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# NONLINEAR OSCILLATIONS OF ELASTIC BEAM INCLUDING DISSIPATION AND THE GALERLKIN METHOD IN THEIR INVESTIGATION



**Досліджена коливальна система, яка моделюється першою змішаною задачею для слабко нелінійного рівняння коливань балки в обмеженій області. Отримані умови існування локального за часовою змінною розв'язку. Особливо виділений коливальний режим із загостренням. Показана можливість застосування до задачі методу Гальоркіна.**

**The paper is devoted to the research of the oscillating system that is described by the first mixed problem for the weakly nonlinear equation of the beam vibrations in a bounded domain. The conditions of the existence of the local, according to a time variable, solution have been obtained. Oscillating blowup regime is especially highlighted. The possibility of the Galerkin method application to the problem is shown.**

**Introduction.** We investigate the solution of the problem on nonlinear transverse vibrations of elastic body subject to the action of dissipative forces in a bounded domain. Such problems have many applications in various technical systems, such as vibration of pipelines, railway lines, drill columns, bridges, electrical wires, optical fiber etc. For the nonlinear oscillations model considered, there is no general analytical methodologies of determining dynamical characteristics of the oscillation process. Thus we propose using the qualitative methods of nonlinear boundary value problems theory to obtain the well-posedness conditions for the problem's local solution in time variable. The methodology of the qualitative study of nonlinear oscillations under the action of dissipative forces is based on the general principles of the nonlinear boundary value problems theory, such as monotony method and Galerkin method.

The one-dimensional nonlinear equation of fifth order with second derivative in time variable of the form $u_{tt} + au_{txxxx} + bu_{xxxx} + G(u) = f$, where $G$ is a certain nonlinear function, generalizes the beam vibration model (see [1] and the bibliography). Most of the applied problems obviously have the action of generalized forces of internal dissipation in oscillating system. In particular, the flexural waves in Voigt-Kelvin bar are described by the fifth order linear equation of the form mentioned [2, c. 60]) which considers the influence of dissipative forces on a dynamic process. The paper focuses on a qualitative study of mathematic model of nonlinear oscillations under the action of dissipative forces. The problem entails considerable mathematical difficulties and is an urgent technical and engineering problem [3], which has been solved in general for very narrow class of problems. In [1], there have been studied the existence of solution of mixed problems in a bounded domain for the system of linear equations where one of the unknown functions describes the vertical displacement of the beam. Problems for nonlinear wave equations of the beam vibration type have been considered, in particular, in [4, 5]. There should be noted that the mathematical aspects of qualitative theory of evolutional partial differential equations of odd order have been a subject of the research of many scientists from second half of 20$^{th}$ century (see, e.g., [6 - 10] and the bibliography). *The aim of this paper* is to investigate a behaviour of the solution of the mixed problem for weakly nonlinear fifth order equation in a bounded domain in spatial variables (beam vibration equation under the action of dissipative forces in the oscillating system) as well as to obtain the boundedness conditions for generalized solution in a finite time moment.

**Problem statement.** Let $Q_T = (0,l)\times(0,T)$, where $T<\infty$, $l<\infty$, $\tau\in[0,T]$. In the domain $Q_T$, for nonlinear equation

$$u_{tt} + au_{txxxx} + bu_{xxxx} + a_0u_t + b_0u = c_0|u|^{p-2}u + f(x,t)\,, \quad p>2 \tag{1}$$

we shall consider the mixed problem with initial conditions

$$u(x,0) = u_0(x)\,, \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x) \tag{3}$$

and boundary conditions

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = 0\,,\ \ u(l,t) = \frac{\partial^2 u(l,t)}{\partial x^2} = 0\,. \tag{4}$$

In equation (1), function $u(x,t)$ is a cross motion of the beam section with $x$ coordinate at arbitrary time moment $t$; $a>0$, $b>0$, $b_0>0$ are constants which can be expressed through geometrical and physico-mechanical parameters of the beam, the constant $a_0>0$ describes the action of resisting forces in the oscillating system (linear case), function $c_0|u|^{p-2}u$ describes the nonlinear elastic forces acting in the system, $f(x,t)$ is the external driving force. Boundary conditions (4) correspond to the model of a beam with fixed pivoting supports on the ends $x=0$ and $x=l$.

From now on we shall assume $p' = p/(p-1)$ and use the cancellation

$$\|v\|_r = \|v\|_{L^r(0,l)} = \left(\int_0^l |v|^r\,dx\right)^{1/r}, \quad r\in[1,+\infty)\,.$$

Concerning the right-hand side of equation (1) ind the initial data, we assume the following conditions to hold:

**(F)** $f,\ f_t \in L^2((0,l)\times(0,\tau_0))$ for any $\tau_0>0$.

**(U)** $u_0 \in H_0^2(0,l)\cap H^4(0,l)\cap L^{2p-2}(0,l)$**;** $u_1\in H_0^2(0,l)\cap H^4(0,l)$.

The function $u\in C([0,T_0];H_0^2(0,l))$ such that $u_t\in C([0,T_0];H_0^2(0,l))$, $u_{tt}\in L^\infty((0,T_0);L^2(0,l))\cap L^2((0,T_0);H_0^2(0,l))$, where $T_0$ is an arbitrary number from the interval $(0,T)$, is called *generalized solution* of problem (1)-(4) in the domain $Q_T$, if it satisfies initial conditions (2)-(3) and the integral equality

$$\int_0^l \left[u_{tt}v + au_{txx}v_{xx} + bu_{xx}v_{xx} + a_0u_tv + b_0uv - c_0|u|^{p-2}uv - fv\right]dx = 0 \tag{5}$$

for almost all $t\in(0,T_0]$ and for any $v\in H_0^2(0,l)\cap L^p(0,l)$. If $T=+\infty$, then the solution is called *global*. If $T<+\infty$, then the solution is called *local*.

**Existence of the oscillation mode without blow-up (local solution of the problem).** ***Main result*****:** *Under the conditions* **(F), (U),** *one can specify a number* $0<T_0<+\infty$ *depending on the coefficients, the right-hand side of the equation and the initial data, such that the generalized solution* $u$ *of problem* (1)-(4) *in the domain* $Q_{T_0}$ *exists.*

Considering the separability of the Banach space $V(0,l) = H_0^2(0,l)\cap L^{2p-2}(0,l)\cap H^4(0,l)$, we shall take in this space a countable set $\{\omega^k\}_{k\in\mathrm{N}}$ such that any finite count of elements of this set is linearly independent and the closure of its linear shell in $V(0,l)$ coincides with $V(0,l)$. Note that $\{\omega^k\}_{k\in \mathrm{N}}$ could be selected as orthonormal in the space $L^2(0,l)$. Concider the functions $u^N(x,t)=\sum_{k=1}^{N} c_k^N(t)\omega^k(x),\ N=1,2...,$

where $c_1^N, c_2^N, ...c_N^N$ are solutions of the corresponding Cauchy problems

$$\int_0^l \left[u_{tt}^N \omega^k + a u_{txx}^N D^\beta \omega_{xx}^k + b u_{xx}^N \omega_{xx}^k + a_0 u_t^N \omega^k + u^N \omega^k - c_0 \left|u^N\right|^{p-2} u^N \omega^k - f\omega^k\right]dx = 0, \qquad (6)$$

$$c_k^N(0) = u_{0,k}^N, \qquad c_{k,t}^N(0) = u_{1,k}^N, \qquad (7)$$

$$u_0^N(x) = \sum_{k=1}^{N} u_{0,k}^N \omega^k,\ \left\|u_0^N - u_0\right\|_{V(0,l)} \to 0,$$

$$u_1^N(x) = \sum_{k=1}^{N} u_{1,k}^N \omega^k,\ \left\|u_1^N - u_1\right\|_{H_0^2(0,l))\cap H^4(0,l)} \to 0,\ N\to\infty, \quad t\in[0,T].$$

On the basis of Caratheodory theorem [11, c. 54], there exists a continuous solution of problem (6), (7), which has an absolutely continuous derivative in $t$ on a certain interval $[0,t_0)$. The estimates obtained below will imply $t_0 = T$, and besides, $T$ will be determined later. Multiplying (6) by $c_{k,t}^N$, summing up over $k$ from $1$ to $N$ and integrating over $t$ from $0$ to $\tau < T$, we obtain

$$\frac{1}{2}\int_0^l \left(u_t^N(x,\tau)\right)^2 dx + \int_0^l\int_0^\tau \left[a\left(u_{txx}^N\right)^2 + b u_{xx}^N u_{txx}^N + \right.$$

$$\left. + a_0\left(u_t^N\right)^2 + b_0 u^N u_t^N - c_0\left|u^N\right|^{p-2} u^N u_t^N - f(x,t)u_t^N\right]dxdt = \frac{1}{2}\int_0^l \left(u_1^N\right)^2 dx. \qquad (8)$$

Let's estimate integrals (8). We have

$$\int_0^l\int_0^\tau b u_{xx}^N u_{txx}^N dxdt \ge \frac{b}{2}\int_0^l \left(u_{xx}^N\right)^2 dx - \frac{b}{2}\int_0^l \left(u_{0,xx}^N\right)^2 dx, \quad \int_0^l\int_0^\tau f(x,t)u^N dxdt \le \int_0^l\int_0^\tau \left[\frac{1}{2}\left(u^N\right)^2 + \frac{1}{2}\left(f(x,t)\right)^2\right]dxdt.$$

Further estimating yields

$$\int_0^l\int_0^\tau c_0\left|u^N\right|^{p-2} u^N u_t^N dxdt \le c_0 C_1 \int_0^l\int_0^\tau \left|u^N\right|^p dxdt + C_2\int_0^l\int_0^\tau \left|u_t^N\right|^p dxdt \le$$

$$\le C_3\int_0^l \left|u_0^N\right|^p dx + C_2\int_0^l\int_0^\tau \left|u_t^N\right|^p dxdt \le C_4\int_0^l \left|u_0^N\right|^p dx + C_5\int_0^\tau\left(\int_0^\tau \left|u_t^N\right|^p dx\right)^{p/2} dt,$$

where $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ are positive constants which do not depend on $N$.

In view of the estimates above, from (8) we obtain the inequality

$$\int_0^l \left(u_t^N\right)^2 dx + \int_0^l \left(u_{xx}^N\right)^2 dx + \int_0^l\int_0^\tau \left(u_{txx}^N\right)^2 dxdt \le C_6\int_0^l \left|u_0^N\right|^p dx + C_7\int_0^l \left(u_1^N\right)^2 dx + C_8\int_0^l \left(u_{0,xx}^N\right)^2 dx +$$

$$+ C_9\int_0^l\int_0^\tau \left(f(x,t)\right)^2 dxdt + C_{10}\int_0^l\int_0^\tau \left(u^N\right)^2 dxdt + C_{11}\int_0^l\int_0^\tau \left(u_{xx}^N\right)^2 dxdt + C_{12}\int_0^\tau\left(\int_0^\tau \left|u_t^N\right|^p dx\right)^{p/2} dt,\ \tau\in(0,T), \quad (9)$$

where $C_6$ - $C_{12}$ are positive constants which do not depend on $N$. Using the Bellman-Gronwall lemma, from (9) we obtain

$$\int_0^l \left(u_t^N\right)^2 dx + \int_0^l \left(u_{xx}^N\right)^2 dx + \int_0^l\int_0^\tau \left(u_{txx}^N\right)^2 dxdt \le M_1 + M_2 \int_0^\tau \left(\int_0^\tau \left|u_t^N\right|^p dx\right)^{p/2} dt, \tag{10}$$

$\tau \in (0,T)$, the positive constants $M_1$, $M_2$ depend on the coefficients, the right-hand side of the equation and the initial data, and they do not depend on $N$. Applying the Bihari lemma [12, p. 110] to inequality (10), we obtain

$$\int_0^l \left(u_t^N\right)^2 dx + \int_0^l \left(u_{xx}^N\right)^2 dx + \int_0^l\int_0^\tau \left(u_{txx}^N\right)^2 dxdt \le \frac{2M_1}{[2-(p-2)M_1^{(p-2)/2} M_2 T]^{2/(p-2)}} \tag{11}$$

under $T < \dfrac{2}{(p-2)M_1^{(p-2)/2} M_2}$. Hence, considering the theorem conditions, from (11) we obtain

$$\left\|u^N\right\|_{L^\infty((0,T_1);H_0^2(0,l))} \le M_3, \quad \left\|u_t^N\right\|_{L^2((0,T_1);H_0^2(0,l))\cap L^\infty((0,T_1);L^2(0,l))} \le M_3, \tag{12}$$

where the positive constant $M_3$ does not depend on $N$, $T_1 \in (0,T)$.

Further differentiating the integral equality (5) over $t$, multiplying the result equality by $c_{k,tt}^N$, summing up over $k$ from 1 to $N$ and finally integrating over $t$ from $0$ to $\tau$, $\tau \in (0,T_1]$, we obtain:

$$\frac{1}{2}\int_0^l \left(u_{tt}^N(x,\tau)\right)^2 dx + \int_0^l\int_0^\tau \Big[ a\left(u_{ttxx}^N\right)^2 + b u_{txx}^N u_{ttxx}^N + a_0\left(u_{tt}^N\right)^2 +$$

$$+ b_0 u_t^N u_{tt}^N - c_0(p-1)\left|u^N\right|^{p-2} u_t^N u_{tt}^N - f_t(x,t) u_{tt}^N \Big] dxdt = \frac{1}{2}\int_0^l \left(u_{tt}^N(x,0)\right)^2 dx. \tag{13}$$

Using the conditions above, let us estimate the summands of (13). Note that in the same way as it is described above, one can obtain the following inequality:

$$\int_{\Omega_\tau} |u_{tt}^N|^2\, dx + \int_{Q_\tau} \sum_{|\alpha|=2} |D^\alpha u_{tt}^N|^2\, dxdt \le M_4 + M_5 \int_0^\tau \left(\int_\Omega |u_{tt}^N|^2\, dx\right)^{p/2} dt, \tag{14}$$

$\tau \in (0,T_1)$, and the positive constants $M_4$, $M_5$ do not depend on $N$. Applying the Bihari lemma to inequality (14), we obtain

$$\int_0^l \left(u_{tt}^N\right)^2 dx + \int_0^l\int_0^\tau \left(u_{ttxx}^N\right)^2 dxdt \le \frac{2M_4}{[2-(p-2)M_4^{(p-2)/2} M_5 T]^{2/(p-2)}} \tag{15}$$

under $T < \dfrac{2}{(p-2)M_4^{(p-2)/2} M_5}$. Hence, from (15) we have

$$\left\|u_{tt}^N\right\|_{L^\infty((0,T_2);L^2(0,l))\cap L^2((0,T_2);H_0^2(0,l))} \le M_6, \tag{16}$$

where the positive constant $M_6$ does not depend on $N$, $T_2 \in (0,T)$. Assume that

$T_0 = \mathbf{min}\left\{\dfrac{2}{(p-2)M_1^{(p-2)/2} M_2}; \dfrac{2}{(p-2)M_4^{(p-2)/2} M_5}\right\}$. Note that

$$\int_0^l\int_0^{T_0} \left|\left|u^N\right|^{p-1}\right|^{p'} dxdt \le M_7, \tag{17}$$

where $T_0 \in (0,T)$, the positive constant $M_7$ does not depend on $N$. From (12), (16), (17) it follows that there exists a subsequence $\{u^{N_m}\}_{N_m \in \mathrm{N}} \subset \{u^N\}_{N \in \mathrm{N}}$ (for convenience, denote it again by $\{u^N\}$) such that

$$u^N(\cdot,T_0) \to * - weakly\ in\ L^2(0,l), \quad u^N \to u\ * - weakly\ in\ L^\infty((0,T_0);H_0^2(0,l)),$$
$$u^N \to u \quad weakly\ in\ L^p((0,T_0);L^p(0,l)), \quad u_t^N \to u_t\ * - weakly\ in\ L^\infty((0,T_0);L^2(0,l)),$$
$$u_t^N \to u_t\ \ weakly\ in\ L^2((0,T_0);H_0^2(o,l)), \quad u_{tt}^N \to u_{tt}\ * - weakly\ in\ L^\infty((0,T_0);L^2(0,l)),$$
$$u_{tt}^N \to u_{tt}\ \ weakly\ in\ L^2((0,T_0);H_0^2(0,l)), \quad \left|u^N\right|^{p-2} u^N \to \chi\ \ weakly\ in\ L^{p'}(Q_{T_0}),\ N \to \infty.$$

Since $\{u_t^N\}$ is bounded in $L^2((0,T_0);H_0^2(0,l))$, and $\{u^N\}$ is bounded in $L^p(Q_{T_0})$ and $H_0^2(0,l) \subset L^p(0,l)$ compactly, therefore on the basis of [13, c.70] $u^N \to u$ strongly in $L^p(Q_{T_0})$ and almost everywhere in $Q_{T_0}$. Therefore $\chi = |u|^{p-2} u$. It is easy to show that the function $u$ satisfies the integral equality (5) and the initial conditions (2)-(3).

On Figure 1, there shown a dependence of the critical value $T_0$ on generalized parameters of the oscillating system $M^*$, $M^{**}$ under various values of the exponent $p$, which describes nonlinearly elastic characteristics of the medium: a) $p = 2.1$; b) $p = 3$; c) $p = 5$.

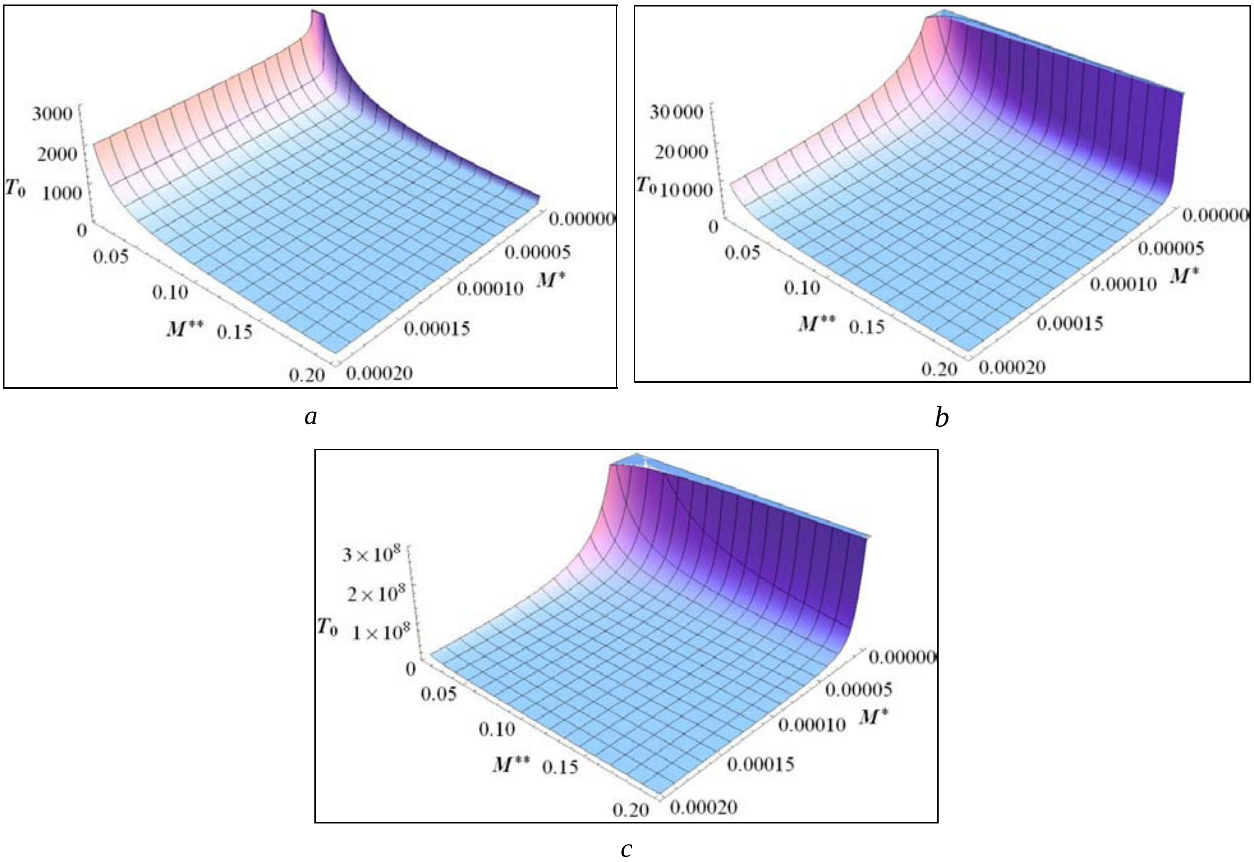


*Fig. 1. Dependence of the value $T_0$ on generalized parameters of the oscillating system under various values of the exponent $p$*

***Remark***. In the case of negativity of the energy functional at the initial time moment $E(0) = \frac{1}{2}\int_0^l \left[u_1^2(x) + b(u_{0,xx})^2 + b_0(u_0)^2\right]dx - \frac{1}{p}\int_0^l c_0 |u_0|^p\, dx < 0$ and under assumption that $c_0 > 0$, there exists no global solution of problem (1)-(4) [14], i.e. there exists a blow-up oscillation mode [15]. In other words, there exists such a finite $T^* > 0$, for which $\lim\limits_{t \to T^* - 0} \int\limits_0^l \left[ u_t^2 + (u_{xx})^2 + |u|^p \right] dx = +\infty.$

**Conclusions.** The main result of the proposed work is extending the class of nonlinear partial differential equations modeling the oscillation processes in elastic media, for which one can obtain the conditions of boundedness for the solution of mixed problem at a finite time moment as well as the conditions of existence of a blow-up oscillation mode. It is important to note that the nonlinear wave equation of beam vibration type studied in the paper, has a lot of applications while developing mathematical models of physico-mechanical processes, in particular, in nonlinear oscillation theory. The qualitative results of analysis of the oscillating system that we have obtained in the paper, show the possibility of applying Galerkin method to the problem considered as well as they allow further applying various approximate methods while investigating the dynamical characteristics of the solutions.

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