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PROPAGATION OF ELASTIC-TOUGH-PLASTIC WAVE IN A ROD OF IDEALLY PLASTIC MATERIAL WHICH POSSESSES THE PROPERTY OF YIELD DELAY

Received: September 11, 2015 / Revised: November 20, 2015 / Accepted: December 14, 2015

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Abstract. The great number of theoretical and experimental investigations indicate that it is necessary to consider plastic deformation of metals as a process which proceeds in certain time interval. However, in such approaches, contradictive points of view emerge concerning investigation of wave processes in materials which possess peculiar properties. In order to describe different mechanical effects, new models of plastic medium are being improved. The phenomenon of emergence of a tooth yield in the strain diagram belongs to such properties of materials. In this article, investigation of propagation of elastic-tough-plastic wave is conducted in a semi-infinite rod; it is conducted on the basis of the electromechanical model of ideal elastic-plastic material with yield delay. The dynamic criterion of plasticity, which is stated by the author is used here. The solving of the problem is conducted according to the statement where constant force is abruptly exerted on a butt of an unloaded rod. The main solution is obtained in stresses, both for elastic and plastic domains of the rod. It is established that characteristic curves which separate an elastic domain from a plastic one and vice versa are frontal lines of weak rupture. On the basis of calculations, fields of stresses, strains, rates of displacements of cross-sections of the rod in a plastic domain are determined. For a fixed instant of time, graphs of changes of stresses and displacements of points on the whole segment of a disturbed rod are plotted. The obtained results have been verified as to exactness of satisfaction to the boundary condition.

Introduction

There exist many scientific works which are dedicated to study of propagation of elastic-plastic waves in rectilinear rods. The number of published works about wave propagation in materials which possess the property of yield delay is considerably less. Later on, the interest in such investigations decreased. The emergence of published articles about improvement of models which describe the phenomenon of yield delay became random events [7]. In the author's opinion, a false assertion about instantaneous jump-like transition from overstressed state into plastic state is put into works in which problems of wave processes in materials are considered. In overwhelming majority of cases, problems of butt impact in a rod are stated as those in which its terminal cross-sections gain constant speed. A certain stress whose value is greater than the static limit of yield corresponds to this speed. In all the cross-sections of a rod for which the time of yield delay is up the stress abruptly decreases to the static yield strength. Such assertion of the model [4] causes doubt and rejection, because the stress which emerges at an end of a rod due to the action of the striker does not vanish. It is not elastic unloading of corresponding cross-sections of a rod, but their transition into plastic state that leads to the rate of displacement of the points of the plastic domain that proceeds.

Let us try to describe the peculiarities of wave propagation in materials with yield delay with a help of electromechanical model of elastic-tough-plastic medium [8]. To solve the stated problem, it is expedient to choose a uniformly distributed constant force which is abruptly exerted on the plane of a rod butt as a disturber.

Main assumption, statement of problem, and equation of motion in elastic domain

Let us consider the process of elastic-plastic state propagation in a semi-infinite rod of ideally plastic material which has the property of yield delay. The solution of the stated problem is to be conducted in Lagrangean coordinate system. Let us put the origin of x, y, z Cartesian coordinate system at a butt of the

rod. The axis of the rod is taken for x -axis, the y, z axes are put in the plane of the butt. The following main assumptions are to be based on.

1) In an arbitrary cross-section of a rod the stress-strain state is homogeneous, and it is determined from the boundary conditions.

2) It is taken into account that relative that relative deformation strains are small.

3) The influence of forces of inertia from transversal deformation of a rod are to be neglected.

4) The density of a rod does not change during the process of deformation.

Let the investigated rod be at rest till the time instant $t = 0$ and have constant F_0 area of its cross-section. Then, at $t > 0$, a constant compressive force P , which is uniformly distributed over its surface, is exerted at the beginning $x = 0$ of the rod. It is assumed that this force causes compressive stress σ_{xx} at the butt of the rod, this stress being greater than the yield point stress σ_y in static load. The delay time τ during which a straight elastic wave propagates along the axis of the rod corresponds to this stress. To determine the stress and deformation in the domain of elastic wave, the differential equation of motion of continuum and Hook's law are used. In the framework of theory of rods, the following system of equations is obtained:

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad \sigma_{xx} = E \varepsilon_{xx}, \quad (1)$$

where σ_{xx} , ε_{xx} are the components of tensors of stresses and deformations along the x -axis; ρ is the density of the rod; E is the Young's modulus of the material of the rod.

The solution of the equations system (1) is known, but traditionally it is written in displacements. In our case, for investigation of the state of rod in plastic domain, it is by far simpler to write the equation of dynamic behavior of material is stressed. Therefore, we reduce the equations system (1) to that of differential equations for the stress σ_{xx} for the elastic domain as well.

Let us differentiate the motion equation of the system (1) with respect to x , and the Hook's law twice with respect to time; we obtain:

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2 \partial x}, \quad \frac{\partial^2 \sigma_{xx}}{\partial t^2} = E \frac{\partial^2 \varepsilon_{xx}}{\partial t^2}. \quad (2)$$

Taking into account the fact that $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, excluding the strain ε_{xx} from the equations system (2), we have:

$$\frac{\partial^2 \sigma_{xx}}{\partial t^2} = c^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2}, \quad c^2 = \frac{E}{\rho}. \quad (3)$$

The equation (3) indicates that the speed of propagation of the normal stresses σ_{xx} wave in the rod is equal to c . The general solution of the equation (3) in D'Alembert's form is the following:

$$\sigma_{xx}(x, t) = \sigma_{xx}^* \left(t - \frac{x}{c} \right) + \sigma_{xx}^{**} \left(t + \frac{x}{c} \right), \quad (4)$$

where σ_{xx}^* , σ_{xx}^{**} are arbitrary functions which are determined from initial conditions and from boundary conditions, which are written like this:

$$\begin{aligned} t = 0, \quad x > 0, \quad \sigma_{xx} = \frac{\partial \sigma_{xx}}{\partial t} = 0, \\ t > 0, \quad x = 0, \quad \sigma_{xx} = -\sigma^d. \end{aligned} \quad (5)$$

Here σ^d is the actual compressive stress at the butt of the rod. It is determined in this way: $\sigma^d = \frac{P}{F}$, where F is the cross-section area of the rod after deformation. Besides, let us introduce a

conditional stress on the butt of the rod for the elastic domain $\sigma_y^e = \frac{P}{F_0}$.

It is known that the displacement of points of rod in elastic domain is determined by the quadrature:

$$u(x, t) = \frac{1}{E} \int_{ct}^x \sigma_{xx}(x, t) dx. \tag{6}$$

Since the displacement is continuous, from the expression (4), at the wave front $x = ct$, we have the equality:

$$\frac{1}{E} [\sigma_{xx}^*(0) + \sigma_{xx}^{**}(2t)] = 0, \quad t > 0. \tag{7}$$

This equality is held true for arbitrary values of the argument of the function $\sigma_{xx}^{**} : t + \frac{x}{c} > 0$. thus, during the mentioned time t , the function σ_{xx}^{**} is constant. Having included this constant into the arbitrary function σ_{xx}^* , we obtain:

$$\sigma_{xx}(x, t) = \sigma_{xx}^* \left(t - \frac{x}{c} \right). \tag{8}$$

To determinate this function, the boundary conditions (5) are used:

$$x = 0, \quad \sigma_{xx}(x, t) = \sigma_{xx}^*(t) = -\sigma^d. \tag{9}$$

The relations (9) hold true for arbitrary positive argument. This means that:

$$\sigma_{xx}(x, t) = -\sigma^d, \quad t \geq \frac{x}{c}. \tag{10}$$

Using (6) for determination of the displacement $u(x, t)$, we obtain:

$$u(x, t) = -\frac{\sigma^d}{E} \int_{ct}^x dx = -\frac{\sigma^d}{E} c \left(t - \frac{x}{c} \right). \tag{11}$$

Correspondingly, for the velocities V_x of the points of the rod in elastic domain we have:

$$V_x = \frac{\partial u(x, t)}{\partial t} = \frac{\sigma^d}{E} c. \tag{12}$$

The deduced relations (11), (12) are analogical to those obtained in [9], where the dynamic equation of motion is written in displacements. On the basis of introduced assumptions, the obtained results satisfy the condition of compatibility of deformations.

In the plane x, t (Fig. 1), we have the following situation. Below the front of the main wave $x = ct$, the domain of rest is situated. A jump of the stress σ_{xx} , strains ϵ_{xx} , velocity of particles to the values which are determined on the basis of the solution of the equation (3) takes place at the front. During the delay time τ , from the end of the rod, only elastic wave, which transmits the stress σ^d which is set at the end of the rod, is propagating. As all cross-sections of the rod before deformation were in equal states, the delay times for them are equal. This means that in the coordinates x, t the material of the rod will be in elastic state inside the strip which is determined from the inequality:

$$c(t - \tau) \leq x \leq ct. \tag{13}$$

The upper boundary of the strip is a boundary of transition of the rod from elastic state to plastic one. In the curve, the continuity with respect to stresses and deformations remains, and their rates have discontinuities. Therefore, this curve can be considered as the front of a wave of weak discontinuity, which displaces at the velocity of the wave (Fig. 1). Behind this front, the material is in plastic state. To determine the parameters of motion of the medium in plastic domain, let us write the equation dynamic behavior of the material which possess the property of yield delay.

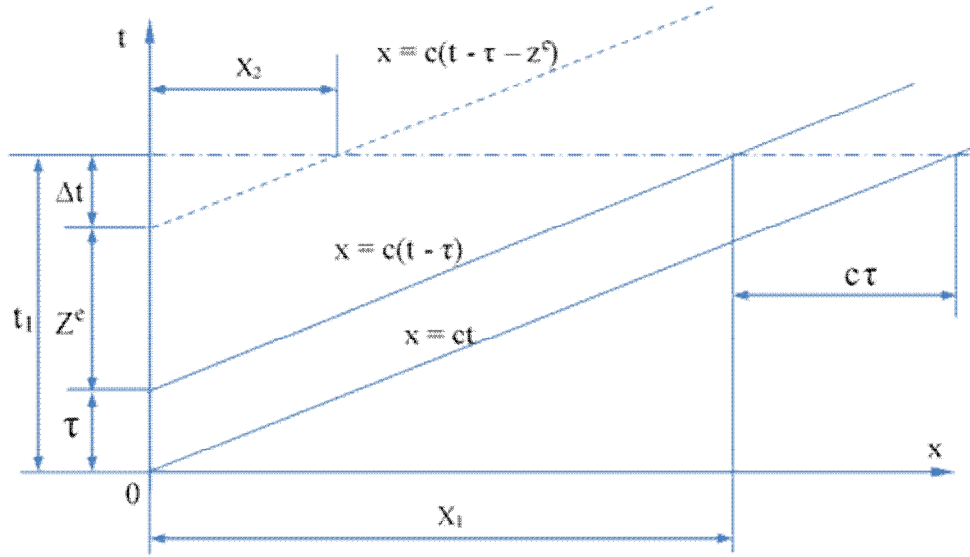


Fig. 1. Distribution of state domains of rod in the plane x, t

Determinative equations of state and dynamic behavior of material of rod in plastic domain

In [1; 9] it is assumed that in investigation of the process of one-dimensional stress wave propagation in rods, the motion of the particles in the direction perpendicular to the x -axis is to be neglected. However, axial deformation is always accompanied by change of cross-section area of a rod. Therefore, instead of actual stress, it is convenient to use so-called conventional stress. The conventional stress is defined as the ration of the force which acts in the cross-section of the rod to the initial area of the cross-section. Later-on, the conventional stress will be called simply stress, excepted the cases when it is necessary to determine actual stress.

Let us consider a domain of the rod which have been transferred into plastic state. For the domain, the equations system (11) holds true; and let us assume that the deformations consist of elastic and plastic parts, i.e.

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p. \quad (14)$$

The material of the rod will be in plastic state for that domain of the variables x, t till the following condition holds true

$$S_{ij}^d S_{ij}^d > \frac{2}{3} \sigma_s^2, \quad (15)$$

where S_{ij}^d are the components of the deviator of actual stresses.

On the basis of ideas of the work [8], the tensor \mathfrak{E}_{ij}^p of velocity of plastic deformation is related to the tensor of stresses by the following dynamic condition of plasticity of material in yielding state

$$\left(S_{ij} - \frac{2}{3} \mu \mathfrak{E}_{ij}^p \right) \left(S_{ij} - \frac{2}{3} \mu \mathfrak{E}_{ij}^p \right) = \frac{2}{3} \left[\sigma_s + k \left(\frac{3}{2} \mathfrak{E}_{ij}^p \mathfrak{E}_{ij}^p \right)^{1/n} \sqrt{\frac{3}{2} \frac{S_{ij}^* S_{ij}^*}{\sigma_s} - 1} \right]^2, \quad (16)$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}, \quad \mathfrak{E}_{ij} = \mathfrak{E}_{ij}^e - \frac{1}{3} \delta_{ij} \mathfrak{E}_{kk}^e,$$

where S_{ij}, S_{ij}^* are the components of the deviator of stresses in the plastic domain and at the boundary of transition from elastic to plastic domain, respectively; σ_{ij} is the component of the tensor of stresses;

$\mathfrak{E}_{ij}^e, \mathfrak{E}_{ij}$ are the components of the tensor and the deviator, respectively, of velocities of elastic

deformations; μ is the coefficient of the toughness; k and n are the constants of the material; δ_{ij} is Kronecker's symbol.

According to the associative law of yield, the components ϵ_{ij}^p are determined by the equations [1]:

$$\epsilon_{ij}^p = \phi \frac{\partial f}{\partial \sigma_{ij}}, \tag{17}$$

where ϕ is the scalar multiplier; f is the plastic potential, which in the given case is determined by the function (16).

Taking into account plastic incompressibility of the material, having substituted (16) into (17), we obtain:

$$\epsilon_{ij}^p = 2\psi \left(S_{ij} - \frac{2}{3} \mu \epsilon_{ij}^p \right) \tag{18}$$

Having solved the equation (18), we have:

$$\epsilon_{ij}^p = \frac{2\phi S_{ij}}{1 + \frac{4}{3} \mu \phi}. \tag{19}$$

After excluding ϵ_{ij}^p from (16) and (19), we obtain:

$$2\phi = \frac{3}{2\mu} \left[\frac{\sqrt{\frac{3}{2} S_{ij} S_{ij}}}{\sigma_s + k \left(\frac{3}{2} \epsilon_{ij}^p \epsilon_{ij}^p \right)^{1/n} \sqrt{\frac{\frac{3}{2} S_{ij}^* S_{ij}^*}{\sigma_s} - 1}} - 1 \right]. \tag{20}$$

Substituting (20) into (19), we write:

$$\epsilon_{ij}^p = \frac{3}{2\mu} \left[1 - \frac{\sigma_s + k \left(\frac{3}{2} \epsilon_{ij}^p \epsilon_{ij}^p \right)^{1/n} \sqrt{\frac{\frac{3}{2} S_{ij}^* S_{ij}^*}{\sigma_s} - 1}}{\sqrt{\frac{3}{2} S_{ij} S_{ij}}} \right] S_{ij}. \tag{21}$$

In the case of one-axial tension-compression, when $\sigma_{xx} \neq 0$, for the components of s_{ij} , we have:

$s_{xx} = \frac{2}{3} \sigma_{xx}$, $s_{yy} = s_{zz} = -\frac{1}{3} \sigma_{xx}$. So, for $s_{ij} s_{ij}$ and $s_{ij}^* s_{ij}^*$, we have:

$$s_{ij} s_{ij} = \frac{2}{3} \sigma_{xx}^2, \quad s_{ij}^* s_{ij}^* = \frac{2}{3} (\sigma_{xx}^d)^2. \tag{22}$$

Analogically we write for the components of ϵ_{ij}^e : $\epsilon_{xx}^e = \frac{2}{3} (1+\nu) \epsilon_{xx}^e$, $\epsilon_{yy}^e = \epsilon_{zz}^e = -\frac{1}{3} (1+\nu) \epsilon_{xx}^e$,

where ν is Poisson coefficient.

Now, we determine $\epsilon_{ij}^e \epsilon_{ij}^e$:

$$\epsilon_{ij}^e \epsilon_{ij}^e = \frac{2}{3} (1+\nu)^2 (\epsilon_{xx}^e)^2. \tag{23}$$

Having substituted (22) and (23) into (21), we obtain:

$$\mathfrak{E}_{xx}^p = \frac{1}{\mu} \left[\sigma_{xx} - \sigma_s - k(1+\nu)^{2/n} (\mathfrak{E}_{xx}^e)^{2/n} \sqrt{\frac{\sigma^d}{\sigma_s} - 1} \right], \quad \mathfrak{E}_{yy}^p = \mathfrak{E}_{zz}^p = -\frac{1}{2} \mathfrak{E}_{xx}^p. \quad (24)$$

Taking into account the relation (14) and Hook's law, for rates of deformations and stresses in one-axial tension-compression we write:

$$\mathfrak{E}_{xx}^p = \mathfrak{E}_{xx} - \mathfrak{E}_{xx}^e, \quad \mathfrak{E}_{xx}^e = \frac{\partial v_x}{\partial x}, \quad \mathfrak{E}_{xx}^e = \frac{\mathfrak{E}_{xx}}{E}. \quad (25)$$

where v_x is the velocity of the particles in plastic domain of the rod along the x -axis.

In the considered problem, all the parameters of motion depend on the x -coordinate and the time t ; therefore, taking into account (25), the determinative equations (24) of the state of the rod in plastic domain are reduced to the following form:

$$\sigma_{xx} = \Phi (\mathfrak{E}_{xx})^{2/n} - \frac{\mu}{E} \mathfrak{E}_{xx} + \mu \mathfrak{E}_{xx} + \sigma_s. \quad (26)$$

$$\text{Here, the following substitution is made } \Phi = \frac{k(1+\nu)^{2/n}}{E^{2/n}} \sqrt{\frac{\sigma^d}{\sigma_s} - 1}.$$

To derive the equation which describes dynamic behavior of the material of the rod in plastic domain, we differentiate the relation (26) with respect to t , and the equation (1) of motion with respect to x ; hence we obtain:

$$\frac{\partial \mathfrak{S}_{xx}}{\partial t} = \frac{2}{n} \Phi \frac{\partial^2 \mathfrak{S}}{\partial t^2} \left(\frac{\partial \mathfrak{S}_{xx}}{\partial t} \right)^{2/n-1} - \frac{m}{E} \frac{\partial^2 \mathfrak{S}_{xx}}{\partial t^2} + m \frac{\partial^3 u}{\partial t^2 \partial x}, \quad \frac{\partial^3 u}{\partial t^2 \partial x} = \frac{1}{r} \frac{\partial^2 \mathfrak{S}_{xx}}{\partial x^2}. \quad (27)$$

Having excluded the displacement u from the system of equations (27), we obtain:

$$\frac{\mu}{\rho} \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \left[\frac{2}{n} \Phi \left(\frac{\partial \sigma_{xx}}{\partial t} \right)^{2/n-1} - \frac{\mu}{E} \right] \frac{\partial^2 \sigma_{xx}}{\partial t^2} - \frac{\partial \sigma_{xx}}{\partial t} = 0. \quad (28)$$

It is sought for differential equation which determine the parameters of the part of the rod which has turned into plastic state after the yield delay by the time τ . Analysis indicates that the equation (28) belongs to elliptic type equations; therefore, a local extension of the domain of variables x, t into initial data is always possible. Let us remind once more that the stress in the equation (28) belong to conventional stresses. To find the solution of the equation (28), we introduce the following independent variables:

$$z_1 = t - \tau - \frac{x}{c}, \quad z_2 = t - \tau + \frac{x}{c}. \quad (29)$$

We present the general solution of (28) in this form:

$$\sigma_{xx}(t, x) = \sigma_{1xx}(z_1) + \sigma_{2xx}(z_2), \quad (30)$$

where $\sigma_{1xx}, \sigma_{2xx}$ are arbitrary functions.

Now we find partial derivatives with respect to the new variables:

$$\begin{aligned} \frac{\partial \mathfrak{S}_{xx}}{\partial t} &= \frac{\partial \mathfrak{S}_{1xx}(z_1)}{\partial z_1} + \frac{\partial \mathfrak{S}_{2xx}(z_2)}{\partial z_2}, & \frac{\partial^2 \mathfrak{S}_{xx}}{\partial t^2} &= \frac{\partial^2 \mathfrak{S}_{1xx}(z_1)}{\partial z_1^2} + \frac{\partial^2 \mathfrak{S}_{2xx}(z_2)}{\partial z_2^2}, \\ \frac{\partial \mathfrak{S}_{xx}}{\partial x} &= -\frac{1}{c} \frac{\partial \mathfrak{S}_{1xx}(z_1)}{\partial z_1} + \frac{1}{c} \frac{\partial \mathfrak{S}_{2xx}(z_2)}{\partial z_2}, & \frac{\partial^2 \mathfrak{S}_{xx}}{\partial x^2} &= \frac{1}{c^2} \frac{\partial^2 \mathfrak{S}_{1xx}(z_1)}{\partial z_1^2} + \frac{1}{c^2} \frac{\partial^2 \mathfrak{S}_{2xx}(z_2)}{\partial z_2^2}. \end{aligned} \quad (31)$$

Having substituted the determined derivatives into the initial equation (28), and taking into account that $E = rc^2$, we have:

$$\frac{2}{n} \Phi \left[\frac{\partial s_{1xx}(z_1)}{\partial z_1} + \frac{\partial s_{2xx}(z_2)}{\partial z_2} \right]^{2/n-1} \left[\frac{\partial^2 s_{1xx}(z_1)}{\partial z_1^2} + \frac{\partial^2 s_{2xx}(z_2)}{\partial z_2^2} \right] - \left[\frac{\partial s_{1xx}(z_1)}{\partial z_1} + \frac{\partial s_{2xx}(z_2)}{\partial z_2} \right] = 0. \quad (32)$$

At the front of transition from elastic state into plastic one, when $x = c(t - \tau)$, the equality $\sigma_{xx}(t, x) = \sigma_y^e = const$ must be satisfied, therefore, from (30) we write:

$$\sigma_{1xx}(0) + \sigma_{2xx}[2(t - \tau)] = \sigma_y^e, \quad t \geq \tau. \quad (33)$$

However, here it is necessary that the equality $\sigma_{2xx}(z_2) = const$ be satisfied. Having included this constant into the arbitrary function $\sigma_{1xx}(z_1)$ and having made the substitution of the designation $z_1 = z = t - \tau - \frac{x}{c}$, we obtain:

$$\sigma_{xx}(t, x) = \sigma_{xx}(z_1) = \sigma_{xx}(z) = \sigma_{xx} \left(t - \tau - \frac{x}{c} \right). \quad (34)$$

The presentation of the solution of the equation (28) in the form (30) enables us to reduce the partial differential equation (28) to an ordinary differential equation in the variables σ_{xx} , z . Thus, from (32) we obtain:

$$\frac{2}{n} \Phi \left(\frac{d\sigma_{xx}}{dz} \right)^{2/n} \frac{d^2\sigma_{xx}}{dz^2} - \left(\frac{d\sigma_{xx}}{dz} \right)^2 = 0. \quad (35)$$

To find its solution, we write the initial conditions:

$$\sigma_{xx}(0) = \sigma_y^e, \quad \frac{d\sigma_{xx}}{dz}(0) = \mathfrak{E}_{xx}(0). \quad (36)$$

Investigation of process of disturbance propagation in plastic domain of rod

As in the front of the interface of elastic and plastic domains the stress σ_y^e is known, the necessity, for the certain reasons, to determine $\mathfrak{E}_{xx}(0)$ arises. For this purpose, we write in actual stresses the equation (26) of plastic state at the boundary of the domains where $z = 0$:

$$\sigma_{xx}^d = \Phi \left(\mathfrak{E}_{xx}^d \right)^{2/n} - \frac{\mu}{E} \mathfrak{E}_{xx}^d + \mu \mathfrak{E}_{xx} + \sigma_s. \quad (37)$$

To determine \mathfrak{E}_{xx}^d from the equation (37), we express \mathfrak{E}_{xx} in terms of the stress σ_{xx}^d and of the rates of \mathfrak{E}_{xx}^d . In the course of deformation of the rod's domain which turns from elastic into plastic state, its cross-section area changes from the initial F_0 in non-stressed state to F in the state which corresponds to the stress σ_{xx}^d . Taking into account incompressibility of plastic deformation, we write the volumetric deformation of the rod in the following form:

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_{xx}^e + 2\varepsilon_{yy}^e = \frac{\sigma_{xx}^d}{E} (1 - 2\nu) \quad \text{i.e.} \quad \varepsilon_{xx} + 2\varepsilon_{yy} = \frac{\sigma_{xx}^d}{E} (1 - 2\nu). \quad (38)$$

Whence, assuming for convenience the stress and strain to be absolute values, we change the corresponding signs for the opposite ones; then we write:

$$2\varepsilon_{yy} = \varepsilon_{xx} - \frac{\sigma_{xx}^d}{E} (1 - 2\nu). \quad (39)$$

Besides, according to the introduced assumptions, homogeneous stress state takes place all over the cross-section area of the rod. Therefore, for the transversal strain ε_{yy} , the following equality is true:

$$2\varepsilon_{yy} = \frac{F}{F_0} - 1 \quad \text{or} \quad 2\varepsilon_{yy} = \frac{\sigma_y^e}{\sigma_{xx}^d} - 1. \quad (40)$$

Equating the right sides of the equations (39) and (40), we write:

$$\varepsilon_{xx} = \frac{\sigma_{xx}^d}{E} (1 - 2\nu) + \frac{\sigma_y^e}{\sigma_{xx}^d} - 1. \quad (41)$$

Taking into account the fact that for small strains [2] $\mathfrak{E}_{xx} = \frac{\partial \varepsilon_{xx}}{\partial t}$, differentiating the equality (41) with respect to t , we obtain:

$$\mathfrak{E}_{xx} = \frac{\mathfrak{E}_{xx}^d}{E} (1 - 2\nu) - \frac{\sigma_y^e}{(\sigma_{xx}^d)^2} \mathfrak{E}_{xx}^d. \quad (42)$$

After the substitution of (42) into (37), we obtain:

$$\sigma_{xx}^d = \Phi \left(\mathfrak{E}_{xx}^d \right)^{2/n} - \mu \left[\frac{2\nu}{E} + \frac{\sigma_y^e}{(\sigma_{xx}^d)^2} \right] \mathfrak{E}_{xx}^d + \sigma_s. \quad (43)$$

From this equation, setting the actual stress σ_{xx}^d , which acts in the cross-section of the rod at $z = 0$, we obtain $\mathfrak{E}_{xx}^d(0)$. To conduct the corresponding numerical calculations, the relation between actual and conventional stresses in the domain (13) is established. It is the domain of the rod which is in elastic state.

Therefore, by means of substitution of $\varepsilon_{xx} = \frac{\sigma_{xx}^d}{E}$ into the equation (41) we obtain:

$$\sigma_y^e = \sigma_{xx}^d \left(1 + \frac{2\nu \sigma_{xx}^d}{E} \right). \quad (44)$$

All our calculations will be conducted for these constants of material which were used in the work [8]. The actual stress at the interface of elastic and plastic domains of the rod, which is equal to the initial stress applied to the bult, we assume to be equal to $\sigma_{xx}^d = \sigma^d = 3,55 \cdot 10^8 \text{ Pa}$. Let us choose the following: $E = 2,1 \cdot 10^{11} \text{ Pa}$, $\sigma_s = 2,15 \cdot 10^8 \text{ Pa}$, $\mu = 1,2 \cdot 10^8 \text{ Pa} \cdot \text{s}$, $k = 1,5 \cdot 10^8 \text{ Pa} \cdot \text{s}^{1/12}$, $\rho = 7800 \frac{\text{kg}}{\text{m}^3}$, $n = 24$, $\nu = 0,25$.

After the substitution of the value of σ_{xx}^d and of corresponding constants of the material into the equation (44), we obtain $\sigma_y^e = 3,553 \cdot 10^8 \text{ Pa}$, and for (43) we have:

$$\sigma_{xx}^d - 1,4 \cdot 10^7 \left(\left| \mathfrak{E}_{xx}^d \right| \right)^{1/12} + 120 \left[2,38 \cdot 10^{-6} + \frac{3,553 \cdot 10^{14}}{(\sigma_{xx}^d)^2} \right] \mathfrak{E}_{xx}^d - 2,15 \cdot 10^8 = 0. \quad (45)$$

By means of solving this equation for the chosen stress $\sigma_{xx}^d = 3,55 \cdot 10^8 \text{ Pa}$, we determine $\mathfrak{E}_{xx}^d(0)$ at the boundary of elastic and plastic domains. The carried out calculations give us $\mathfrak{E}_{xx}^d(0) = -2,0877 \cdot 10^8 \frac{\text{Pa}}{\text{s}}$. The necessary for solving the equation (35) initial condition (36) of the rate conventional stress at $z = 0$ can be determined from the formula (44) after differentiation of the latter. We obtain:

$$\mathfrak{E}_{xx}(0) = \mathfrak{E}_{xx}^d(0) \left(1 + \frac{4\nu \sigma_{xx}^d}{E} \right) = -2,091 \cdot 10^8 \frac{\text{Pa}}{\text{s}}. \quad (46)$$

After substitution of corresponding values of all the constants of the material, the equation (35) assumes the form:

$$1,17 \cdot 10^6 \left(\left| \frac{d\sigma_{xx}}{dz} \right| \right)^{1/2} \frac{d^2 \sigma_{xx}}{dz^2} - \left(\frac{d\sigma_{xx}}{dz} \right)^2 = 0. \quad (47)$$

Its solution is presented in the form of graph of $\sigma_{xx}(z)$, which is shown in Fig. 2. to get the general idea of change of stress in a rod with the front of a wave of transition into plastic state (the wave moves at a speed of $c = \sqrt{\frac{E}{\rho}} = 5100 \frac{m}{s}$) we make the following substitution: $z \rightarrow t - \tau - \frac{x}{c}$.

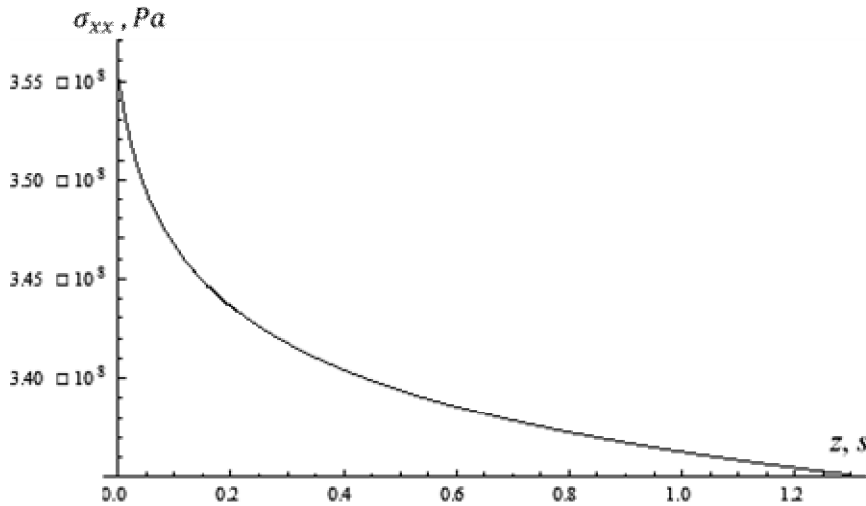


Fig. 2. Graphic presentation of solution of dynamic behavior of rod in plastic domain

Investigations indicate [10] that the time of yield delay in the case of abruptly applied constant stress is approximately equal to the time during which the stress is achieved in constant rate load; i.e. the time during which the material is in “overloaded” state will be determined in the following way:

$$\tau = \frac{\sigma^d - \sigma_s}{\dot{\sigma}_{xx}}. \quad (48)$$

On the other hand, proceeding from the conditions of plasticity (16) of one-axial tension, the yielding of material begins when the following equality is satisfied [8]:

$$\sigma^d - \sigma_s = \frac{k^2}{\sigma_s} (1 + \nu)^{4/n} \left(\frac{\dot{\sigma}_{xx}}{E} \right)^{4/n}. \quad (49)$$

Excluding the rate of stress change from the equations (48) and (49), we obtain:

$$\tau = \frac{k^{n/2} (1 + \nu)}{E \sigma_s^{n/4} (\sigma^d - \sigma_s)^{n/4 - 1}}. \quad (50)$$

Having substituted the values of the corresponding constants and of the stress σ^d into (50), we have $\tau = 1,453 \cdot 10^{-4} s$.

Having carried out the corresponding substitution in the solution of the equation (47) namely, $\sigma_{xx}(z) \rightarrow \sigma_{xx} \left(t - \tau - \frac{x}{c} \right)$, we obtain the field of stresses of the rod in the plastic domain which is shown in Fig. 3. The calculations are made for the following domain of the variables x, t :

$$t \in [\tau, 1,24 + \tau], \quad x \in [0, 3300], \quad c(t - 0,62 - \tau) \leq x \leq c(t - \tau). \quad (51)$$

To determine the other parameters of the state of the rod in the plastic domain, we write the determinative equation (26) in the following form:

$$\dot{\epsilon}_{xx} = \frac{\sigma_{xx}}{\mu} + \frac{1}{E} \dot{\sigma}_{xx} - \frac{\Phi}{\mu} (\dot{\epsilon}_{xx})^{2/n} - \frac{\sigma_s}{\mu}. \quad (52)$$

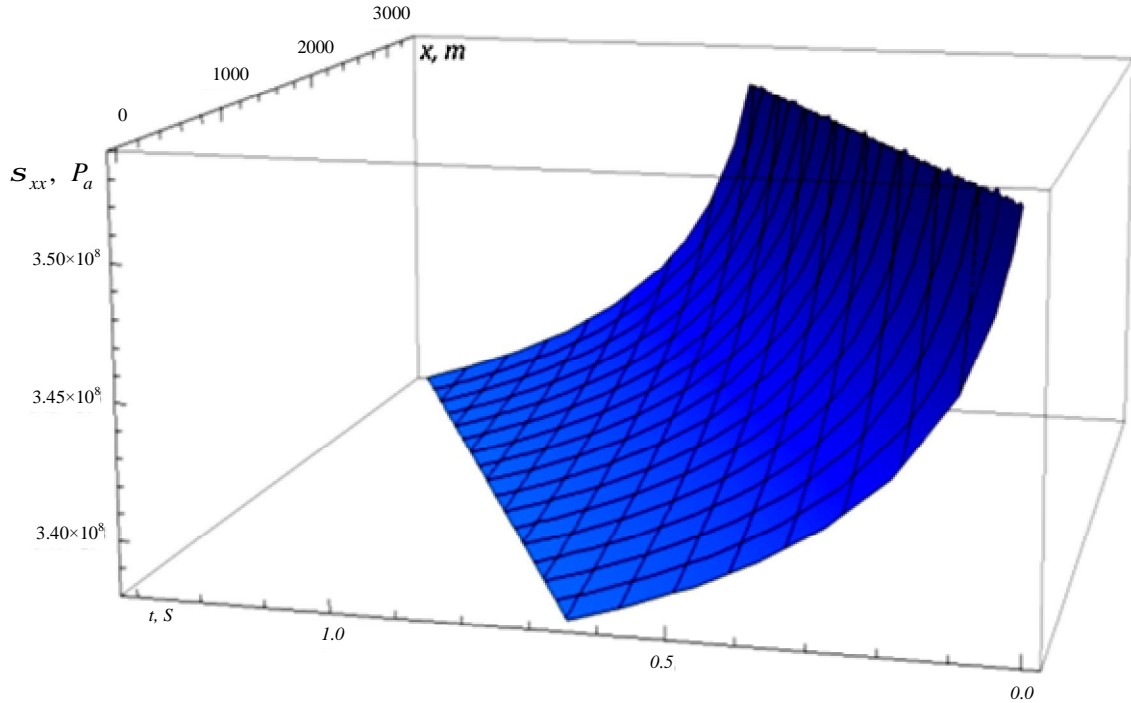


Fig. 3. Field of stresses in plastic domain

After the substitution of all the chosen constants of the material into (52), we obtain:

$$\dot{\epsilon}_{xx} = 8,33 \cdot 10^{-9} \sigma_{xx} + 4,762 \cdot 10^{-12} \dot{\sigma}_{xx} - 0,117 (\dot{\epsilon}_{xx})^{1/12} - 1,7917. \quad (53)$$

The field of rates of strains $\dot{\epsilon}_{xx} \left(t - \tau - \frac{x}{c} \right)$ is determined with a help of the equation (53), into which the obtained on the basis of (47) solution of $\sigma_{xx}(z)$ and the determined rate of the stresses $\dot{\sigma}_{xx}$ are to be substituted. After calculations and the corresponding change, we obtain the general chart of strain rates, it is shown in Fig. 4. The calculation confirms the expected results that at the front of transition from elastic to plastic domain the jump of strain rate from 0 to the value of $\dot{\epsilon}_{xx}(0) = 0,589452 \frac{1}{s}$ takes place.

It is necessary to note that at the interface between elastic and plastic domains the continuity with respect to strain, velocity of parts of the rod and their displacement remain. Therefore, from the equation (11), in the line $x = c(t - \tau)$ (Fig. 1), we determine absolute value of the compressive strain:

$$\epsilon_{xx}(0) = -\frac{\partial u(x,t)}{\partial x} = \frac{\sigma^d}{E} = 0,00169. \quad (54)$$

Correspondingly, for velocities of points in elastic domain, from the equation (12) we have:

$$V_{0x} = \frac{\sigma^d}{E} c = 8,62 \frac{m}{s}. \quad (55)$$

Displacements u_0 of points of the rod at the interface of the domains we determine from the formulae (6) and (10) in the following way:

$$u_0 = -\frac{\sigma^d}{E} \int_{ct}^{c(t-\tau)} dx = \frac{\sigma^d}{E} c\tau = 1,2534 \cdot 10^{-3} m. \quad (56)$$

The distribution of the velocities v_x of points of the rod in plastic domain is determined by means of the use of the solution of (53) and the second relation of (25). Taking into account the established law of signs, introducing the variable z , we write:

$$v_x(x, t) = V_{0x} + c \int_0^{t-\tau-\frac{x}{c}} \epsilon_{xx}(z) dz. \tag{57}$$

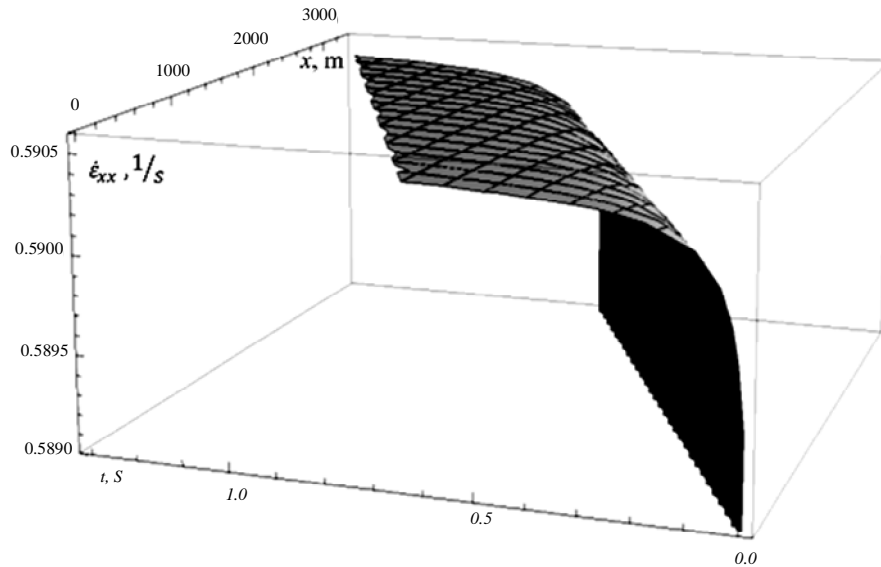


Fig. 4. Field of strain rates in plastic domain

After integration of (57), we obtain the field of velocities of points of the rod in plastic domain (Fig. 5).

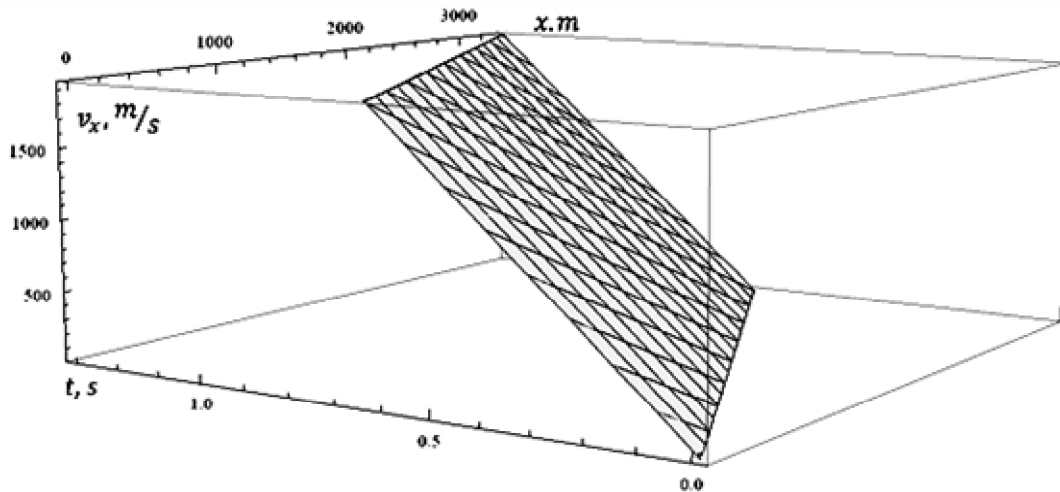


Fig. 5. Distribution of velocities of particles of rod in plastic domain

The general picture of distribution of points of the rod in plastic domain we obtain from the following relation:

$$u(x, t) = u_0 + \int_0^{t-\tau-\frac{x}{c}} v_x dz. \tag{58}$$

The results of calculation for the domain of the variables x, t (51) are shown in Fig. 6.

The field of deformations of the rod is obtained through differentiation of the relation (58) with respect to x . Note that the deformation of particles of the rod can be presented in the following form:

$$\epsilon_{xx}(z) = \frac{1}{c} \frac{du}{dz}. \tag{59}$$

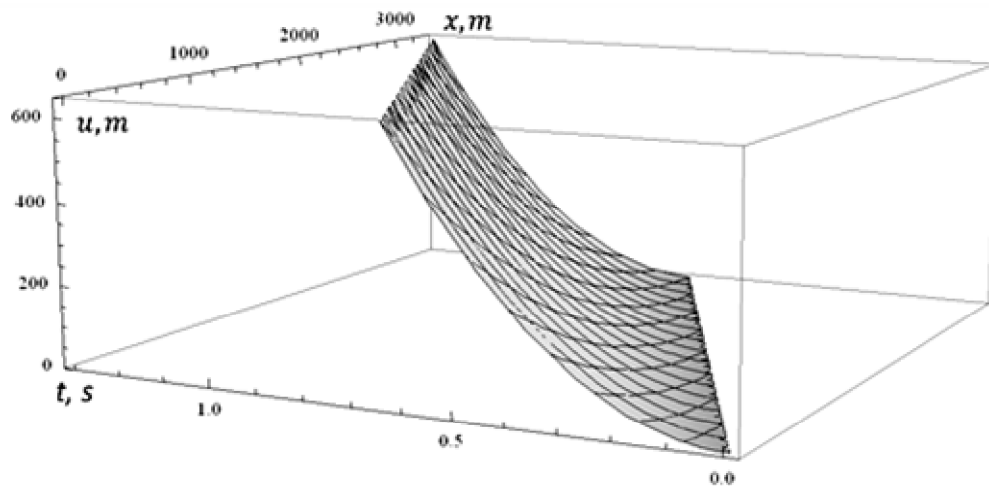


Fig. 6. Distribution of displacements of points of rod behind the front of delimitation between elastic and plastic domains

After corresponding operations and calculations, we find the distribution of strains of particles of the rod in plastic domain (Fig. 7).

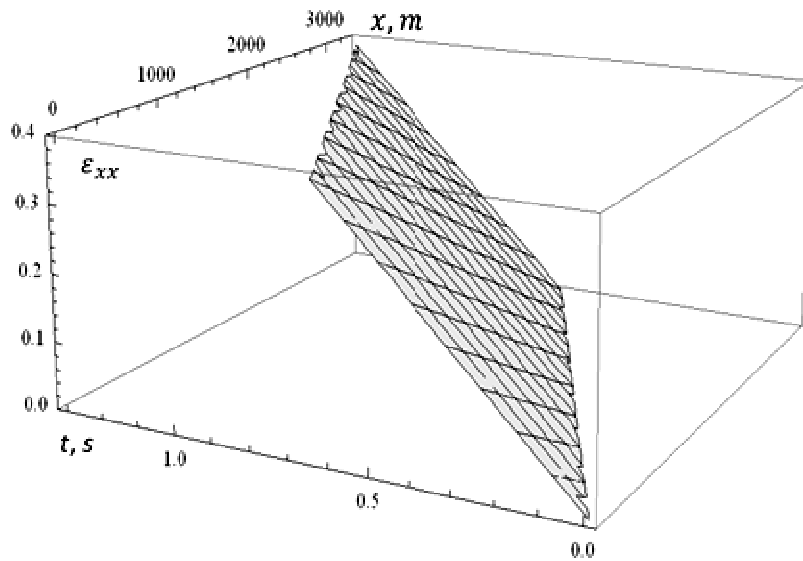


Fig. 7. Field of strain of rod in plastic domain

In the domain of plastic disturbance, the parameters of state, which are determined on the basis of the solution of the equation (47), are given in Fig. 2 – Fig. 7. However, the obtained results are true only in the case when the condition (15) of plasticity is satisfied, i.e. the inequality $\sigma_{xx}^d > \sigma_s$ is observed. Let us determine the domain of the variables x, t , for which the solution of the equation (47) satisfies the condition (15). It is convenient to conduct this analysis with the use of the variable z .

Let us make a transition from the conventional stress σ_{xx} which acts at the butt of the rod, which is obtained from the solution of the equation (47), to the actual stress σ_{xx}^d . For this purpose, we substitute the conventional stress into the relation (39). Together with the equation (40) and the transformed equation (39), we write:

$$2\varepsilon_{yy} = \varepsilon_{xx} - \frac{\sigma_{xx}}{E}(1 - 2\nu), \quad 2\varepsilon_{yy} = \frac{\sigma_y^e}{\sigma_{xx}^d} - 1. \quad (60)$$

Having excluded $2\varepsilon_{yy}$ from (60), we have:

$$\sigma_{xx}^d = \frac{\sigma_y^e}{1 + \varepsilon_{xx} - \frac{\sigma_{xx}}{E}(1 - 2\nu)}. \quad (61)$$

After the substitution of the solutions for $\sigma_{xx}(z)$ from (47) and $\varepsilon_{xx}(z)$ from (59) into (61), we obtain the law of change of actual stress at the butt of the rod (Fig. 8).

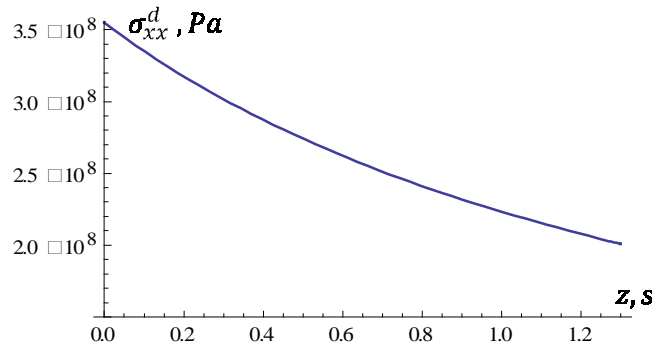


Fig. 8. Law of change of actual stresses in butt cross-section of rod

To determine the value of the variable $z = z^e$ at which the transition of the material of the rod from plastic into elastic state takes place, we make the substitution $\sigma_{xx}^d = \sigma_s$ in the formula (61). Calculations indicate that this takes place when $z^e = 1,1037 s$. Since all cross-sections of the rod change their state from plastic to elastic at the same value of z^e , the interface of these domains in the plane of the coordinates x, t is determined by the following equation:

$$x = c(t - \tau - z^e). \quad (62)$$

The straight line (62) in Fig. 1 is represented by the dotted line. In all points over the line (62), the material is in elastic state with constant values of parameters. Note that, unlike in the established point of view [3–6], the line (62) is not a wave front of elastic unloading, it is a boundary of transition from plastic into elastic state. The domain of plastic deformations is determined from the following inequality:

$$c(t - \tau - z^e) \leq x \leq c(t - \tau). \quad (63)$$

On the basis of the obtained results, graphs of change of stress and strain of points of the rod are plotted for the time instant $t_1 = z^e + \tau + \Delta t = 1,304 s$ (Fig. 1). Here, it is assumed that $\Delta t = 0,2 s$. Let us designate the coordinate of the cross-section of the rod which is at the boundary of transition from elastic to plastic domain by x_1 . Then:

$$x_1 = c(z^e + \Delta t) = 6649 m. \quad (64)$$

Having substituted the time instant t_1 into the equation (62), we obtain the coordinate x_2 of the cross-section at the boundary of transition of the part of the rod from plastic into elastic state:

$$x_2 = c\Delta t = 1020 m. \quad (65)$$

For this cross-section, on the basis of the solution of the equation (47), we obtain:

$$\begin{aligned} \sigma_{xx}(t_1, x_2) &= 335,83 \cdot 10^6 Pa, & u(t_1, x_2) &= 1843,35 m, \\ v_x(t_1, x_2) &= 3332 \frac{m}{s}, & \varepsilon_{xx}(t_1, x_2) &= 0,6533. \end{aligned} \quad (66)$$

In the domain of the rod where $x \leq x_2$, a zone of constant values of parameters emerges; these parameters are equal to the following values:

$$\sigma_{xx}(t, x) = \sigma_{xx}(t_1, x_2), \quad v_x(t, x) = v_x(t_1, x_2), \quad \varepsilon_{xx}(t, x) = \varepsilon_{xx}(t_1, x_2).$$

For simple reasons, the shifts of points of the rod at the time instant t_1 , in the segment $x \in [0, x_2]$, can be presented in the form $u = u(t_1, x) = u(t_1, x_2) + (x_2 - x)\varepsilon_{xx}(t_1, x_2)$, which in terms of numbers leads to the following equation:

$$u = 1843,35 - (x - 1020) \cdot 0,6533. \quad (67)$$

On the basis of the carried out analysis and calculations for all the domains, graphs of stresses and displacements of particles of the rod are plotted for the time instant $t = t_1$; they are shown in Fig. 9 and Fig. 10, respectively.

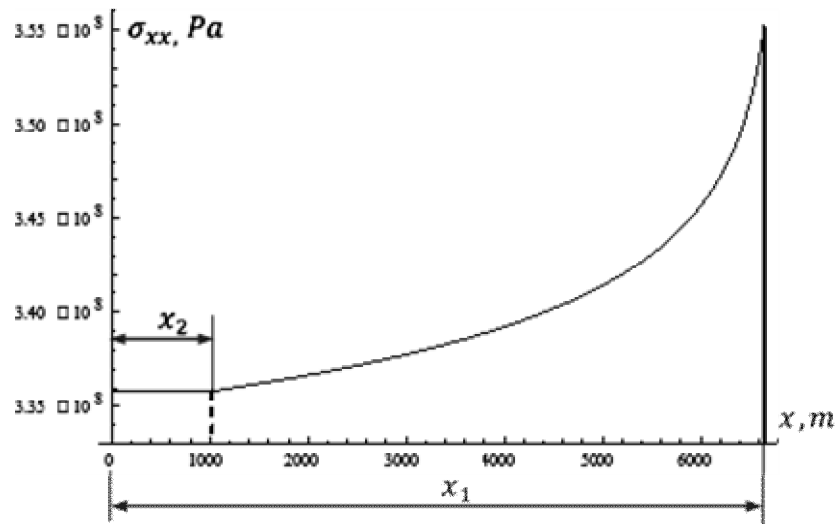


Fig. 9. Distribution of stresses in rod for time instant $t = t_1$

In fig. 9, for $x = x_1$, the elastic domain of the rod behind the front of the main wave of the length $\Delta x = c \cdot \tau = 0,7415 \text{ m}$ is delimited by heavy dotted line (Fig. 1).

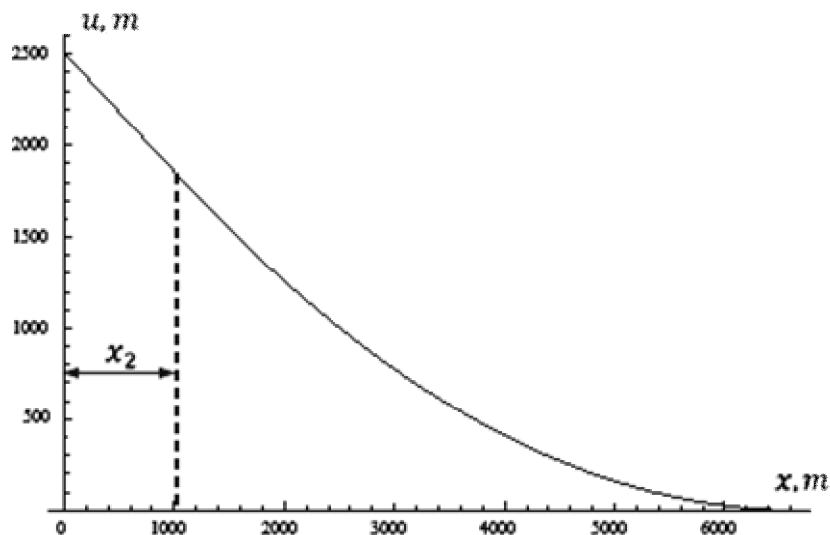


Fig. 10. Distribution of displacements in rod for time instant $t = t_1$

Estimation of obtained results concerning exactness of boundary condition satisfaction

All investigations of dynamic behavior of a rod of material which possesses the phenomenon of yield delay are based on the solution of the equation (47). However, from this equation, conventional stress is determined. The transition from the obtained conditional stress to actual one is given by the relation (61).

Let us make a comparison between the actual stress σ_{xx}^d at the bult of the rod and the true stress σ_{xx}^{di} , which arises from the boundary condition.

Having made the substitution $\sigma_{xx}^d \rightarrow \sigma_{xx}^{di}$ in the equation (41), we write it in the following form:

$$\frac{(1-2\nu)}{E}(\sigma_{xx}^{di})^2 - (1+\epsilon_{xx})\sigma_{xx}^{di} + \sigma_y^e = 0 \tag{68}$$

Here, ϵ_{xx} is the strain at the bult of the rod, which is obtained from the solution of (59). From the equation (68), we determine σ_{xx}^{di} , we have:

$$\sigma_{xx}^{di} = \frac{E}{2(1-2\nu)} \left[1 + \epsilon_{xx} - \sqrt{(1 + \epsilon_{xx})^2 - \frac{4\sigma_y^e}{E}(1-2\nu)} \right] \tag{69}$$

After substituting the values of the constants of the material and parameters into (68), we obtain:

$$\sigma_{xx}^{di} = \left[1 + \epsilon_{xx} - \sqrt{(1 + \epsilon_{xx})^2 - 3,384 \cdot 10^{-3}} \right] 2,1 \cdot 10^{11} \tag{70}$$

With a help of the criterion $\chi = \frac{\sigma_{xx}^d}{\sigma_{xx}^{di}}$, let us estimate the extent to which the actual stress at the bult

of the rod coincides with the true stress which is determined from (70). On the basis of the carried out calculations, graph of the dependence of χ on the variable z is plotted; it is shown in Fig. 11.

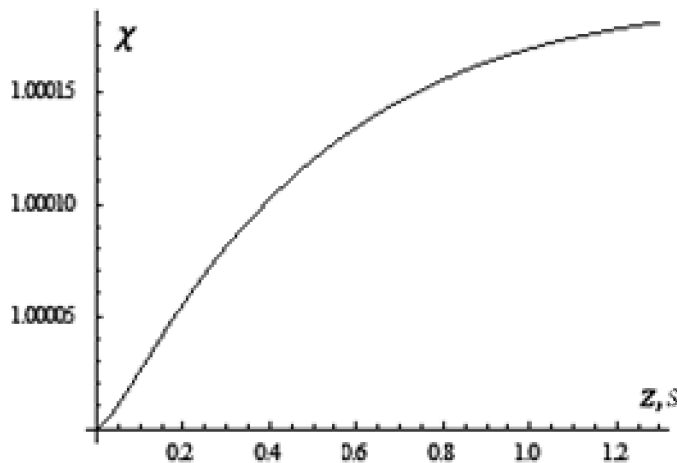


Fig. 11. Graph of estimation of coincidence of stress with true stress

As can be seen from Fig. 11, the boundary conditions at the butt of the rod are satisfied with high accuracy; this is within the established in the problem statements assumptions.

In the investigation of dynamic behavior of the elastic-plastic rod, we deliberately reject the traditional [11] dimensionless form of variable parameters recording in the stated determinative equations. Substitution of corresponding values of materials constants enables us to visually show peculiar distinctions of results, which are obtained within the framework of plastic medium.

Conclusions

The carried out investigation of propagation of elastic-tough-plastic waves in a rod whose material possesses the property of yield delay disproves the false assertion of instantaneous jump-like transition from overstressed elastic state to plastic state. It is established that the characteristic curves which delimit elastic domain from plastic one and separate plastic domain from elastic one are lines of the front of the wave of weak discontinuity. For the plastic domain of the variables x, t , fields of stresses, strains, rates of strains, velocities of particles, and displacements are obtained. A graphic picture of distribution of stresses

and displacements is drawn for a fixed instant of time. Investigations of dynamic behavior of the rod on the basis of the electromechanical model have indicated certain peculiarities in propagation of disturbance. It is established that the differential equations of the state of the rod in plastic domain are of elliptic type. The obtained results satisfy the boundary conditions with high accuracy.

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