbf{A}_j^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(i)}\right)+\mathbf{B}_j^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(i)}\right)+\\&+2\int_{-\infty}^{\infty}\left(\mathbf{A}_j^{\mathrm{T}}\operatorname{Re}\left[\mathbf{d}_jg_j'\left(x_1\right)\right]+\mathbf{B}_j^{\mathrm{T}}\operatorname{Re}\left[\mathbf{c}_jg_j'\left(x_1\right)\right]\right)\ln\left(x_1-z_\beta^{(i)}\right)dx_1=\\&=\mathbf{A}_j^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(i)}\right)+\mathbf{B}_j^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(i)}\right)-\boldsymbol{\lambda}_jP_t\left(z_\beta^{(i)}\right),\end{aligned} \tag{34}$$

where the complex vector constants $\boldsymbol{\lambda}_j$ are defined by the following equations

$$\boldsymbol{\lambda}_j=\mathbf{A}_j^{\mathrm{T}}\operatorname{Re}\left[\mathbf{d}_j\right]+\mathbf{B}_j^{\mathrm{T}}\operatorname{Re}\left[\mathbf{c}_j\right]. \tag{35}$$

Denoting the Cauchy integrals of the Stroh complex functions as

$$\mathbf{q}_j\left(z_\beta^{(i)}\right)=\int_{\Gamma_j}\left\langle\frac{d\tau_*^{(j)}}{\tau_*^{(j)}-z_\beta^{(i)}}\right\rangle\mathbf{f}^{(j)}\left(\tau_*^{(j)}\right),\quad\bar{\mathbf{q}}_j\left(z_\beta^{(i)}\right)=\int_{\Gamma_j}\left\langle\frac{d\bar{\tau}_*^{(j)}}{\bar{\tau}_*^{(j)}-z_\beta^{(i)}}\right\rangle\overline{\mathbf{f}^{(j)}\left(\tau_*^{(j)}\right)}, \tag{36}$$

based on Eqs. (26)–(31), (33), and (34) one can obtain that

$$\mathbf{f}^{(1)}\left(z_*^{(1)}\right)=\mathbf{f}_\infty^{(1)}\left(z_*^{(1)}\right)+\frac{1}{2\pi i}\left[\mathbf{q}_1\left(z_*^{(1)}\right)+\sum_{\beta=1}^{4}\mathbf{I}_\beta\left(\mathbf{A}_1^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(1)}\right)+\mathbf{B}_1^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(1)}\right)\right)-\left\langle P_t\left(z_*^{(1)}\right)\right\rangle\boldsymbol{\lambda}_1\right]; \tag{37}$$

$$\bar{\mathbf{A}}_1^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(1)}\right)+\bar{\mathbf{B}}_1^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(1)}\right)=-\bar{\mathbf{q}}_1\left(z_\beta^{(1)}\right)+\bar{\boldsymbol{\lambda}}_1P_t\left(z_\beta^{(1)}\right); \tag{38}$$

$$\mathbf{A}_2^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(1)}\right)+\mathbf{B}_2^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(1)}\right)=\mathbf{q}_2\left(z_\beta^{(1)}\right)+\boldsymbol{\lambda}_2P_t\left(z_\beta^{(1)}\right); \tag{39}$$

$$\mathbf{f}^{(2)}\left(z_*^{(2)}\right) = \mathbf{f}_\infty^{(2)}\left(z_*^{(2)}\right) + \frac{1}{2\pi i}\left[\mathbf{q}_2\left(z_*^{(2)}\right) - \sum_{\beta=1}^{4}\mathbf{I}_\beta\left(\mathbf{A}_2^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(2)}\right) + \mathbf{B}_2^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(2)}\right)\right) + \left\langle P_t\left(z_*^{(2)}\right)\right\rangle \boldsymbol{\lambda}_2\right]; \quad (40)$$

$$\mathbf{A}_1^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(2)}\right) + \mathbf{B}_1^{\mathrm{T}}\mathbf{p}\left(z_\beta^{(2)}\right) = -\mathbf{q}_1\left(z_\beta^{(2)}\right) + \boldsymbol{\lambda}_1 P_t\left(z_\beta^{(2)}\right); \quad (41)$$

$$\bar{\mathbf{A}}_2^{\mathrm{T}}\mathbf{m}\left(z_\beta^{(2)}\right) + \bar{\mathbf{B}}_2^{\mathrm{T}}p\left(z_\beta^{(2)}\right) = \bar{\mathbf{q}}_2 v\left(z_\beta^{(2)}\right) + \bar{\boldsymbol{\lambda}}_2 P_t\left(z_\beta^{(2)}\right), \quad (42)$$

where $\mathbf{I}_1 = \mathrm{diag}\,[1,0,0,0]$, $\mathbf{I}_2 = \mathrm{diag}\,[0,1,0,0]$, $\mathbf{I}_3 = \mathrm{diag}\,[0,0,1,0]$, and $\mathbf{I}_4 = \mathrm{diag}\,[0,0,0,1]$.

Eqs. (38), (39), (41), and (42) allow to express the improper integrals (33) over the infinite bimaterial interface through the integrals over the finite contours $\Gamma_j$:

$$\begin{aligned}
\mathbf{m}\left(z_\beta^{(1)}\right) &= \left(\bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1} - \mathbf{A}_2\mathbf{B}_2^{-1}\right)^{-\mathrm{T}}\left(\bar{\mathbf{B}}_1^{-\mathrm{T}}\mathbf{y}_1\left(z_\beta^{(1)}\right) - \mathbf{B}_2^{-\mathrm{T}}\mathbf{y}_2\left(z_\beta^{(1)}\right)\right);\\
\mathbf{p}\left(z_\beta^{(1)}\right) &= \left(\bar{\mathbf{B}}_1\bar{\mathbf{A}}_1^{-1} - \mathbf{B}_2\mathbf{A}_2^{-1}\right)^{-\mathrm{T}}\left(\bar{\mathbf{A}}_1^{-\mathrm{T}}\mathbf{y}_1\left(z_\beta^{(1)}\right) - \mathbf{A}_2^{-\mathrm{T}}\mathbf{y}_2\left(z_\beta^{(1)}\right)\right);\\
\mathbf{y}_1\left(z_\beta^{(1)}\right) &= -\bar{\mathbf{q}}_1\left(z_\beta^{(1)}\right) + \bar{\boldsymbol{\lambda}}_1 P_t\left(z_\beta^{(1)}\right);\\
\mathbf{y}_2\left(z_\beta^{(1)}\right) &= \mathbf{q}_2\left(z_\beta^{(1)}\right) + \boldsymbol{\lambda}_2 P_t\left(z_\beta^{(1)}\right);
\end{aligned} \quad (43)$$

$$\begin{aligned}
\mathbf{m}\left(z_\beta^{(2)}\right) &= \left(\bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1} - \mathbf{A}_1\mathbf{B}_1^{-1}\right)^{-\mathrm{T}}\left(\bar{\mathbf{B}}_2^{-\mathrm{T}}\mathbf{y}_3\left(z_\beta^{(2)}\right) - \mathbf{B}_1^{-\mathrm{T}}\mathbf{y}_4\left(z_\beta^{(2)}\right)\right);\\
\mathbf{p}\left(z_\beta^{(2)}\right) &= \left(\bar{\mathbf{B}}_2\bar{\mathbf{A}}_2^{-1} - \mathbf{B}_1\mathbf{A}_1^{-1}\right)^{-\mathrm{T}}\left(\bar{\mathbf{A}}_2^{-\mathrm{T}}\mathbf{y}_3\left(z_\beta^{(2)}\right) - \mathbf{A}_1^{-\mathrm{T}}\mathbf{y}_4\left(z_\beta^{(2)}\right)\right);\\
\mathbf{y}_3\left(z_\beta^{(2)}\right) &= \bar{\mathbf{q}}_2\left(z_\beta^{(2)}\right) + \bar{\boldsymbol{\lambda}}_2 P_t\left(z_\beta^{(2)}\right);\\
\mathbf{y}_4\left(z_\beta^{(2)}\right) &= -\mathbf{q}_1\left(z_\beta^{(2)}\right) + \boldsymbol{\lambda}_1 P_t\left(z_\beta^{(2)}\right).
\end{aligned} \quad (44)$$

Substituting Eqs. (43), (44) into Eqs. (37) and (40) one can obtain the integral formulae for the Stroh complex functions for a bimaterial, which do not contain improper integrals over the infinite path (bimaterial interface):

$$\begin{aligned}
\mathbf{f}^{(1)}\left(z_*^{(1)}\right) = \mathbf{f}_\infty^{(1)}\left(z_*^{(1)}\right) + \frac{1}{2\pi i}\Bigg[&\mathbf{q}_1\left(z_*^{(1)}\right) + \sum_{\beta=1}^{4}\mathbf{I}_\beta\left(\mathbf{G}_1^{(1)}\bar{\mathbf{q}}_1\left(z_\beta^{(1)}\right) + \mathbf{G}_2^{(1)}\mathbf{q}_2\left(z_\beta^{(1)}\right)\right) + \\
&+ \left\langle P_t\left(z_*^{(1)}\right)\right\rangle\left(\mathbf{G}_2^{(1)}\boldsymbol{\lambda}_2 - \mathbf{G}_1^{(1)}\bar{\boldsymbol{\lambda}}_1 - \boldsymbol{\lambda}_1\right)\Bigg] \quad \forall\, \mathrm{Im}\, z_*^{(1)} > 0;
\end{aligned} \quad (45)$$

$$\begin{aligned}
\mathbf{f}^{(2)}\left(z_*^{(2)}\right) = \mathbf{f}_\infty^{(2)}\left(z_*^{(2)}\right) + \frac{1}{2\pi i}\Bigg[&\mathbf{q}_2\left(z_*^{(2)}\right) - \sum_{\beta=1}^{4}\mathbf{I}_\beta\left(\mathbf{G}_1^{(2)}\mathbf{q}_1\left(z_\beta^{(2)}\right) + \mathbf{G}_2^{(2)}\bar{\mathbf{q}}_2\left(z_\beta^{(2)}\right)\right) + \\
&+ \left\langle P_t\left(z_*^{(2)}\right)\right\rangle\left(\mathbf{G}_1^{(2)}\boldsymbol{\lambda}_1 - \mathbf{G}_2^{(2)}\bar{\boldsymbol{\lambda}}_2 + \boldsymbol{\lambda}_1\right)\Bigg] \quad \forall\, \mathrm{Im}\, z_*^{(2)} > 0,
\end{aligned} \quad (46)$$

where

$$\begin{aligned}
\mathbf{G}_1^{(1)} &= -\left[\mathbf{A}_1^{\mathrm{T}}\left(\bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1} - \mathbf{A}_2\mathbf{B}_2^{-1}\right)^{-\mathrm{T}}\bar{\mathbf{B}}_1^{-\mathrm{T}} + \mathbf{B}_1^{\mathrm{T}}\left(\bar{\mathbf{B}}_1\bar{\mathbf{A}}_1^{-1} - \mathbf{B}_2\mathbf{A}_2^{-1}\right)^{-\mathrm{T}}\bar{\mathbf{A}}_1^{-\mathrm{T}}\right];\\
\mathbf{G}_2^{(1)} &= -\left[\mathbf{A}_1^{\mathrm{T}}\left(\bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1} - \mathbf{A}_2\mathbf{B}_2^{-1}\right)^{-\mathrm{T}}\mathbf{B}_2^{-\mathrm{T}} + \mathbf{B}_1^{\mathrm{T}}\left(\bar{\mathbf{B}}_1\bar{\mathbf{A}}_1^{-1} - \mathbf{B}_2\mathbf{A}_2^{-1}\right)^{-\mathrm{T}}\mathbf{A}_2^{-\mathrm{T}}\right];\\
\mathbf{G}_1^{(2)} &= -\left[\mathbf{A}_2^{\mathrm{T}}\left(\mathbf{A}_1\mathbf{B}_1^{-1} - \bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1}\right)^{-\mathrm{T}}\mathbf{B}_1^{-\mathrm{T}} + \mathbf{B}_2^{\mathrm{T}}\left(\mathbf{B}_1\mathbf{A}_1^{-1} - \bar{\mathbf{B}}_2\bar{\mathbf{A}}_2^{-1}\right)^{-\mathrm{T}}\mathbf{A}_1^{-\mathrm{T}}\right];\\
\mathbf{G}_2^{(2)} &= -\left[\mathbf{A}_2^{\mathrm{T}}\left(\mathbf{A}_1\mathbf{B}_1^{-1} - \bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1}\right)^{-\mathrm{T}}\bar{\mathbf{B}}_2^{-\mathrm{T}} + \mathbf{B}_2^{\mathrm{T}}\left(\mathbf{B}_1\mathbf{A}_1^{-1} - \bar{\mathbf{B}}_2\bar{\mathbf{A}}_2^{-1}\right)^{-\mathrm{T}}\bar{\mathbf{A}}_2^{-\mathrm{T}}\right].
\end{aligned} \quad (47)$$

According to [7], based on Eqs. (4), (5), (6) the Cauchy integrals (36) can be expressed through the physical boundary functions as

$$\begin{aligned}
\mathbf{q}_j\big(z_\beta^{(i)}\big) &= \int\limits_{\Gamma_j} \left\langle \ln\big(\tau_*^{(j)}(s) - z_\beta^{(i)}\big)\right\rangle \mathbf{A}_j^{\mathrm{T}}\tilde{\mathbf{t}}(s)ds - \int\limits_{\Gamma_j} \left\langle \frac{n_2(s) - p_*^{(j)} n_1(s)}{\tau_*^{(j)}(s) - z_\beta^{(i)}}\right\rangle \mathbf{B}_j^{\mathrm{T}}\tilde{\mathbf{u}}(s)ds - \\
&- \int\limits_{\Gamma_j} \left\langle \ln\big(\tau_*^{(j)}(s) - z_\beta^{(i)}\big)\right\rangle \big(\boldsymbol{\lambda}_j n_2(s) - \boldsymbol{\rho}_j n_1(s)\big)\theta(s)ds - \int\limits_{\Gamma_j} \left\langle \mathbf{f}^*\big(\tau_*^{(j)}(s) - z_\beta^{(i)}\big)\right\rangle \boldsymbol{\mu}_j h_n(s)ds; \\
\bar{\mathbf{q}}_j\big(z_\beta^{(i)}\big) &= \int\limits_{\Gamma_j} \left\langle \ln\big(\bar{\tau}_*^{(j)}(s) - z_\beta^{(i)}\big)\right\rangle \bar{\mathbf{A}}_j^{\mathrm{T}}\tilde{\mathbf{t}}(s)ds - \int\limits_{\Gamma_j} \left\langle \frac{n_2(s) - \bar{p}_*^{(j)} n_1(s)}{\bar{\tau}_*^{(j)}(s) - z_\beta^{(i)}}\right\rangle \bar{\mathbf{B}}_j^{\mathrm{T}}\tilde{\mathbf{u}}(s)ds - \\
&- \int\limits_{\Gamma_j} \left\langle \ln\big(\bar{\tau}_*^{(j)}(s) - z_\beta^{(i)}\big)\right\rangle \big(\bar{\boldsymbol{\lambda}}_j n_2(s) - \bar{\boldsymbol{\rho}}_j n_1(s)\big)\theta(s)ds - \int\limits_{\Gamma_j} \left\langle f^*\big(\bar{\tau}_*^{(j)}(s) - z_\beta^{(i)}\big)\right\rangle \bar{\boldsymbol{\mu}}_j h_n(s)ds,
\end{aligned} \tag{48}$$

where $\tilde{t}_i = \tilde{\sigma}_{ij} n_j$ is the extended traction vector, and

$$\boldsymbol{\mu}_j = \frac{1}{k_t^{(j)}}\left(\mathbf{A}_j^{\mathrm{T}}\,\mathrm{Im}\,[\mathbf{d}_j] + \mathbf{B}_j^{\mathrm{T}}\,\mathrm{Im}\,[\mathbf{c}_j]\right), \quad \boldsymbol{\rho}_j = \mathbf{A}_j^{\mathrm{T}}\,\mathrm{Re}\left[p_t^{(j)}\mathbf{d}_j\right] + \mathbf{B}_j^{\mathrm{T}}\,\mathrm{Re}\left[p_t^{(j)}\mathbf{c}_j\right]. \tag{49}$$

According to Eqs. (25) and (48), expressions (45) and (46) for the Stroh complex functions are the integral formulae relating the latter at the internal points of the thermoelectroelastic bimaterial with the boundary values of the temperature $\theta$, the heat flux $h_n$, the extended displacement vector $\tilde{\mathbf{u}}$ and the traction vector $\tilde{\mathbf{t}}$ at the contours $\Gamma_j$. Consequently, Eqs. (4), (45), (46), and (48) allow to derive the Somigliana type integral formula for a thermoelectroelastic bimaterial

$$\tilde{\mathbf{u}}(\boldsymbol{\xi}) = \begin{cases} 2\,\mathrm{Re}\left[\mathbf{A}_1\mathbf{f}^{(1)}\big(Z_*^{(1)}(\boldsymbol{\xi})\big) + \mathbf{c}_1 g_1\big(Z_t^{(1)}(\boldsymbol{\xi})\big)\right] & (\forall\boldsymbol{\xi} \in S_1), \\ 2\,\mathrm{Re}\left[\mathbf{A}_2\mathbf{f}^{(2)}\big(Z_*^{(2)}(\boldsymbol{\xi})\big) + \mathbf{c}_2 g_2\big(Z_t^{(2)}(\boldsymbol{\xi})\big)\right] & (\forall\boldsymbol{\xi} \in S_2). \end{cases} \tag{50}$$

Based on Eqs. (4), (45), (46), and (48) one can also derive the integral formula for the extended stress tensor at the arbitrary internal point of a thermoelectroelastic bimaterial:

$$\tilde{\boldsymbol{\sigma}}_j(\boldsymbol{\xi}) = \begin{cases} 2\,\mathrm{Re}\left[\mathbf{B}_1\left\langle \delta_{2j} - \delta_{1j}p_*^{(1)}\right\rangle \mathbf{f}'\big(Z_*^{(1)}(\boldsymbol{\xi})\big) + \mathbf{d}_1\big(\delta_{2j} - \delta_{1j}p_t^{(1)}\big)g_1'\big(Z_t^{(1)}(\boldsymbol{\xi})\big)\right] & (\boldsymbol{\xi} \in S_1), \\ 2\,\mathrm{Re}\left[\mathbf{B}_2\left\langle \delta_{2j} - \delta_{1j}p_*^{(2)}\right\rangle \mathbf{f}'\big(Z_*^{(2)}(\boldsymbol{\xi})\big) + \mathbf{d}_2\big(\delta_{2j} - \delta_{1j}p_t^{(2)}\big)g_2'\big(Z_t^{(2)}(\boldsymbol{\xi})\big)\right] & (\boldsymbol{\xi} \in S_2). \end{cases} \tag{51}$$

By means of the Sokhotskii-Plemelj formula based on Eqs. (50), (51) one can easily obtain boundary integral equations for displacement and stress fields in a bimaterial with thermally insulated interface.

## 4. Conclusions

The extended Stroh formalism combined with the complex variable approach produce a powerful tool for solution of 2D problems of thermoelectroelasticity. For instance, the application of the Cauchy integral formula allows obtaining boundary integral equations for bimaterial solids, which identically account for the given boundary conditions at the material interface. The integral formulae for the Stroh complex functions, and, consequently, the integral representations of the temperature, extended stress and displacement fields, are obtained in a straightforward manner without any preliminary assumptions. The kernels of the integral equations can be derived in the closed form. Besides, obtained integral formulae do not contain domain integrals, which are the main advances comparing to existing ones, which are derived by means of the reciprocity techniques.


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## Двовимірні інтегральні формули та рівняння для термоелектропружного біматеріалу із теплоізольованою межею поділу


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У роботі на основі методів теорії функції комплексної змінної запропоновано прямий підхід до побудови інтегральних формул та рівнянь плоскої термоелектропружності для анізотропного біматеріалу з теплоізольованою межею поділу. Отримані співвідношення не містять інтегралів по області та залежать лише від фізично значимих крайових функцій, таких як температура, тепловий потік, розширені переміщення та напруження, що є основними перевагами цих співвідношень.



**Ключові слова:** *термоелектропружний, анізотропний, біматеріал, інтегральні формули*

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