M$\mathbf{M^{odeling}_{athematical}}C\mathbf{^{omputing}}$

On three facts of reticences in the classical mathematical modeling of elastic materials

Rushchitsky J. J.

S.P.Timoshenko Institute of Mechanics
3 Nesterov str., 03680, Kyiv, Ukraine

Three facts of reticences (passing over in silence, an absence of comments) in the procedures of mathematical modeling of elastic materials are described and commented. The first fact consists in a reticence of one of the first steps in the mentioned above procedure — an assumption that the kinematics of deformation is described by the linear approximation of motion of material continuum, namely by gradients of deformation. In the paper, a novel nonlinear approach to this procedure is offered. The second and third facts are associated with constitutive relations. The second fact consists in the absence of necessary comments relative to determination of smallness of strains and gradients of displacements (absence of comments relative to a criterion of applicability of the linear model) because the criterion $|u_{i,k}| \ll 1$ is sufficiently abstract. It is shown that there exists a based on the nonlinear Cauchy relations approximate procedure of determination of threshold values of strains and gradients of deformations starting with which a nonlinearity of process appears. The third fact consists in the absence of comments relative to essential differences between the nonlinear constitutive equations, which are written for the ordered pairs "Lagrange stress tensor – Cauchy-Green strain tensor" and "Kirchhoff stress tensor – gradients of displacements". It is shown on an example of the shear stress and the Murnaghan model of nonlinear elastic deformation that deviation from the corresponding straight lines of linear deformation for different pairs differs in many times in the range of small strains and small gradients of displacements. The general estimate of facts of reticences looks positive, because for one part of scientists-mechanicians the reticences form the comfort feeling of monolithic character of the classical theory of elasticity, whereas for another part the reticences form a space for development of the theory of elasticity.

Keywords: *mathematical modeling of elastic materials, reticences in classical nonlinear theory of elasticity, smallness of strains and gradients of displacements, nonlinear kinematic parameters*

2000 MSC: 74B20

UDC: 539.3

1. The first reticence in the mathematical modeling of elastic materials

1.1. The classical description of kinematics of deformation in mechanics of materials

The classical mathematical model of deformation as a transition of a body (piece of material) from the undeformed state to the deformed state is based on the continuum representation of this body in the form of some geometrical domain, the density in each point-particle of its material is given as the continuous function of coordinates. The foundations of based on this model theory are stated to a greater or lesser extent in many well-known in solid mechanics books [1–14]. Most abstractedly, the kinematic aspects of deformation of body are stated in the Truesdell and Noll works [12–14].

The primary notion is the body $\mathcal{D}$. The motion of body $\overrightarrow{x} = \overrightarrow{\chi}_\kappa(\overrightarrow{X}, t)$ is described as a mapping χ_κ of the reference configuration $\overrightarrow{\kappa}(\mathcal{D})$ on the actual configuration $\chi(\mathcal{D}, t)$ (a mapping of the point $X \equiv (X_1, X_2, X_3)$ of the body in the reference configuration on the point $x \equiv (x_1, x_2, x_3)$ of the body in the actual configuration). In coordinates, the expression $\overrightarrow{x} = \overrightarrow{\chi}_\kappa(\overrightarrow{X}, t)$ has a form $x_m =$

$= (\chi_\kappa)_m(X_1, X_2, X_3, t)$. The mapping χ_κ is called the deformation of body $\mathcal{D}$ relative to the reference configuration. Here traditionally the Cartesian coordinate system $OX_1X_2X_3$ is introduced, and the point-particle M of the body is characterized by the radius-vector $\overrightarrow{OM} = \overrightarrow{R} = (X_1, X_2, X_3)$. If the point-particle M in the undeformed state passes after deformation into the point M^*, which is defined by the radius-vector $\overrightarrow{OM^*} = \overrightarrow{r} = (x_1, x_2, x_3)$, then the quantity

$$\overrightarrow{OM} - \overrightarrow{OM^*} = \overrightarrow{R} - \overrightarrow{r} = \overrightarrow{u}(x_1, x_2, x_3) \tag{1}$$

is called the vector of displacement of the point-particle M.

A measure of deformation is introduced as follows: first the point N with the radius-vector $\overrightarrow{ON} = \overrightarrow{R} + \Delta\overrightarrow{R} = (X_1 + \Delta X_1, X_2 + \Delta X_2, X_3 + \Delta X_3)$ and corresponding to its point N^* with the radius-vector $\overrightarrow{ON^*} = \overrightarrow{r} + \Delta\overrightarrow{r} = (x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$ are considered; then it follows from (1)

$$\overrightarrow{R} + \Delta\overrightarrow{R} = \overrightarrow{ON^*} = \overrightarrow{r} + \Delta\overrightarrow{r} + \overrightarrow{u}(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3); \tag{2}$$

further the function $\overrightarrow{u}(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$ is represented by the Taylor series in the neighborhood of the point (x_1, x_2, x_3)

$$u_m(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) = \sum_{n=0}^{\infty} \frac{\partial^n u_m}{(\partial x_1)^{n_1}(\partial x_1)^{n_2}(\partial x_1)^{n_3}}(\Delta x_1)^{n_1}(\Delta x_2)^{n_2}(\Delta x_3)^{n_3}, \tag{3}$$

where $n_1 + n_2 + n^3 = n$; in the representation (3), the linear approximation is saved

$$u_m(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) = u_m(x_1, x_2, x_3) + \frac{\partial u_m}{\partial x_k}\Delta x_k. \tag{4}$$

As a result, the formula (3) takes the form

$$dX_m\overrightarrow{E}_m = dx_m\overrightarrow{e}_m + u_{m,k}dx_k\overrightarrow{e}_m, \tag{5}$$

where $\overrightarrow{E}_m$, $\overrightarrow{e}_m$ are the orts of Cartesian coordinates $OX_1X_2X_3, Ox_1x_2x_3$, respectively. The introduced by the formula (5) measure of deformation has a geometrical sense of changing the lengths of the vector-radiuses of the point-particle under deformation of the body. It is used in the classical theory of deformation (both linear and nonlinear) and looks natural under condition that the approximations of the nonlinear displacement from (3) following after the linear one are already not essential.

Remark 1. Let us draw an attention to that despite some classical books on nonlinear mechanics mention the fact of introduction of the measure of deformation as the linear approximation of displacement in the form of nonlinear function even there this fact is not commented. For example, Truesdell [12, chapter IX, section 1] only mentioned that "it seems to be useful to introduce displacements in continuum mechanics, if only both displacements and their gradients are small in some sense".

The next step in constructing the geometrical part (kinematics) of the model of deformation of material as some continuum consists in introduction of the strain tensor. Toward this end, the change of squared lengths of radius-vectors of the point-particle under deformation is considered. As a result, the expression is obtained

$$(dX_m)^2 \equiv (dL)^2 = \Big(dx_m + \frac{\partial u_m}{\partial x_k}dx_k\Big)\Big(dx_m + \frac{\partial u_m}{\partial x_i}dx_i\Big) = ((dl)^2 \equiv (dx^m)^2) + \Big(2\frac{\partial u_i}{\partial x_k} + \frac{\partial u_n}{\partial x_k}\frac{\partial u_n}{\partial x_i}\Big)dx_kdx_i,$$

that further is represented in the form $(dL)^2 - (dl)^2 = \Big[2(\partial u_i/\partial x_k) + (\partial u_n/\partial x_k)(\partial u_n/\partial x_i)\Big]dx_kdx_i$, and then the expression $\Big[2(\partial u_i/\partial x_k) + (\partial u_n/\partial x_k)(\partial u_n/\partial x_i)\Big]dx_kdx_i$ is divided on symmetric and anti-

symmetric parts

$$\varepsilon_{ik} = (1/2)\Big[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_n}{\partial x_k}\frac{\partial u_n}{\partial x_i}\Big)\Big], \omega_k = (1/2)\Big[\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\Big]. \tag{6}$$

Finally, ε_{ik} is called the Cauchy-Green strain tensor, and in the case of the linear model the formula (6) is simplified

$$\varepsilon_{ik} = (1/2)\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big). \tag{7}$$

Remark 2. Lur'e [8, chapter 2, section 1, subsection 1.1] mentioned: "There are any necessity to use in the linear theory of elasticity the listed measures of deformation; while the deformation of massive and weakly deforming bodies being considered, then this theory is based on the quite acceptable assumption on an essential smallness of elements of the matrix of tensor $\nabla\vec{u} : |\partial u_k/\partial a_i| \ll 1$. By that, the successive neglecting of squares and products of tensor components as compared with their first degrees".

Remark 3. Relationships (7) are called very frequently in the linear theory of elasticity the Cauchy relations. In the nonlinear approach, when deformation can be finite, the representation (6) is assumed to be sufficient.

Remark 4. In many cases, it seems more convenient to describe the strain tensor (6) by its first algebraic invariants (basis invariants)

$$\begin{aligned}
I_1 &= g_i\varepsilon_{ik} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \\
I_2 &= \varepsilon_{ik}\varepsilon_{ki} = (\varepsilon_{11})^2 + (\varepsilon_{22})^2 + (\varepsilon_{33})^2 + 2(\varepsilon_{12})^2 + +2(\varepsilon_{23})^2 + 2(\varepsilon_{31})^2, \\
I_3 &= \varepsilon_{ik}\varepsilon_{nk}\varepsilon_{in} = (\varepsilon_{11})^3 + (\varepsilon_{22})^3 + (\varepsilon_{33})^3 + 3\varepsilon_{11}(\varepsilon_{12})^2 + 3\varepsilon_{11}(\varepsilon_{31})^2 \\
&\quad + 3\varepsilon_{22}(\varepsilon_{12})^2 + 3\varepsilon_{33}(\varepsilon_{23})^2 + 3\varepsilon_{33}(\varepsilon_{31})^2 + 6\varepsilon_{12}\varepsilon_{23}\varepsilon_{31}.
\end{aligned} \tag{8}$$

Thus, the first reticence in the mathematical modeling of materials can be thought as a tacit introduction of linear approximation in description of kinematics of deformation and an absence of comments of the ignoring the nonlinear description of the measure of nonlinear deformation.

1.2. A few words on the nonlinear description of the measure of nonlinear deformation

Consider again the representation of displacement in the form of Taylor series (3) and save not two (as it is made in the classical theory of elasticity), but three first terms

$$u_m(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) = \vec{u}_m(x_1, x_2, x_3) + \frac{\partial u_m}{\partial x_k}\Delta x_k + \frac{\partial^2 u_m}{\partial x_k \partial x_i}\Delta x_k \Delta x_k. \tag{9}$$

When the approximate representation (9) being substituted into (2), then

$$dX_m\vec{E}_m = dxm\,\vec{e}_m + u_{m,k}dx_k\,\vec{e}_m + \frac{\partial^2 u_m}{\partial x_k \partial x_i}dx_k dx_k\,\vec{e}_m. \tag{10}$$

The introduced by formula (10) measure of deformation saves the geometrical sense of changing the lengths of radius-vectors of the point-particle under deformation of the body. Further, to define the strain tensor, it is necessary to analyze the expression

$$(dL)^2 - (dl)^2 = (dX_m)^2 - (dx_m)^2 = \Big(2\frac{\partial u_i}{\partial x_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_i}\Big)dx_k dx_i +$$
$$+ \Big(\frac{\partial^2 u_n}{\partial x_k \partial x_i} + 2\frac{\partial u_m}{\partial x_n}\frac{\partial^2 u_m}{\partial x_k \partial x_i}\Big)dx_n dx_k dx_i + \frac{\partial^2 u_m}{\partial x_k \partial x_i}\frac{\partial^2 u_m}{\partial x_n \partial x_p}dx_k dx_i dx_n dx_p. \tag{11}$$

The right hand side in (11) contains three summands: the first summand is used in the classical theory of elasticity to determine the strain tensor ε_{ik} and rotation tensor ω_{ik} (tensors of the second rank), the second and third ones can be a basis to determine the new tensors $\varepsilon_{ikn} = (\partial^2 u_n / \partial x_k \partial x_i) + 2(\partial u_m / \partial x_n)(\partial^2 u_m / \partial x_k \partial x_i)$ and $\varepsilon_{iknp} = (\partial^2 u_m / \partial x_k \partial x_i)(\partial^2 u_m / \partial x_n \partial x_p)$, which characterize the finite deformations (tensors of the third and fourth ranks). The formula (11) can be commented in the following way.

Comment 1. The successive complication (only on one following term) of the measure of deformation (9) does not complicate the classical nonlinear strain and rotation tensors (tensors of the second rank).

Comment 2. The presented in description of elastic deformation (10) new tensors of the third and fourth ranks were not known earlier in the classical nonlinear theory of elasticity. But such a kind of tensors was arose in the microstructural models of deformation of materials. For example, the nonlinear model of simple microhyperelastic bodies [15,16] includes the third rank tensor called there the tensor of the first moments of stresses. This tensor is involved into equations of moments of momentum, which in this case are already not so simple as in the classical theory of elasticity (they are already not degenerated in the condition of symmetry of the stress tensor).

Comment 3. Tensors $\varepsilon_{ik}, \varepsilon_{ikn}, \varepsilon_{iknp}$ are symmetric by some indexes

$$\varepsilon_{ik} = \varepsilon_{ki}, \varepsilon_{ikn} = \varepsilon_{kin}, \varepsilon_{iknp} = \varepsilon_{kipn}. \tag{12}$$

Thus, formula (11) testifies that the kinematic picture of deformation within the framework of novel representation of the measure of deformation (10) is characterized by three tensors $\varepsilon_{ik}, \varepsilon_{ikn}, \varepsilon_{iknp}$.

This procedure can be continued up to obtaining, for example, the novel nonlinear wave equations within the framework of abovementioned approach. But here only the fact of reticence of possibility of similar approaches is important.

2. Next two reticences in the mathematical modeling of materials

These reticences are associated with constitutive equations in the nonlinear model of elastic deformation. As it is well-known [1–12], the form of representation of constitutive equations depends on a choice of ordered pairs "kinetic parameters – kinematic parameters". Further, two pairs will be considered: the pair "Lagrange stress tensor – Cauchy-Green strain tensor" and the pair "Kirchhoff stress tensor – gradients of displacements".

2.1. To the motion and constitutive equations in the nonlinear theory of elasticity

The general approach in constructing the motion equations of elastic medium consists in using the balance equations (mass, linear momentum, momentum of momentum, energy). In contrast to the kinematic part of classical theory of elasticity, this part is related to the kinetics of motion and needs of introduction of kinetic parameters (forces and stresses).

In conditions of absence of factors of changing the mass and momentum of momentums, it is sufficient to analyze only the equations of balance of linear momentum. The local representation of these equations has the standard for the classical theory of elasticity form

$$\sigma_{ik,k} + F_i = \rho \ddot{u}_i \quad or \quad t_{ki,k} + F_i = \rho \ddot{u}_i, \tag{13}$$

where σ_{ik} is the symmetric Lagrange stress tensor, t_{ik} is the non-symmetric Kirchhoff stress tensor, F_i are components of the resultant external force vector.

Usually, equations (13) are written through the displacements. In this case, the postulation of constitutive equations (relations between stresses and strains) is needed.

It is well-known [5] that two approaches to constructing the constitutive equations exist in the classical theory of elasticity. Both are based on the fundamental property of elastic material of reversibility of a process of deformation after unloading (taking-down the loading that caused the deformation). This property is observed in two coupled phenomena: in the full restorability of the body shape after unloading and the full loss energy by the body stored during the deformation after unloading. The shown two phenomena underlie two approaches to constructing the constitutive equations [5].

The first approach derives from Cauchy. It consists in formulation of one-to-one relationships between stress and strain tensors. If the zeroth stresses correspond to the zeroth strains, then the initial shape of body restores fully after unloading. If before the loading the initial stresses and some initial shape existed, then after unloading the condition of one-to-one depentanizer provides the same stresses and the same shape.

The second approach derives from Green. It consists in representation of potential (internal) energy of elastic deformation in the form of a function of the strain tensor under condition that the components of this tensor define fully the state of body by deformation. In this case, the internal energy is equal to zero under zeroth strains, that is, the elastic body losses fully its energy stored when being deformed.

Let us restrict an analysis to the case of isotropy of elastic properties of deformed body. Then, within the first approach, the constitutive equations can be represented in some general form [3,5]

$$\sigma_{ik} = \psi_0(I_1, I_2, I_3)\delta_{ik} + \psi_1(I_1, I_2, I_3)\varepsilon_{ik} + \psi_2(I_1, I_2, I_3)(\varepsilon_{ik})^2. \tag{14}$$

Here the functions ψ_0, ψ_1, ψ_2 as the functions of basic invariants are also the invariants. They can be assumed to be the moduli of elastic deformations and must be determined by experimental way (that is, first they are defined analytically and then are matched with experimental data).

In the linear model of elastic behavior, relations (14) are simplified essentially and have the name of the Hooke law

$$\sigma_{ik} = \lambda\varepsilon_{mm}\delta_{ik} + \mu\varepsilon_{ik}. \tag{15}$$

In the scientific practice, the functions are used, which are sufficiently simple in an analytical representation. To describe the second approach, let us recall that the division of elastic materials on hyperelastic, elastic, and hypoelastic ones exist, and consider further the case of hyperelastic materials [17]. They are defined as such for which the infinite times differentiable elastic potential $W(\varepsilon_{ik})$ exists, and for which the constitutive equations can be written by the formula

$$\sigma_{ik} = (\partial W/\partial \varepsilon_{ik}). \tag{16}$$

There are sufficiently small group of concrete representations of elastic potentials, two of which can be thought as having some general form: the Signorini potential

$$W = W(I_1^A, I_2^A) = h_2(I_1^A - 3) + h_1(I_2^A - 3) + h_3(I_1^A - 3)^2 \tag{17}$$

(h_1, h_2, h_3 are the Signorini elastic constants, I_1^A, I_2^A are the invariants of Almansi strain tensor) and the Murnaghan potential

$$W = W(I_1, I_2, I_3) = (1/2)\lambda(I_1)^2 + \mu I_1 + (1/3)A(I_1)^3 + BI_1I_2 + (1/3)CI_3 \tag{18}$$

(λ, μ are the Lame elastic constants, A, B, C are the Murnaghan elastic constants).

2.2. On the Murnaghan potential

Let us stop further on the Murnaghan potential (18) and write it through the components of nonlinear Cauchy-Green strain tensor ε_{ik}

$$W = W(\varepsilon_{ik}) = (1/2)\lambda(\varepsilon_{mm})^2 + \mu(\varepsilon_{ik})^2 + (1/3)A\varepsilon_{ik}\varepsilon_{im}\varepsilon_{km} + B(\varepsilon_{ik})^2\varepsilon_{mm} + (1/3)C(\varepsilon_{mm})^3. \tag{19}$$

Remark 5. The formula (19) shows more clearly that this potential is a polynomial with the constant coefficients by the squared and cubed strain tensor components. It follows also from representation (19) that this potential involves only the second and third degrees of the strain tensor components.

Remark 6. The elastic constants are determined for many engineering materials [17–20]. The values of constants ρ, λ, μ, A, B, C for some characteristic metallic materials are shown in the Table 1.

<div align="center">

Table 1.

materials	$\rho \cdot 10^3 \text{kg/m}^3$	$\lambda \cdot 10^{10} \text{Pa}$	$\mu \cdot 10^{10} \text{Pa}$	$A \cdot 10^{11} \text{Pa}$	$B \cdot 10^{11} \text{Pa}$	$C \cdot 10^{11} \text{Pa}$
aluminum	2.70	5.20	2.70	-0.65	-2.05	-3.70
bruss LC	8.50	10.5	3.70	-4.05	$+17.0$	$+2.40$
copper	8.93	10.7	4.80	-2.8	-1.72	-2.40
molybdenum	10.2	15.7	11.0	-0.26	-2.83	$+3.72$
steel Hecla37	7.82	11.1	8.21	-4.45	-2.82	-7.16
tungsten	18.9	7.50	7.30	-1.08	-1.43	-9.08

</div>

Remark 7. The Murnaghan constants are negative for all the shown above metallic materials excluding brass and molybdenum. If to consider the one-dimensional deformation, then the Murnaghan potential takes the form

$$W = W(\varepsilon_{11}) = (1/2)(\lambda + 2\mu)(\varepsilon_{11})^2 + (1/3)(A + 3B + C)(\varepsilon_{11})^3 \tag{20}$$

and the stress (Lagrange) is determined by the formula $\sigma_{11} = (dW/d\varepsilon_{11}) = (\lambda + 2\mu)(\varepsilon_{11}) + (A + 3B + C)(\varepsilon_{11})^2$. It follows from this formula that in the case of one-dimensional deformation of the shown above metallic materials the expression $A + 3B + C$ is negative $A + 3B + C < 0$ and the curve of dependence $\sigma \sim \varepsilon$ lies from below of the straight line $\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11}$ that corresponds to the linear law of deformation. It is said then that expression (20) describes *the soft nonlinearity of deformation*. The brass is characterized by $A + 3B + C = (-0.31 + 1.7 + 2.4)10^{12} > 0$ and the curve of dependence $\sigma \sim \varepsilon$ lies from above of the straight line $\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11}$, therefore it should be referred to the materials with *the hard nonlinearity of deformation*.

The represented above features of the Murnaghan potential in description of nonlinearity turns out to be important in many concrete problems of nonlinear mechanics. So, we focus on the Murnaghan potential in view of its specific position in the nonlinear elasticity — it is the most developed and applicable in the potential's family.

2.3. The motion equations in displacements

The nonlinear motion equations in displacements are found as some generalization of the classic Lame equations [3,8,10,11,17]. In the first approach, the representation (10) is substituted into the balance equation (9)

$$[\psi_0(I_1, I_2, I_3)]_{,k}\delta_{ik} + [\psi_1(I_1, I_2, I_3)]_{,k}\varepsilon_{ik} +$$
$$+ [\psi_1(I_1, I_2, I_3)]_{,k}\varepsilon_{ik,k} + [\psi_2(I_1, I_2, I_3)]_{,k}\varepsilon_{in}\varepsilon_{nk} + \psi_2(I_1, I_2, I_3)(\varepsilon_{in}\varepsilon_{nk})_{,k} + F_i = \rho\ddot{u}_i \rightarrow$$
$$\rightarrow \left(\frac{\partial\psi_0}{\partial I_1}I_{1,k} + \frac{\partial\psi_0}{\partial I_2}I_{2,k} + \frac{\partial\psi_0}{\partial I_3}I_{3,k}\right)\delta_{ik} + \left(\frac{\partial\psi_0}{\partial I_1}I_{1,k} + \frac{\partial\psi_1}{\partial I_2}I_{2,k} + \frac{\partial\psi_1}{\partial I_3}I_{3,k}\right)\varepsilon_{ik} +$$
$$+ [\psi_1(I_1, I_2, I_3)]_{,k}\varepsilon_{ik,k} + \left(\frac{\partial\psi_2}{\partial I_1}I_{1,k} + \frac{\partial\psi_2}{\partial I_2}I_{2,k} + \frac{\partial\psi_2}{\partial I_3}I_{3,k}\right)\varepsilon_{in}\varepsilon_{nk} + \psi_2(I_1, I_2, I_3)(\varepsilon_{in}\varepsilon_{nk})_{,k} + F_i =$$
$$= \rho\ddot{u}_i \tag{21}$$

and then for the known functions ψ_0, ψ_1, ψ_2 the derivatives $\psi_{0,m}, \psi_{1,m}, \psi_{2,m}$ and derivatives of invariants are determined. As a result, for all the known representations, equations (17) can be divided on two parts — the linear one $\big[(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} + F_i - \rho\ddot{u}_i\big]$ (corresponding to the linear Hooke law and Lame equations) and nonlinear one $N_i(u_{n,m}, u_{n,mp}, u_{n,mpq}, \ldots)$(depending on products of displacements gradients and their derivatives with the order of nonlinearity two and more

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} + F_i - \rho\ddot{u}_i = N_i(u_{n,m}, u_{n,mp}, u_{n,mpq}, \ldots). \tag{22}$$

In the second approach, the components of stress tensor are also substituted into the balance equation. But first these components must be determined from the known potential by the formulas $\sigma_{ik} = (\partial W/\partial\varepsilon_{ik})$ or $t_{ik} = (\partial W/\partial u_{ki})$. In the first case, the nonlinear symmetric Lagrange stress tensor has a standard form (only one component is shown below, the structure of dependence for the rest components is similar to the first one)

$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda(\varepsilon_{22} + \varepsilon_{33}) + A\big[(\varepsilon_{11})^2 + (1/3)(\varepsilon_{12}\varepsilon_{12} + \varepsilon_{13}\varepsilon_{13})\big] \tag{23}$$
$$+B\big[3(\varepsilon_{11})^2 + 2\varepsilon_{11}(\varepsilon_{22} + \varepsilon_{33})\big] + C((\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})).$$

The second case (the non-symmetric Kirchhoff stress tensor) will be analyzed more in depth in the next subsection. Note here that the potential should be written through the displacements gradients. As a result, it is found already the exponential function of the higher degrees (the second order of nonlinearity is transformed into the sixth one, the third order — into the ninth and so on).

2.4. On representation of stresses and strains through the displacements gradients

Consider now the announced in the subtitle representation for the case of Murnaghan potential. This potential is found the exponential function of the displacements gradient from the 2nd to the 6th orders. Just such a way was chosen in the nonlinear acoustics [17,21,22]. There, the longitudinal and transverse plane sound waves were studied. To derive the corresponding nonlinear wave equations in terms of displacements, the summands of the 2nd and 3rd orders were saved in the potential, whereas the summands of the 5th- 6th orders were neglected

$$W = (1/2)\big[(\lambda + 2\mu)(u_{1,1})^2 + \big((u_{2,1})^2 + (u_{3,1})^2\big)\big]+ \tag{24}$$
$$+ \big[\mu + (1/2)\lambda + (1/3)A + B + (1/3)C\big](u_{1,1})^3 + (1/2)(\lambda + B)u_{1,1}\big((u_{2,1})^2 + (u_{3,1})^2\big),$$

where the usual for analysis of the plane waves assumption on propagation of waves along the axis Ox_1 was accepted $u_k = u_k(x_1, t)$.

The components of nonlinear Kirchhoff stress tensor had the form (further, only 3 of 9 are shown)

$$t_{11} = (\lambda + 2\mu)u_{1,1} + (3/2)\big[(\lambda + 2\mu) + 2(A + 3B + C)\big](u_{1,1})^2+$$
$$+ (1/2)\big[(\lambda + 2\mu) + (1/2)A + B\big]\big[(u_{2,1})^2 + (u_{3,1})^2\big],$$
$$t_{12} = \mu u_{2,1} + (1/2)\big[\lambda + 2\mu + (1/2)A + B\big]u_{1,1}u_{2,1}, \tag{25}$$
$$t_{12} = \mu u_{2,1} + (1/2)\big[\lambda + 2\mu + (1/2)A + B\big]u_{1,1}u_{2,1},$$

While the relations (25) being substituted into the motion equations $t_{ik,i} + F_i = \rho\ddot{u}_i$, then the quadratically nonlinear wave equations can be obtained for three polarized plane waves

$$\rho\ddot{u}_1 - (\lambda + 2\mu)u_{1,11} = N_1 u_{1,11}u_{1,1} + N_2\big(u_{2,11}u_{2,1} + u_{3,11}u_{3,1}\big), \tag{26}$$
$$\rho\ddot{u}_2 - \mu u_{2,11} = N_2\big(u_{2,11}u_{1,1} + u_{1,11}u_{2,1}\big), \tag{27}$$
$$\rho\ddot{u}_3 - \mu u_{2,11} = N_2\big(u_{3,11}u_{1,1} + u_{1,11}u_{3,1}\big), \tag{28}$$
$$N_1 = 3\big[(\lambda + 2\mu) + 2\big(A + 3B + C\big)\big], \quad N_2 = (\lambda + 2\mu) + (1/2)A + B. \tag{29}$$

Remark 8. The intuitively rational decision to neglect in potential (24) the summands of the order more than third one results in including into the stress tensor and wave equations the nonlinear summands only the second order. Therefore, the fields of stresses and strains are described similarly with exactness to the gradients of displacements of the third order (the third order is neglected).

Comment 4. The question should be taken into account the third-fifth orders in the stress tensor and in wave equations, whereas only the second order is taken into account in the strain tensor, is still open. Taking into account of the third order (cubic nonlinearity) showed [17,22] that the plane transverse waves are modeled better, because in this case the phenomena of nonlinear interaction and self-generation of these waves is described. So, the waves might be studied within the approach of higher approximations. But here the collision arises between description of strains within the quadratic approximation and stresses in the cubic approximation.

2.5. The second reticence (the first reticence around the constitutive equations)

Let us consider now the first reticence relative to the constitutive equations. It is concentrated on the question which strains can be considered as the small ones and which gradients of displacements can be considered as the small ones. Most often in the theory of elasticity, the smallness of gradients is mentioned and the classical inequality $|u_{i,k}| \ll 1$ is written. But this inequality is not commented more definitely for the real materials. It means in the theory of elasticity that strains and gradients correlate fully, that is an equality of small strain to some concrete value results automatically in an equality of gradient to the same value. The primary fact here is the relation between strains and gradients (7), which is known also as the nonlinear Cauchy relations

$$\varepsilon_{ik} = (1/2)\Big[(\partial u_i)/(\partial x_k) + (\partial u_k/\partial x_i) + (\partial u_n/\partial x_k)(\partial u_n/\partial x_i))\Big].$$

To comment this situation, let us consider the simple case of elastic deformation $u_k = u_k(x_1, t)$, mentioned above in connection with the plane waves. The formula (7) is simplified to $\varepsilon_{11} = u_{1,1} + (1/2)\big[(u_{1,1})^2 + (u_{2,1})^2 + (u_{3,1})^2\big]$. This formula permits to establish the simple approximate link between ε_{11} and $u_{1,1} = u_{2,1} = u_{3,1} = v$. Choose the most unfavorable variant of values $u_{1,1}, u_{2,1}, u_{3,1}$ and rewrite (7) in the form $(v)^2 - (2/3)v - (2/3)\varepsilon_{11} = 0$. Then $v = (1/9)\big(1 - \sqrt{1 - 6\varepsilon_{11}}\big)$.

This formula shows that the following difference between v and ε_{11} exists (in percents): $\varepsilon_{11} = 4 \cdot 10^{-2} \to 11.4\%$, $\varepsilon_{11} = 3 \cdot 10^{-2} \to 8.6\%$, $\varepsilon_{11} = 2 \cdot 10^{-2} \to 5.8\%$, $\varepsilon_{11} = 1 \cdot 10^{-2} \to 3.0\%$, $\varepsilon_{11} = 5 \cdot 10^{-3} \to 1.5\%$, $\varepsilon_{11} = 3 \cdot 10^{-3} \to 0.9\%$, $\varepsilon_{11} = 1 \cdot 10^{-3} \to 0.3\%$, $\varepsilon_{11} = 5 \cdot 10^{-4} \to 0.15\%$, $\varepsilon_{11} = 1 \cdot 10^{-4} \to 0.03\%$.

Thus, the threshold value for a small strain can be approximately determined in dependence on understanding what is a small difference between a small strain and a small gradient of displacement: when this difference is about 1 percent, then the threshold value for small strain is about $3 \cdot 10^{-3}$; when this difference is about 0,1 percent, then the threshold value for small strain is about $3 \cdot 10^{-4}$.

Comment 5. The shown approximate estimate has only the geometrical character and does not depend on properties of elastic material.

Comment 6. A knowledge of the threshold value for a small strains permits to decide: we are within the range of small strains or not, but it cannot give the answer: the model of elastic deformation must be nonlinear or not.

2.6. The third reticence (the second reticence relative to the constitutive equations)

To find this answer (mentioned above) in Comment 6, restrict further the analysis to only the Kirchhoff shear stress t_{12} (25)

$$t_{12} = \mu u_{2,1} + (1/2)\big[\lambda + 2\mu + (1/2)A + B\big]u_{1,1}u_{2,1} \tag{30}$$

and the Lagrange shear stress σ_{12}, which under the analogous assumptions can be written in the form

$$\sigma_{12} = 2\mu\varepsilon_{21} + 2(A + 2B)\varepsilon_{11}\varepsilon_{21}. \tag{31}$$

Comment 7. The constitutive equations (29) "Lagrange shear stress – Cauchy-Green shear strain" and (25) "Kirchhoff shear stress – displacement shear gradient" are at least not identical. They must be represented geometrically by different nonlinear curves for one and the same material.

This forms the new reticence relative to constitutive equations for nonlinear models of elastic deformation. Consider the difference between the mentioned above curves for some particular case and choose three engineering materials from the Table 1: aluminum, copper, steel. Assume also the most unfavorable variants of values $\varepsilon_{11} = \varepsilon_{21} = \varepsilon$, $u_{1,1} = u_{2,1} = v$ and consider corresponding formulas

$$\sigma_{12} = 2\mu\varepsilon + 2(A + 2B)(\varepsilon)^2, \tag{32}$$

$$t_{12} = \mu v + \left[\lambda + 2\mu + (1/2)(A + 2B)\right](v)^2. \tag{33}$$

Further, restrict analysis to the range of values of ε and v below of the threshold value $\varepsilon = v = 3 \cdot 10^3$, where values of ε and v are identical with exactness of one percent, and determine for the chosen three materials the deflection (in percents) from the straight lines $\sigma_{12} = 2\mu\varepsilon$ and $t_{12} = \mu v$.

Aluminum:

$$\varepsilon = 0.03;\ 0.02;\ 0.01;\ 0.005;\ 0.001;\ 0.00057 \rightarrow (\% =)51.8;\ 34.5;\ 17.25;\ 8.63;\ 1.73;\ 1.00;$$
$$v = 0.03;\ 0.02;\ 0.01;\ 0.005;\ 0.0048;\ 0.0046 \rightarrow (\% =)6.59;\ 4.40;\ 2.20;\ 1.10;\ 1.06;\ 1.00; \tag{34}$$

Copper:

$$\varepsilon = 0.03;\ 0.02;\ 0.01;\ 0.005;\ 0.001;\ 0.00077 \rightarrow (\% =)38.0;\ 26.0;\ 13.0;\ 6.50;\ 1.30;\ 1.00;$$
$$v = 0.03;\ 0.02;\ 0.01;\ 0.008;\ 0.007;\ 0.0065 \rightarrow (\% =)4.60;\ 3.07;\ 1.54;\ 1.23;\ 1.08;\ 1.00; \tag{35}$$

Steel:

$$\varepsilon = 0.03;\ 0.02;\ 0.01;\ 0.005;\ 0.001;\ 0.00082 \rightarrow (\% =)36.9;\ 24.6;\ 12.3;\ 6.15;\ 1.23;\ 1.00;$$
$$v = 0.03;\ 0.02;\ 0.01;\ 0.005;\ 0.001;\ 0.00082 \rightarrow (\% =)5.50;\ 3.67;\ 1.83;\ 1.28;\ 1.10;\ 1.00; \tag{36}$$

Comment 8. Let us recall that the threshold value of strains and gradients of displacements means here the value, the exceeding of which by them can not be considered as the small ones. Therefore an comparison of the approximately determined deviations of nonlinear curves $\sigma \sim \varepsilon$ and $t \sim v$ from their linear prototypes is realized in this paper for small strains and gradients of displacements. The main result that follows from comparison of deflection of curves $\sigma \sim \varepsilon$ and $t \sim v$ is twofold: 1. The tables (34)–(36) confirm the fact that metals become deformed nonlinear elastically under small strains and the Murnaghan model is able to describe this feature of metals. 2. The curves curves $\sigma \sim \varepsilon$ and $t \sim v$ differ about eight times – if the deflection near the threshold values $\varepsilon = v = 3 \cdot 10^{-2}$ for the curves $\sigma \sim \varepsilon$ are significantly large (from 37% to 52%), then for curves $t \sim v$ the deflections are essentially smaller (from 4.6% to 6.6%).

3. Final conclusions relative to reticences in the mathematical modeling of elastic materials

The general estimate of facts of reticences seems to be positive, because for one part of scientists-mechanicians the reticences form the comfort feeling the monolithic character of the classical theory of elasticity, whereas for another part the reticences form the space for developing the theory. Just this second line of estimating the reticences is presented in this paper. Three facts of reticences are chosen.

The first fact consists in a reticence of one of the first steps in the mentioned above procedure — an assumption that the kinematics of deformation is described by the linear approximation of motion of material continuum, namely by gradients of deformation. In the paper, some novel nonlinear approach to this procedure is offered. The second and third facts are associated with constitutive relations. The second fact consists in absence of necessary comments relative to determination of smallness of strains and gradients of displacements (absence of comments relative to a criterion of applicability of the linear model because the criterion $|u_{i,k}| \ll 1$ is sufficiently abstract. It is shown that the based on the nonlinear Cauchy relations approximate procedure of determination of threshold values of strains and gradients of deformations starting with which a nonlinearity of process appears. The third fact consists in absence of comments relative to essential differences between the nonlinear constitutive equations, which are written for the ordered pairs "Lagrange stress tensor – Cauchy-Green strain tensor" and "Kirchhoff stress tensor – gradients of displacements". It is shown on an example of the shear stress and the Murnaghan model of nonlinear elastic deformation that deviation from the corresponding straight lines of linear deformation for different pairs differs in many times in the range of small strains and small gradients of displacements.

[1] Eringen A. C. Nonlinear Theory of Continuous Media. McGraw Hill, New York (1962).

[2] Fu Y. B. Nonlinear Elasticity: Theory and Applications. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (2001).

[3] Goldenblatt I. I. Nonlinear Problems of the Theory of Elasticity. Nauka, Moscow (1969), (In Russian)

[4] Green A. E., Adkins J. E. Large Elastic Deformations and Nonlinear Continuum Mechanics. Oxford University Press, Clarendon Press, London (1960).

[5] Guz. Fundamentals of the Three-Dimensional Theory of Stability of Deformable Bodies. Springer, Berlin (1999).

[6] Hetnarski R. B., Ignaczak J. The Mathematical Theory of Elasticity. CRC Press, Taylor & Francis Group, Boca Raton (2011).

[7] Holzapfel G. A. Nonlinear Solid Mechanics. A Continuum Approach for Engineering. Wiley, Chichester (2006)

[8] Lur'e A. I. Nonlinear Theory of Elasticity. North-Holland, Amsterdam (1990).

[9] Lur'e A. I. Theory of Elasticity. Springer, Berlin (1999).

[10] Novozhilov V. V. Foundations of the nonlinear theory of elasticity. Graylock Press, Rochester, New York (1953).

[11] Ogden R. W. The Nonlinear Elastic Deformations. Dover, New York (1997).

[12] Truesdell C. A first course in rational continuum mechanics. The John Hopkins University, Baltimore (1972).

[13] Truesdell C., Noll W. The Nonlinear Field Theories of Mechanics. Flugge Handbuch der Physik, Band III / 3. Springer Verlag, Berlin (1965).

[14] Truesdell C., Toupin R. The Classical Field Theories, Flugge Handbuch der Physik, Band III / 1. Springer Verlag, Berlin (1960).

[15] Eringen A. C. Mechanics of Continua. John Wiley, New York (1967).

[16] Eringen A. C., Suhubi E. S. Nonlinear theory of simple microelastic solids. Int. J. Engng. Sci. **2**, n.2, 189–203 (1964).

[17] Rushchitsky J. J. Nonlinear Elastic Waves in Materials. Springer, Heidelberg (2014).

[18] Murnaghan F. D. Finite Deformation in an Elastic Solid. John Wiley, New York (1951, 1967).

[19] Guz A. N., Makhort F. G., Gushcha O. I. Introduction to Electroelasticity. Naukova Dumka, Kiev (1977), (In Russian).

[20] Guz A. N. Elastic Waves in Bodies with Initial Stresses, **1**, General Problems, **2**, Regularities of Propagation. Naukova Dumka, Kiev (1986), (In Russian).

[21] Goldberg Z. A. On interaction of plane longitudinal and transverse waves. Akusticheskii Zhurnal, **6**, n.2), 307-310 (1960), (In Russian and English)

[22] Cattani C., Rushchitsky J. J. Wavelet and wave analysis as applied to materials with micro or nanostructures. Singapore-London: World Scientific Publishing Co. Pte. Ltd. (2007).

Про три факти умовчання в класичному математичному моделюванні пружних матеріалів

Рущицький Я. Я.

Інститут механіки ім. С. П. Тимошенка
вул. Нестерова, 3, 03680, Київ, Україна

Описано і прокоментовано три факти умовчання (відсутності коментарів) в процедурах математичного моделювання пружних матеріалів. Перший факт полягає в некоментуванні одного з перших кроків у згаданій процедурі — припущення, що кінематика деформування описується за допомогою лінійної апроксимації руху матеріального континууму, а саме за допомогою градієнтів деформації. У статті запропоновано певний новий нелінійний підхід в процедурі. Другий і третій факти пов'язані з побудовою визначальних рівнянь. Другий факт полягає у відсутності належних коментарів щодо визначення малості деформацій і градієнтів зміщень (відсутності коментарів щодо критерію застосування лінійної моделі, оскільки критерій $|u_{i,k}| \ll 1$ є досить абстрактним). Показано, що існує базована на нелінійних співвідношеннях Коші наближена процедура визначення порогових значень деформацій і градієнтів зміщень, при яких вже починає виявлятися нелінійність процесу деформування. Третій факт полягає у відсутності коментарів щодо суттєвих відмінностей між нелінійними визначальними рівняннями, записаними для впорядкованих пар «тензор напружень Лягранжа – тензор деформацій Коші-Гріна» та «тензор напружень Кірхгоффа – градієнти зміщень». Показано на прикладі зсувних напружень та моделі нелінійного пружного деформування Мернагана, що відхилення від відповідних прямих лінійного деформування для різних пар відрізняються в діапазоні малих деформацій і малих градієнтів зміщень у багато разів. Загальна оцінка фактів умовчання виглядає позитивною, оскільки для однієї частини вчених-механіків умовчання створює комфортне відчуття монолітності класичної теорії пружності, а для іншої частини умовчання створюють простір для розвитку теорії.

Ключові слова: *математичне моделювання пружних матеріалів, замовчування в класичній теорії пружності, малість деформацій і градієнтів зміщень, нелінійні кінематичні параметри*

2000 MSC: 74B20

УДК: 539.3