MATHEMATICAL MODELING AND COMPUTING, Vol. 1, No. 2, pp. 214–223 (2014)

Modeling local non-homogeneity in electroconductive non-ferromagnetic thermoelastic solid

Nahirnyj T. S.[1,2], Senyk Y. A.[2], Tchervinka K. A.[3]

[1] *Faculty of Mechanical Engineering, University of Zielona Góra*
4 Prof. Szafran str., 65-516 Zielona Góra, Poland
[2] *Centre of Mathematical Modeling of IAPMM named after Ya.S.Pidstryhach*
15 D. Dudaev str., 79005, Lviv, Ukraine
[3] *Ivan Franko National University of Lviv*
1 Universytetska str., 79000 Lviv, Ukraine

(Received 30 November 2014)

We consider the key systems describing steady state of a locally inhomogeneous electroconductive non-ferromagnetic solid within framework of the local gradient approach in thermomechanics. An arbitrarily chosen subdomain of the solid is regarded as a thermodynamically open system that can exchange by mass with environment. It is assumed that this exchange occurs suddenly at the initial time when the body structure is instantly set. The mass sources are introduced into the model to conform the actual and reference body states. The sources are associated with method of body surface forming.

Keywords: *local gradient approach, near-surface non-homogeneity, size effect, electroconductive non-ferromagnetic thermoelastic solid, double electric layer*

2000 MSC: 74A15, 74A50, 74D05, 78A97, 78A30

UDC: 539.3

1. Induction

One of the main objectives of mechanic of solids is developing mathematical models for prediction of deformation, strength and other exploitation parameters of real structural elements and appliances normally found in conditions of intense external action. These mathematical models should sufficiently take into account the properties and structure of the material and the nature of the processes. Taking into account the structural (local) heterogeneity is particularly important due to the wide use of thin films, fibers and other nanoelements in engineering [1]. Such elements own comparable contribution of bulk and surface compounds in the internal energy, which is usually associated with substantial heterogeneity of element structure.

To describe and model surface and size effects proper to nanostructures the surface elasticity [2], strain gradient theory [3], nonlocal theory [4] and others are used. Among the approaches that are focused on the consideration of surface factor one can account for local gradient approach in thermomechanics. The models that are constructed within this approach take into account the heterogeneity (microstructure) of the material and describe various size effects. In [5–9] and others there is presented the basic relations of such models and for the linear approximation there is studied the regularities of nearsurface non-homogeneity in thermoelastic solids, electroconductive non-ferromagnetic solids and solid solutions. In these papers the boundary condition formulation usually uses the constant value of the chemical potential (conjugate parameter to the density) at the body surface. However, this condition imposes substantial restrictions on the use of the approach models and often does not allow adequate size effects describing. In [10] the boundary condition for the density is proposed and actual density deviation from reference value is linked to the roughness of the real surface i.e. its geometric heterogeneity. In [9,11] there is shown the existence of the "nearsurface mass defect" in such models, and the model generalization is presented for its elimination. The mass sources are introduces in consideration corresponding to the method of the surface forming.

© 2014 Lviv Polytechnic National University
CMM IAPMM NASU

In this paper there is derived the key equation system for studying stationary state of electro conductive non-ferromagnetic locally non-homogeneous thermoelastic body taking into account the heterogeneity of the real surface of the body. The multiscale size effects are illustrated using half-space example.

2. Basic relations

We consider a deformable electroconductive non-ferromagnetic solid which can mass-, heat- and charge-exchange with the environment. For basic processes occuring in the body, we accept the deformation process, heat and mass transfer.

The equation of energy E conservation in the local form neglecting convective components of fluxes has the form

$$\frac{\partial E}{\partial \tau} = \boldsymbol{\nabla} \cdot \left(\boldsymbol{\sigma} \cdot \boldsymbol{v} - T\boldsymbol{j}_s - H\boldsymbol{j}_m - \varphi \boldsymbol{j}_\omega - \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} \right) + \sigma_E, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{v}$ is the velocity vector, T, H, φ are the absolute temperature, chemical and thermodynamic electric potentials, $\boldsymbol{j}_s, \boldsymbol{j}_m, \boldsymbol{j}_\omega$ are vectors of entropy, mass and charge fluxes, $\boldsymbol{E}, \boldsymbol{B}$ are vectors of the electric and magnetic fields, σ_E is energy production, τ denotes the time, μ_0 is the magnetic constant.

We assume that the total energy E is represented as the sum of internal energy U, kinetic energy K and energy of electromagnetic field U_e [12]

$$E = U + K + U_e. \tag{2}$$

Along with equation (1) the balance equations for the energy of electromagnetic field

$$U_e = \left(\varepsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2 \right) / 2, \tag{3}$$

momentum of mechanical translational motion $\boldsymbol{k}_v$, entropy S, charge ω and mass, as well as Maxwell's equations must hold [12,13]. In the local form they are

$$\begin{gathered}
\frac{\partial U_e}{\partial \tau} = -\frac{1}{\mu_0} \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) - (\boldsymbol{j}_\omega + \omega \boldsymbol{v}) \cdot \boldsymbol{E}, \\
\frac{\partial \boldsymbol{k}_v}{\partial \tau} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{F}_e, \qquad \frac{\partial S}{\partial \tau} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_s = \sigma_s, \\
\frac{\partial \omega}{\partial \tau} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_\omega = 0, \qquad \frac{\partial \rho}{\partial \tau} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_m = \sigma_m,
\end{gathered} \tag{4}$$

$$\begin{gathered}
\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \tau}, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial \tau} + \mu_0 \left(\boldsymbol{j}_\omega + \omega \boldsymbol{v} \right), \\
\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \omega.
\end{gathered} \tag{5}$$

Here $\boldsymbol{F}_e$ is the ponderomotive force, for which there is held the relation [12]

$$\boldsymbol{F}_e = (\boldsymbol{j}_\omega + \omega \boldsymbol{v}) \times \boldsymbol{B} + \omega \boldsymbol{E}, \tag{6}$$

ρ is the density, $\sigma_m = \sigma_E / H$ are sources of mass, $\boldsymbol{H} = \boldsymbol{B}/\mu_0$ is vector of magnetic H-field.

We note also that the equations for the electromagnetic field are written in the approximation of slowly moving media when the speed $\boldsymbol{v}$ is much less than the speed c of electromagnetic waves in vacuum.

Taking into account relations (3)–(6) and the expression of kinetic energy increment $dK = \boldsymbol{v} \cdot d\boldsymbol{k}_v$ from the total energy balance equation (1), we obtain

$$\frac{\partial U}{\partial \tau} = T\frac{\partial S}{\partial \tau} + H\frac{\partial \rho}{\partial \tau} + \varphi\frac{\partial \omega}{\partial \tau} + \boldsymbol{\sigma} : \frac{\partial \mathbf{e}}{\partial \tau} - T\sigma_s - \boldsymbol{\nabla} T \cdot \boldsymbol{j}_s - \boldsymbol{\nabla} H \cdot \boldsymbol{j}_m - \left(\boldsymbol{\nabla}\varphi - \boldsymbol{E}'\right) \cdot \boldsymbol{j}_\omega, \tag{7}$$

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}. \tag{8}$$

We assume for mass flux such presentation

$$\boldsymbol{j}_m = -g_{mm}\frac{\partial\left(\boldsymbol{\nabla} H\right)}{\partial \tau}, \tag{9}$$

that corresponds to the sudden appearance of the structure of the material at initial time [5,8].

Substituting this relation into (7) we obtain

$$\frac{\partial G}{\partial \tau} = T\frac{\partial S}{\partial \tau} + H\frac{\partial \rho}{\partial \tau} + \varphi\frac{\partial \omega}{\partial \tau} + \boldsymbol{\sigma} : \frac{\partial \mathbf{e}}{\partial \tau} - T\sigma_s - \boldsymbol{\nabla} T \cdot \boldsymbol{j}_s - \left(\boldsymbol{\nabla}\varphi - \boldsymbol{E}'\right) \cdot \boldsymbol{j}_\omega, \tag{10}$$

where

$$G = U - \frac{g_{mm}}{2}\boldsymbol{\nabla} H \cdot \boldsymbol{\nabla} H.$$

Using the classical expression of entropy production for electroconductive body [13]

$$\sigma_s = -\frac{\boldsymbol{\nabla} T}{T} \cdot \boldsymbol{j}_s - \frac{\boldsymbol{\nabla}\varphi - \boldsymbol{E}'}{T} \cdot \boldsymbol{j}_\omega \tag{11}$$

on the base of Onsager principle the kinetic equations of the model are

$$\boldsymbol{j}_s = -\lambda_{ss}\frac{\boldsymbol{\nabla} T}{T} - \lambda_{s\omega}\frac{\boldsymbol{\nabla}\varphi - \boldsymbol{E}'}{T}, \quad \boldsymbol{j}_\omega = -\lambda_{\omega s}\frac{\boldsymbol{\nabla} T}{T} - \lambda_{\omega\omega}\frac{\boldsymbol{\nabla}\varphi - \boldsymbol{E}'}{T}. \tag{12}$$

Analyzing the relation (9), (10) we see that energy G is defined in the space of entropy, density, electric charge and strain tensor

$$G = G\left(S, \rho, \omega, \mathbf{e}\right)$$

and for the energy increase the Gibbs equation is held

$$dG = T\, dS + H\ d\rho + \varphi\ d\omega + \boldsymbol{\sigma} : d\mathbf{e}.$$

Introducing energy F with Legendre transformation

$$F = G - TS - \varphi\,\omega$$

for this energy increase we obtain

$$dF = -SdT + Hd\rho - \omega d\varphi + \boldsymbol{\sigma} : d\mathbf{e}, \tag{13}$$

therefore the new thermodynamic potential is defined in the space of temperature T, density ρ, thermodynamic electric potential φ and strain tensor $\mathbf{e}$

$$F = F(T, \rho, \varphi, \mathbf{e}).$$

For entropy S, chemical potential H, electric potential φ and stress tensor $\boldsymbol{\sigma}$ the following state equations must hold

$$S = -\frac{\partial F}{\partial T}, \quad H = \frac{\partial F}{\partial \rho}, \quad \omega = -\frac{\partial F}{\partial \varphi}, \quad \boldsymbol{\sigma} = \frac{\partial F}{\partial \mathbf{e}}. \tag{14}$$

The further specification of the model demands the state equations and reference state concretization. We accept as reference state the state of uniform isotropic body free of charge and force load with parameters

$$T = T_*, \quad S = S_*, \quad \rho = \rho_*, \quad \omega = 0, \quad \mathbf{e} = 0, \quad \boldsymbol{\sigma} = 0.$$

Further we take energy F as a quadratic function in the space of basic parameter disturbances from the reference state

$$\theta = T - T_*, \quad \rho - \rho_*, \quad \mathbf{e} = \mathbf{e} - 0, \quad \varphi = \varphi - 0, \tag{15}$$

in the form

$$\begin{aligned} F = F_* - S_*\theta + H_*(\rho - \rho_*) + \frac{1}{2}\lambda e^2 + \mu\mathbf{e} : \mathbf{e} - (3\lambda + 2\mu)\, a_t e\theta - (3\lambda + 2\mu)\, a_m e\,(\rho - \rho_*) - \\ - (3\lambda + 2\mu)\, a_\omega \varphi e + \frac{1}{2}\alpha_{mm}(\rho - \rho_*)^2 + \alpha_{mt}(\rho - \rho_*)\,\theta + \alpha_{m\omega}(\rho - \rho_*)\,\varphi + \\ + \frac{c_v}{2T_*}\theta^2 + \alpha_{t\omega}\theta\varphi + \frac{1}{2}\alpha_{\omega\omega}\varphi^2, \end{aligned} \tag{16}$$

where $\lambda, \mu, a_i, \alpha_{ij}$ are the characteristics of material ($i, j = \{t, m, \omega\}$); $e = \mathbf{e} : \mathbf{I}$, $\mathbf{I}$ is identity tensor.

Note that such energy F presentation is justified for relatively small deflection of parameters, including density, relatively reference state, i.e. $|\rho - \rho_*|/\rho_* \ll 1$ must hold. If density dependence is essential that is typical in the case of structured materials especially nanomaterials of high porosity then the representation (16) would need to take into account higher orders of density disturbance. This can also be achieved by postulating depending on the density of the coefficients in this representation.

Formulated above system of equations consisting of the balance equations of momentum, entropy, mass, charge, the expression for the entropy production, Maxwell's equations, state and kinetic equations, along with the formula for the mechanical momentum of translational motion

$$\boldsymbol{k}_v = \rho \boldsymbol{v}$$

and the Cauchy relation for strain tensor

$$\mathbf{e} = \frac{1}{2}\left[\boldsymbol{\nabla} \otimes \boldsymbol{u} + (\boldsymbol{\nabla} \otimes \boldsymbol{u})^T\right]$$

constitute complete system of equations of the model of local non-homogeneous electroconductive non-ferromagnetic thermoelastic body. Here $\boldsymbol{u}$ is the displacement vector, $\otimes$ denotes tensor dyadic product, superscript "T" denotes transposition.

Presentation (9) reduces the mass balance equation to the form

$$-g_{mm}\nabla^2 H + \rho - \rho_* = d_{\sigma m}, \tag{17}$$

where

$$d_{\sigma m} = \int_{-\infty}^{\tau} \sigma_m \, d\tau.$$

Note that the mass source is introduced in order to correctly describe the heterogeneity of nearsurface density. They must satisfy relation

$$\int\limits_{(V)} (\rho - \rho_*)\, dV = \int\limits_{(V)} d_{\sigma m}\, dV, \tag{18}$$

where (V) is the region of space occupied by the considered body. The parameters of mass source distribution are to be proposed with account for the method of forming the body surface.

3. The solid with microstructure. Equilibrium state

We now consider the model approximation where the disturbance of density from reference value is relatively small, i.e. $|\rho - \rho_*|/\rho_* \ll 1$, that corresponds to accounting for the microstructure of the material. In this case we assume that the coefficients in the presentation (16) are constant. On the basis of (14) and (16) write the explicit form of the equations of state

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + \{\lambda e - (3\lambda + 2\mu)\left[a_m(\rho - \rho_*) + a_t\theta + a_\omega\varphi\right]\}\ \mathbf{I},$$

$$S = S_* + (3\lambda + 2\mu)a_t e - \alpha_{mt}(\rho - \rho_*) - \alpha_{t\omega}\varphi + \frac{c_v}{T_*}\theta,$$

$$H = H_* - (3\lambda + 2\mu)\, a_m e + \alpha_{mm}(\rho - \rho_*) + \alpha_{mt}\theta + \alpha_{m\omega}\varphi,$$

$$\omega = (3\lambda + 2\mu)\, a_\omega e - \alpha_{t\omega}\theta - \alpha_{m\omega}(\rho - \rho_*) - \alpha_{\omega\omega}\varphi. \tag{19}$$

Taking as the key functions the displacement vector $\boldsymbol{u}$, thermodynamic electric potential φ, the disturbances of temperature θ and density$\rho - \rho_*$, we get a linear system of equations to describe the stationary state of the solid

$$\mu\nabla^2\boldsymbol{u} + (\lambda + \mu)\,\boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - (3\lambda + 2\mu)\left[a_m\boldsymbol{\nabla}\,(\rho - \rho_*) + a_t\boldsymbol{\nabla}\theta + a_\omega\boldsymbol{\nabla}\varphi\right] = 0,$$

$$\lambda_{ss}\nabla^2\theta + \lambda_{s\omega}\nabla^2\varphi - \frac{\lambda_{s\omega}}{\varepsilon_0}\left[(3\lambda + 2\mu)\, a_\omega\boldsymbol{\nabla}\cdot\boldsymbol{u} - \alpha_{t\omega}\theta - \alpha_{m\omega}(\rho - \rho_*) - \alpha_{\omega\omega}\varphi\right] = 0,$$

$$\alpha_{mm}\nabla^2(\rho - \rho_*) + \alpha_{mt}\nabla^2\theta + \alpha_{m\omega}\nabla^2\varphi - (3\lambda + 2\mu)a_m\nabla^2\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \frac{1}{g_{mm}}\,(\rho - \rho_*) = -\frac{1}{g_{mm}}d_{\sigma m},$$

$$\lambda_{\omega s}\nabla^2\theta + \lambda_{\omega\omega}\nabla^2\varphi - \frac{\lambda_{\omega\omega}}{\varepsilon_0}\left[(3\lambda + 2\mu)\, a_\omega\boldsymbol{\nabla}\cdot\boldsymbol{u} - \alpha_{t\omega}\theta - \alpha_{m\omega}(\rho - \rho_*) - \alpha_{\omega\omega}\varphi\right] = 0. \tag{20}$$

This system of equations along with the equation (18) has to be complemented with condition of electro-neutrality of the body

$$\int\limits_{(V)} \omega\, dV = 0. \tag{21}$$

Note that last three equations of (20) depend on first invariant of strain tensor $e = \mathbf{e} : \mathbf{I} = \boldsymbol{\nabla}\cdot\boldsymbol{u}$ and mechanical component of solution may be removed from these equations resulting in coefficient correction. In the case of third equation it does not cause changes in equation order. Acting by operator "div" on the first equation of (20), we obtain

$$(\lambda + 2\mu)\,\nabla^2\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - (3\lambda + 2\mu)\left[a_m\nabla^2(\rho - \rho_*) + a_t\nabla^2\theta + a_\omega\nabla^2\varphi\right] = 0.$$

Using this relation, the third equation of (20) is converted to the form

$$\left(\alpha_{mm} - \frac{(3\lambda + 2\mu)^2\, a_m^2}{\lambda + 2\mu}\right)\nabla^2(\rho - \rho_*) + \left(\alpha_{mt} - \frac{(3\lambda + 2\mu)^2\, a_m a_t}{\lambda + 2\mu}\right)\nabla^2\theta +$$
$$+ \left(\alpha_{m\omega} - \frac{(3\lambda + 2\mu)^2\, a_m a_\omega}{\lambda + 2\mu}\right)\nabla^2\varphi - \frac{1}{g_{mm}}\,(\rho - \rho_*) = -\frac{1}{g_{mm}}d_{\sigma m}, \tag{22}$$

that is written for density, temperature, electric potential and contains no strain component.

If we neglect the influence of temperature on the thermodynamic electric potential and mechanical fields, the system of equations for studying the steady state of local non-homogeneous non-ferromagnetic electroconductive elastic body has the form

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\,\boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - (3\lambda+2\mu)\,[a_m\boldsymbol{\nabla}\,(\rho-\rho_*) + a_\omega\boldsymbol{\nabla}\varphi] = 0,$$

$$\alpha_{mm}\nabla^2(\rho\,-\,\rho_*) + \alpha_{m\omega}\nabla^2\varphi - (3\lambda+2\mu)a_m\nabla^2\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \frac{1}{g_{mm}}\,(\rho-\rho_*) = -\frac{1}{g_{mm}}d_{\sigma m},$$

$$\nabla^2\varphi - \frac{1}{\varepsilon_0}\,[(3\lambda+2\mu)\,a_\omega\boldsymbol{\nabla}\cdot\boldsymbol{u} - \alpha_{m\omega}(\rho\,-\,\rho_*) - \alpha_{\omega\omega}\varphi] = 0 \tag{23}$$

along with integral conditions (18), (21).

If one of the solving functions is stress tensor $\boldsymbol{\sigma}$, then the first equation of (23) is to be replaced with the following equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0,$$

$$\boldsymbol{\nabla}\times\left\{\frac{1}{2\mu}\boldsymbol{\sigma} - \left[\frac{\lambda}{2\mu\,(3\lambda+2\mu)}\sigma - a_m\,(\rho-\rho_*) - a_\omega\varphi\right]\mathbf{I}\right\}\times\boldsymbol{\nabla} = 0, \tag{24}$$

that stand for equilibrium equation (Cauchy's first law of motion) and compatibility condition respectively.

The second equation of (23) in the form that does not contain either displacement, strain or stress components is

$$\left(\alpha_{mm} - \frac{(3\lambda+2\mu)^2a_m^2}{\lambda+2\mu}\right)\nabla^2(\rho\,-\,\rho_*) + \left(\alpha_{m\omega} - \frac{(3\lambda+2\mu)^2a_ma_\omega}{\lambda+2\mu}\right)\nabla^2\varphi -$$
$$- \frac{1}{g_{mm}}\,(\rho-\rho_*) = -\frac{1}{g_{mm}}d_{\sigma m}. \tag{25}$$

If the thermodynamic electric potential influence on density is neglected, then this equation may be written in the form

$$\nabla^2(\rho\,-\,\rho_*) - \xi^2\,(\rho-\rho_*) = -\xi^2 d_{\sigma m}. \tag{26}$$

If the stresses are used as key functions then the last equation of (23) has the form

$$\nabla^2\varphi + \frac{1}{\varepsilon_0}\left\{\left[\alpha_{\omega\omega} - 3\,(3\lambda+2\mu)\,a_\omega^2\right]\varphi + [\alpha_{m\omega} - 3\,(3\lambda+2\mu)\,a_\omega a_m]\,(\rho-\rho_*) - a_\omega\sigma\right\} = 0. \tag{27}$$

If the deformation influence on thermodynamic electric potential is neglected, then this equation may be written in the form

$$\nabla^2\varphi - \chi^2\varphi = -\frac{\alpha_{m\omega}}{\varepsilon_0}(\rho-\rho_*). \tag{28}$$

Thus besides the system (23) as a key equation system the relations (24), (25), (27) or (24), (26), (28) can be conveniently used. Still the conditions (18), (21) are to be assured.

Analyzing the system of equations (23), (24) and further ones we see that when neglecting the influence of strain and stress on the thermodynamic electric potential the study of equilibrium state of non-homogeneous non-ferromagnetic electroconductive solid is reduced to a sequential finding of density and thermodynamic electric potential through the interconnected system of equations and following study of stress-strain state. If additionally we neglect the influence of the thermodynamic electric potential on the density, the study of equilibrium state of the body is reduced to a sequential finding of the density and the thermodynamic electric potential on the basis of inhomogeneous Helmholtz equations with the following examination of the stress-strain state.

4. Non-ferromagnetic electroconductive elastic half-space

We consider an isotropic non-ferromagnetic electroconductive elastic half-space that in Cartesian coordinate system $\{x, y, z\}$ occupies domain $x > 0$. Let the surface $x = 0$ be free of mechanical load and the constant nonzero values of density disturbance $\rho_a - \rho_*$ and thermodynamic electric potential φ_a are specified at it. At infinity $x \to +\infty$ the reference state is realized.

The key system of equations in the considered one-dimensional case takes the form

$$\frac{d\sigma_{xx}}{dx} = 0, \quad \frac{d^2\sigma_{yy}}{dx^2} = \frac{d^2\sigma_{zz}}{dx^2} = a_0 \frac{d^2}{dx^2}\left(\rho - \rho_* + a_0^{\omega}\varphi\right),$$

$$\frac{d^2(\rho - \rho_*)}{dx^2} - \xi^2\left(\rho - \rho_*\right) = -\xi^2 d_{\sigma m},$$

$$\frac{d^2\varphi}{dx^2} - \chi^2\varphi = -\frac{\alpha_{m\omega}}{\varepsilon_0}(\rho - \rho_*). \tag{29}$$

The boundary conditions are

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}|_{x=0} = \mathbf{0}, \quad \rho|_{x=0} = \rho_a, \qquad \varphi|_{x=0} = \varphi_a,$$

$$\boldsymbol{\sigma} \underset{x\to+\infty}{\longrightarrow} \mathbf{0}, \quad \rho \underset{x\to+\infty}{\longrightarrow} \rho_*, \quad \varphi \underset{x\to+\infty}{\longrightarrow} 0, \tag{30}$$

and the integral conditions are

$$\int_0^{+\infty} (\rho - \rho_*)\, dx = \int_0^{+\infty} d_{\sigma m} dx,$$

$$\int_0^{+\infty} \omega dx \equiv \int_0^{+\infty} \left\{a_\omega\sigma - \left[\alpha_{m\omega} - (3\lambda + 2\mu)a_m a_\omega\right](\rho - \rho_*) - \left[\alpha_{\omega\omega} - (3\lambda + 2\mu)a_\omega^2\right]\varphi\right\} dx = 0. \tag{31}$$

The mass source distribution $d_{\sigma m}$ we model with density distribution that sets up in the body where no mass sources are specifies, i.e. with solution of homogeneous second equation of (29). We assume also that characteristic size of mass source distribution may be different from the characteristic size of structural non-homogeneity ξ^{-1} and we denote it with ζ^{-1}. So we take

$$d_{\sigma m} = -\xi^{-2} m_s(\rho_a - \rho_*)\exp(-\zeta x), \tag{32}$$

and second equation of (29) has the form

$$\frac{d^2(\rho - \rho_*)}{dx^2} - \xi^2\left(\rho - \rho_*\right) = m_s(\rho_a - \rho_*)\exp(-\zeta x).$$

The solution of this equation that satisfies conditions (30), (31) is

$$\rho = \rho_* + (\rho_a - \rho_*)\frac{\zeta\exp(-\xi x) - \xi\exp(-\zeta x)}{\zeta - \xi},$$

and $m_s = -\xi(\zeta + \xi)$.

The solution of the last equation in (29) that satisfies (30) and (21) is

$$\varphi = \frac{\rho_a - \rho_*}{\zeta - \xi}\frac{\alpha_{m\omega}}{\chi\varepsilon_0}\left(\frac{\zeta\xi(\zeta^2 - \xi^2)\exp(-\chi x)}{(\xi^2 - \chi^2)(\zeta^2 - \chi^2)} + \frac{\chi\zeta\exp(-\xi x)}{\chi^2 - \xi^2} + \frac{\xi\chi\exp(-\zeta x)}{\zeta^2 - \chi^2}\right),$$

and charge distribution within the accepted assumptions is

$$\omega = (\rho_a - \rho_*)\alpha_{m\omega}\frac{\zeta\xi}{\zeta-\xi}\left(\frac{\chi(\zeta^2-\xi^2)\exp(-\chi x)}{(\xi^2-\chi^2)(\zeta^2-\chi^2)} + \frac{\xi\exp(-\xi x)}{\chi^2-\xi^2} + \frac{\zeta\exp(-\zeta x)}{\zeta^2-\chi^2}\right).$$

The stresses are

$$\sigma_{xx} = 0, \quad \sigma_{yy} = \sigma_{zz} = a_0\frac{\rho_a-\rho_*}{\zeta-\xi}\left(a\frac{\zeta\xi}{\chi}\frac{(\zeta^2-\xi^2)\exp(-\chi x)}{(\xi^2-\chi^2)(\zeta^2-\chi^2)} + \right.$$
$$\left. + \left(1+\frac{a}{\chi^2-\zeta^2}\right)\zeta\exp(-\xi x) - \left(1+\frac{a}{\chi^2-\xi^2}\right)\xi\exp(-\zeta x)\right).$$

Here $a = a_0^\omega\dfrac{\alpha_{m\omega}}{\varepsilon_0}$.

Results of numerical analysis are shown in Fig. 1, 2. In the first figure the density ρ/ρ_* is plotted for $\zeta/\xi = 0,5; 1,25; 3$ (graphs 1–3 respectively). In the second one can see charge ω/ω_0 ($\omega_0 = \rho_*\alpha_{m\omega}$) distribution. Graph 1 is drawn for $\zeta/\xi = 1.25$, $\chi/\xi = 45$, graph 2 is drawn for $\zeta/\xi = 0.5$, $\chi/\xi = 15$ and graph 3 is drawn for $\zeta/\xi = 1.25$, $\chi/\xi = 15$.

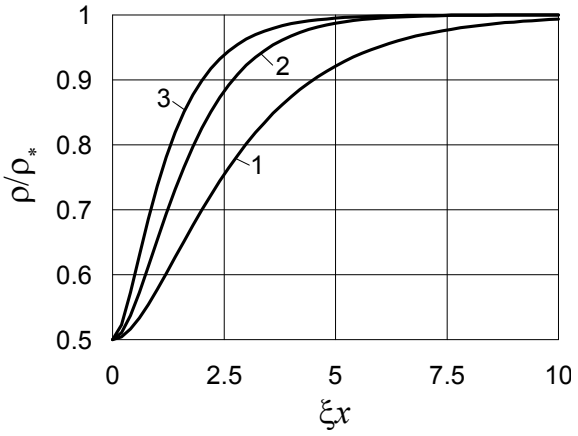

Fig. 1. Density in local non-homogeneous half-space.

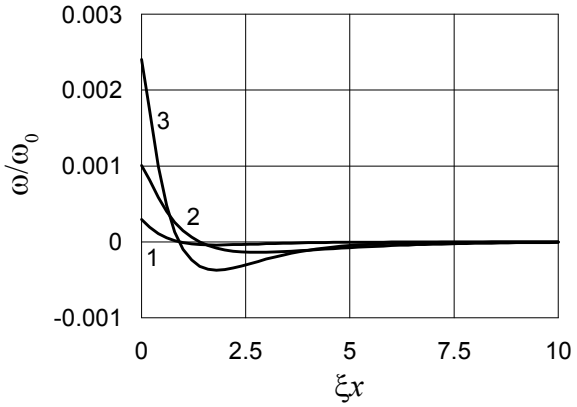

Fig. 2. Charge in local non-homogeneous half-space.

From analysis of these plots one can see that model parameters ξ^{-1}, ζ^{-1}, χ^{-1} are the characteristic sizes of structural, surface and charge-related non-homogeneities. The non-homogeneities are associated with surface and exponentially reduce to zero with withdrawal from the surface. The charge distribution demonstrates the double electric layer that exists near surface of the body. On this base one could argue that for parameter $\alpha_{m\omega}$ the following inequality is held: $\alpha_{m\omega} < 0$.

5. Conclusion

Using the methods of irreversible thermodynamics there are formulated the systems of equations describing the steady state of electroconductive non-ferromagnetic thermoelastic solid. The model takes into account the structure of the material and the heterogeneity of the near-surface density that suddenly arose at the initial time. The emergence of the structure of the material is taken into account by postulating appropriate component of the mass flux vector. The introduced sources of mass allow considering the "near-surface mass defect" that is inherent to the known models being built within the local gradient approach. If the influence of strain and stress on thermodynamical electric potential is neglected then the study of steady state of non-ferromagnetic electroconductive elastic body is reduced to a sequential finding of density and thermodynamic electric potential through interconnected system of equations with the following study of the stress-strain state of the body. The density, charge and stresses distributions in the free half-space are non-homogeneous and feature three characteristic sizes.

[1] Shaofan Li, Xin-Lin Gao. Handbook of Micromechanics and Nanomechanics. CRC Press, 1256 p. (2013).

[2] Gurtin M. E., Murdoch A. I. A continuum theory of elastic material surfaces. Arch. Ration. Mech. Anal. **57**, 291–323 (1975).

[3] Aifantis E. C. Strain gradient interpretation of size effects. Int. J. Fract. **95**, 299–314 (1999).

[4] Eringen A. C. Nonlocal Continuum Field Theories. Springer, New York (2002).

[5] Burak Y. I., Nagirnyi T. S. Mathematical modeling of local gradient processes in inertial thermomechanical systems. Int. Appl.Mech. **28**, 775 (1992).

[6] Burak Y., Nahirnyj T., Tchervinka K. Local Gradient Thermomechanics. In: Richard B. Hetnarski (eds) Encyclopedia of Thermal Stresses, Springer Reference: 2794–2801 (2014).

[7] Nagirnyi T. S., Tchervinka K. A. Interface phenomena and interaction energy at the surface of electroconductive solids. Comput. Meth. Sci. Technol. **14**, 105 (2008).

[8] Nagirnyi T. S., Tchervinka K. A. Thermodynamical models and methods of thermomechanics taking into account near-surface and structural nonhomogeneity. Bases of nanomechanics I. Lviv, SPOLOM, (2012) (in ukrainian).

[9] Nagirnyi T. S., Tchervinka K. A. Basics of mechanics of local non-homogeneous elastic bodies. Bases of nanomechanics II. Lviv, Rastr–7, 167 p. (2014) (in ukrainian).

[10] Nahirnyj T. S., Chervinka K. A., Boiko Z. V. On the choice of boundary conditions in problems of the local gradient approach in thermomechanics. J. Math. Sci. **186**, 130 (2012).

[11] Nahirnyj T. S., Tchervinka K. A. On steady state description for electroconductive non-ferromagnetic local non-homogeneous solid I Int. XX Ukr. Conf. "Modern Problems of Applied mathematics and informatics", Lviv, 7–9 April 2014, 110–111 (2014).

[12] Burak Y. I., Halapats B. P., Gnidets' B. M. Physical and mechanical processes in conductive bodies. Kyiv, Nauk. Dumka, 229 (1978) (in ukrainian).

[13] Glansdorff P., Prigogine I. Thermodynamic Theory of Structure, Stability and Fluctuations. Wiley, New York (1971).

Моделювання локальної неоднорідності у електропровідному неферомагнітному термопружному тілі

Нагірний Т. С.[1,2], Сеник Ю. А.[1], Червінка К. А.[3]

[1] *Центр математичного моделювання ІППММ ім. Я. С. Підстригача*
вул. Дж. Дудаєва, 15, 79005, Львів, Україна
[2] *Механіко-машинобудівний факультет, Університет міста Зелена Гура*
вул. проф. Шафрана 4, 65-516, Зелена Гура, Польща
[3] *Кафедра математичного моделювання,*
Львівський національний університет імені Івана Франка
вул. Університетська, 1, Львів 79000, Україна

Розглянуто ключові системи рівнянь, що описують рівноважний стан локально неоднорідного електропровідного неферомагнітного термопружного тіла у рамках локально градієнтного підходу у термомеханіці. Довільно виділена під область тіла розглядається як термодинамічно відкрита система, що може обмінюватись масою з оточенням. Припускається, що такий обмін відбувається миттєво у початковий момент часу при встановленні структури тіла. У розгляд введено джерела маси, що дозволило узгодити актуальний та відліковий стани у моделі. Джерела маси пов'язано із способом формування поверхні тіла.

Ключові слова: *локально градієнтний підхід, приповерхнева неоднорідність, розмірний ефект, електропровідне неферомагнітне термопружне тіло, подвійний електричний шар*

2000 MSC: 74A15, 74A50, 74D05, 78A97, 78A30

УДК: 539.3