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ABOUT COMMUNICATION OF MANY-MULTIPLE COVERING OPTIMIZATION PROBLEMS OF BOUNDED SETS AND PROBLEMS OF MULTIPLEX-PARTITIONING

There is proposed a modification of optimality criterion in the continuous problem of optimal multiplex-partitioning of a bounded set from *n*-dimensional Euclidean space, which allows in the result of its solving receive the smallest radius of the multiple covering by balls of this set.

Keywords: continuous problem of multiple covering, optimal k-multiple covering by balls, Voronoi diagrams of higher orders, multiplex-partitioning of sets

Entrance

The problems of multiple ball covering of 2-dimensional area occur in different scopes of human activity. A wide range of its practical applications has shown, ex., in [1 - 3].

Largely models of covering problems studied in the sciencific literature are discrete. Known problems 0-1 covering and multiple covering as integer linear programming problems [4, 5]. Problems, where covered set are continuous in the scientific literature called continuous covering problems [2 - 7, etc.]. Well learnt is the problem of single covering by circles with limited part of square, whose common name is the *p*-centers problem. For it are proposed different heuristic algorithms and algorithms, based by Voronoi-diagram. A large bibliography about the development of algorithms for solving the *p*-centers problem was given in [8]. In [9] are proposed use the higher order Voronoi areas [10] and smoothing of objective function for search of descent direction in solving min-max problems, which just are mathematical models for continuous problems optimal covering. In [2,3,5-7] are presented numerical algorithms for problems of solving multiple covering bounded sets N by balls minimum radius.

In [11] are presented mathematical formulations so-called continuous problems of optimal multiplex set partitioning. These problems describe he situation, when is necessary to break stated area by regions, which cover customers with the same k nearest neighbor service centers from N existing (or possible). It is assumed, that customers from every area can be served by any of nearest k centers. In models proposed in [11], partitioning criteria is liniar and consist of minimize total expenses to provision or receipt of some service.

Obviously, the continuous set covering problems and multiplex partitioning problems are cognate in their interpretations. The goal of this work is demonstration connection between stated problems in differnt their formulations, possibility define the radius of ball-covering in process of solving continuous linear problems of multiplex set partitioning. Also explore view of quality multiplex set partitioning criteria, that in proper placing centers task could be get placing same as placing on balls minimum radius centers covered k –multiple stated set.

Mathematical formulations of continuous ball covering problems and multiplex set partitioning problems

At first give mathematical formalization of continuous problems of multiple covering of bounded set, which are more constructive in respect of solving algorithms development [3,5].

Let Ω – bounded, is Lebesgue measurable on a closed set in space E_n, $\tau_i = \left(\tau_i^{(1)},...,\tau_i^{(n)}\right) \in \Omega$, for all $i = 1,\ldots,N$, – some points, called «centers» (they can be fixed or be subject to definition). $B(\tau_i, R) = \{x \in E_n : c(x,\tau_i) \le R\}$ – ***c***-ball R radius with center in point τ_i from Ω, where $c(x,\tau_i)$ – metrics.

The problem of search radius *N* circles, which creating *k*–multiple ***c***-ball set covering, consist of search value

$$\bar{R} = \sup_{x\in\Omega} \min_{\lambda(x)\in\Lambda_N^k} \max_{i=\overline{1,N}} c(x,\tau_i)\lambda_i(x), \quad (1)$$

where $\Lambda_N^k = \left\{ \lambda = (\lambda_1,...,\lambda_N) : \lambda_i = 0 \vee 1, \ i = \overline{1,N}; \sum_{i=1}^{N} \lambda_i = k \right\}$.

The problem of minimum *k*–multiple ***c***-ball covering is written in such way: find

$$\bar{R}(\lambda^*(\cdot),\tau_*^N) = \inf_{(\tau_1,...,\tau_N)\in\Omega^N} \sup_{x\in\Omega} \min_{\lambda(x)\in\Lambda_N^k} \max_{i=\overline{1,N}} c(x,\tau_i)\lambda_i(x), \quad (2)$$

and vector-function $\lambda^*(\cdot)$: $\forall x \in \Omega$ $\lambda^*(x) \in \Lambda$, and vector $\tau_*^N = (\tau_1^*,...,\tau_N^*) \in \Omega^N \subset E_n^N$, in which in (2) is achieved bottom limit.

Next present mathematical formalization of continuous linear optimal multiplex sets partitionig problem [11]. Intrduce next designations: $\mathrm{N} = \{1,2,...,N\}$ – all centers indexes set; $\mathrm{M}(\mathrm{N},k)$ – set of all k-size subsets of set N, $|\mathrm{M}(\mathrm{N},k)| = C_N^k = L$; $\sigma_l = \{j_1^l, j_2^l,..., j_k^l\}$, $l = \overline{1,L}$, – elements of set $\mathrm{M}(\mathrm{N},k)$. With every element σ_l of set $\mathrm{M}(\mathrm{N},k)$ will assotiate some subset Ω_{σ_l} by points from Ω, $l = 1,2,...,L$. In turn, with this subset Ω_{σ_l} will assotiate centers set $\{\tau_{j_1^l}, \tau_{j_2^l},...,\tau_{j_k^l}\}$.

The totality of Lebesgue measurable subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, \ldots \Omega_{\sigma_L}$ from $\Omega \subset E_n$ will call k **-order partitioning of set** Ω on it subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, \ldots \Omega_{\sigma_L}$, disjoint, if

$$\bigcup_{i=1}^{L} \Omega_{\sigma_i} = \Omega, \ \mathrm{mes}\left(\Omega_{\sigma_i} \cap \Omega_{\sigma_j}\right) = 0, \ \sigma_i \in \mathrm{M}(\mathrm{N},k), \ i \ne j, \ i, j = 1,\ldots,L,$$

where $\mathrm{mes}(\cdot)$ means Lebesgue measure. The subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, \ldots \Omega_{\sigma_L}$ of set Ω call k **-order subsets** of this set.

Let $\Sigma_\Omega^{N,k}$ is the class of all possible k-order partitionings of set Ω on it disjoint subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, \ldots \Omega_{\sigma_L}$:

$$\Sigma_\Omega^{N,k} = \left\{ \bar{\omega} = \{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\} : \bigcup_{i=1}^{L} \Omega_{\sigma_i} = \Omega; \mathrm{mes}\left(\Omega_{\sigma_i} \cap \Omega_{\sigma_j}\right) = 0, i \ne j, \sigma_i, \sigma_j \in \mathrm{M}(\mathrm{N},k), \ i, j = 1,\ldots,L \right\}.$$

Task A1- k **.** Continuous linear problem of optimal k-order partitioning of set $\Omega \subset E_n$ on it disjoint subcets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, \ldots, \Omega_{\sigma_L}$ (can consist empty), with fixed centers $\tau_1,...,\tau_N$ without restrictions:

$$F\left(\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\}\right) \to \min_{\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\}\in\Sigma_\Omega^{N,k}},$$

$$F\left(\left\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\right\}\right)=\sum_{l=1}^{L}\int_{\Omega_{\sigma_l}}\sum_{i\in\sigma_l}\left(c(x,\ \tau_i)/w_i+a_i\right)\rho(x)dx,$$

where $x=\left(x^{(1)},...,x^{(n)}\right)\in\Omega$; $\tau=\left(\tau_1,...,\tau_i,...,\tau_N\right)\in\Omega^N$, coordinates $\tau_i^{(1)},...,\tau_i^{(n)}$ of center τ_i, $i=1,...,N,$ are fixed; functions $c\left(x,\tau_i\right)$ − bounded, define on $\Omega\times\Omega$, measurable by argument x by any fixed $\tau_i=\left(\tau_i^{(1)},...,\tau_i^{(n)}\right)$ from Ω for all $i=1,..,N$; $\rho(x)$ − bounded, measurable, not-negative on set Ω function; $w_i>0, a_i\geq 0,\ i=\overline{1,N}$, − stated numbers.

k-order partitioning $\overline{\omega}^*=\left\{\Omega^*_{\sigma_1},\ldots,\Omega^*_{\sigma_L}\right\}$ of set $\Omega\subset E_n$, reaches minimum value of funtional F, will call **optimal solving of task A1-**k.

If in task **A1-**k centers coordinates $\tau_1,...,\tau_N$ are unknown in advance, and it must be defined together with k-order partitioning $\overline{\omega}^*=\left\{\Omega^*_{\sigma_1},\Omega^*_{\sigma_2},\ldots,\Omega^*_{\sigma_L}\right\}$ of set $\Omega\subset E_n$, then get new task.

Task A2-k**.** The continuous linear problem of optimal k-order partitioning of set $\Omega\subset E_n$ on it disjoint subsets $\Omega_{\sigma_1},\Omega_{\sigma_2},\ldots,\Omega_{\sigma_L}$ (can consist empty), unlimited in centers placing $\tau_1,...,\tau_N$:

$$F\left(\left\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\right\},\left\{\tau_1,...,\tau_N\right\}\right)\to\min_{\substack{\left\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\right\}\in\Sigma_\Omega^{N,k}\\ \left\{\tau_1,...,\tau_N\right\}\in\Omega^N}},$$

where functional has view

$$F(\overline{\omega},\tau^N)=F\left(\left\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\right\},\left\{\tau_1,...,\tau_N\right\}\right)=\sum_{l=1}^{L}\int_{\Omega_{\sigma_l}}\sum_{i\in\sigma_l}\left(c(x,\ \tau_i)/w_i+a_i\right)\rho(x)dx, \quad (3)$$

where $x=\left(x^{(1)},...,x^{(n)}\right)\in\Omega$; $\tau=\left(\tau_1,...,\tau_i,...,\tau_N\right)\in\Omega^N$; functions and parameters the same as in task **A1-**k.

The pair $\left(\overline{\omega}^*,\tau_*^N\right)=\left(\left\{\Omega^*_{\sigma_1},\ldots,\Omega^*_{\sigma_L}\right\},\left\{\tau_1^*,\ldots,\tau_N^*\right\}\right)$, provided minimum value of functional (1),will call **optimal solving of task A2-**k.

The concept of method solving the problem of optimal multiplex-partitioning of set with fixed centers

By analogy with methodology of solving the continuous linear problems of optimal set partitioning [2] (wich are individual cases tasks **A1-**k or **A2-**k) output task **A1-**k white as next task infinity-dimensional mathimatical programming with boolean variables.

Let $\overline{\omega}=\left\{\Omega_{\sigma_1},...,\Omega_{\sigma_l},\ldots\Omega_{\sigma_L}\right\}$ – some partitioning k-order of set Ω. For every point $x\in\Omega_{\sigma_l}$, $l=\overline{1,L}$, set by conformity N-dimensiona vector $\lambda^l(x)=(\lambda_1^l(x),...,\lambda_N^l(x))$, which coordinates define by such way:

$$\lambda_i^l(x)=\begin{cases}1,\ x\in\Omega_{\sigma_l}\ \&\ i\in\sigma_l,\\ 0\ \mathit{in\ other\ case}\end{cases}\quad i=1,...,N,\ l=1,...,L, \quad (4)$$

where $\sigma_l \in \mathrm{M}(\mathrm{N},k)$, $\sigma_l = \{j_1^l, j_2^l, ..., j_k^l\}$ – set of indexes of centers $\{\tau_{j_1^l}, \tau_{j_2^l}, ..., \tau_{j_k^l}\}$, which are associated with subset Ω_{σ_l}.

Vector-function $\lambda^l(x) = (\lambda_1^l(x), ..., \lambda_N^l(x))$, wich is define on set Ω, with coordinates, which are set by formula (4), will call characteristic vector-function of subset Ω_{σ_l}, which is consisted in partitioning k-order of set Ω.

Task **A1-**k is reformulated concerning characteristic vector-functions of subsets, which constitute k-order partitioning of set Ω.

Task B1-k. $\min\limits_{\lambda(\cdot)\in\Gamma_0^k} I(\lambda(\cdot))$, $I(\lambda(\cdot)) = \int\limits_\Omega \sum\limits_{l=1}^{L} \left(\sum\limits_{i=1}^{N} (c(x,\ \tau_i)/w_i + a_i)\lambda_i^l(x) \right) \rho(x)dx$,

$$\Gamma_0^k = \Big\{ \lambda(x) = (\lambda^1(x), ..., \lambda^l(x), ..., \lambda^L(x)) :\ \lambda^l(x) = (\lambda_1^l(x), ..., \lambda_N^l(x));$$

$$\lambda_i^l(x) = 0 \vee 1 \ for\ x \in \Omega, i = \overline{1,N},\ l = \overline{1,L};\ \sum_{i=1}^{N} \lambda_i^l(x) = k, l = \overline{1,L}, \text{ м.в. for } x \in \Omega \ \Big\};$$

$\tau = \left(\tau_1, ..., \tau_N\right) \in \underbrace{\Omega \times ... \times \Omega}_{N} = \Omega^N$ – the stated vector.

The optimal solving of task **B1-**k is reached on the vector-function $\lambda^*(x) = (\lambda_*^1(x), ..., \lambda_*^l(x), ..., \lambda_*^L(x))$, each component $\lambda^l(x)$, $l = \overline{1,L}$, of which is calculated by formula: almost everywhere for $x \in \Omega$

$$\lambda_{*i}^l(x) = \begin{cases} 1, \text{ if } c(x,\tau_i)/w_i + a_i \le c(x,\tau_j)/w_j + a_j, \\ \quad \text{simultaneously } \forall i \in \sigma_l,\ j \in \mathrm{N} \setminus \sigma_l, \text{ and then } x \in \Omega_{\sigma_l}^*, \quad i = \overline{1,N}. \\ 0 \ \text{ in other cases,} \end{cases} \tag{5}$$

The functional of task **B1-**k with $\lambda(\cdot) = \lambda^*(\cdot)$ is written as such way:

$$I(\lambda^*(\cdot)) = \int\limits_\Omega \min_{l=\overline{1,L}} \left(\sum_{i\in\sigma_l} \big(c(x,\ \tau_i)/w_i + a_i \big) \right) \rho(x)dx.$$

For task **B2-**k, equal **A2-**k and written concerning characteristic functions of subsets, costituted partitioning k-order of stated set Ω, the optimal solving can be get by next formulas: almost everywhere for $x \in \Omega$

$$\lambda_{*i}^l(x) = \begin{cases} 1, \text{ if } c(x,\tau_{*i})/w_i + a_i \le c(x,\tau_{*j})/w_j + a_j, \\ \qquad \forall i \in \sigma_l, \mathrm{j} \in \mathrm{N} \backslash \sigma_l \text{ , and then } x \in \Omega_{*\sigma_l}, \quad i = 1, ..., N, \quad l = \overline{1,L}, \\ 0 \ \text{ in other cases,} \end{cases}$$

as $\tau_{*1}, ..., \tau_{*N}$ selected optimum solving of task

$$G(\tau) \to \min_{\tau\in\Omega^N}, \tag{6}$$

where

$$G(\tau) = \int\limits_\Omega \min_{\sigma_l\in\mathrm{M}(\mathrm{N},k)} \sum_{i\in\sigma_l} [c(x,\tau_i) + a_i]\rho\big(x\big)dx. \tag{7}$$

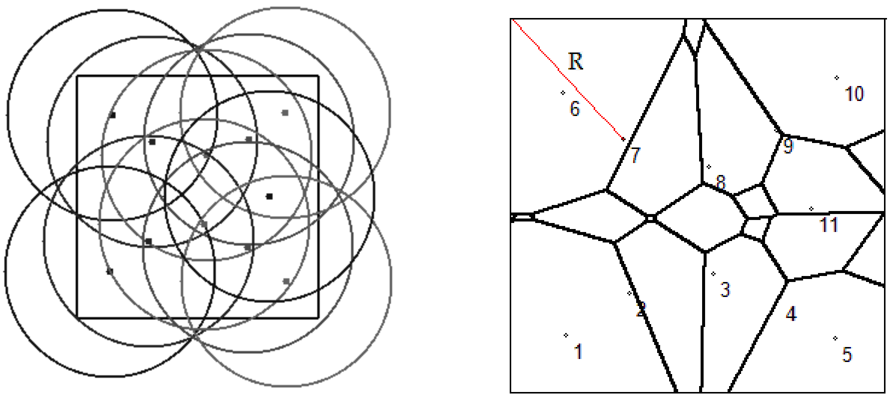
The comparing of results of solving continuous covering problems and multiplex set partitioning

In the time of implementation numeric algorithm for solving the multiplex set partitioning task **A1-** k , as notated above, is possible in same time define covering radius proper multiple of this set as value max distance between center and the farthest point in subset k -order, calculated by (1).

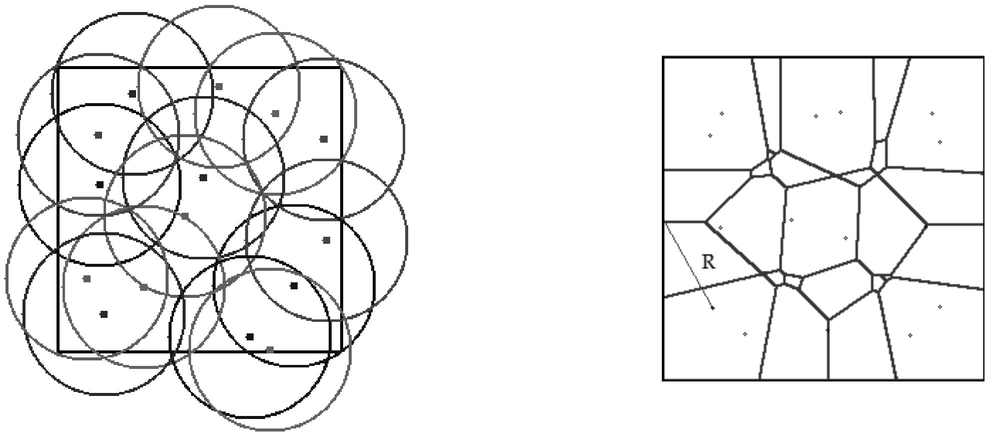
On Pic. 1 are presented two-multiple ball covering and optimality duplex partitioning of forsquare $\Omega = [0,1]\times[0,1]$ with N=15 fixed points with such parameters: $c(x,\tau_i)$ – the Euclidean metrics, $a_i = 0, w_i = 1, i = \overline{1,N}$; $\rho(x) = 1\ \forall x \in \Omega$. The covering radius value in both tasks is R=0.4353. Hereinafter in partitioning caovering radius is marked by thin line. On pic. 2, 3 shown optimal two-multiple and three-multiple ball coverings, and optimal partitionig of set $\Omega = [0,1]\times[0,1]$ by proper order with placing N=15 and 17 centers respectively. Calculated radius of two-multiple covering is: R=0.28714 and R=0.2961 (Pic. 2); R=0.33264 and R=0.33232 (Pic. 3).

The radiuses of ball covering, calcilated in process solving the continuous tasks of optimal duplex partitioning with N=13,15, 17, 19, 21 are in the table 1. For comparing in table 1 are presented the results of solving proper optimal 2-multiple ball covering tasks using algorithms from [2] and [6].

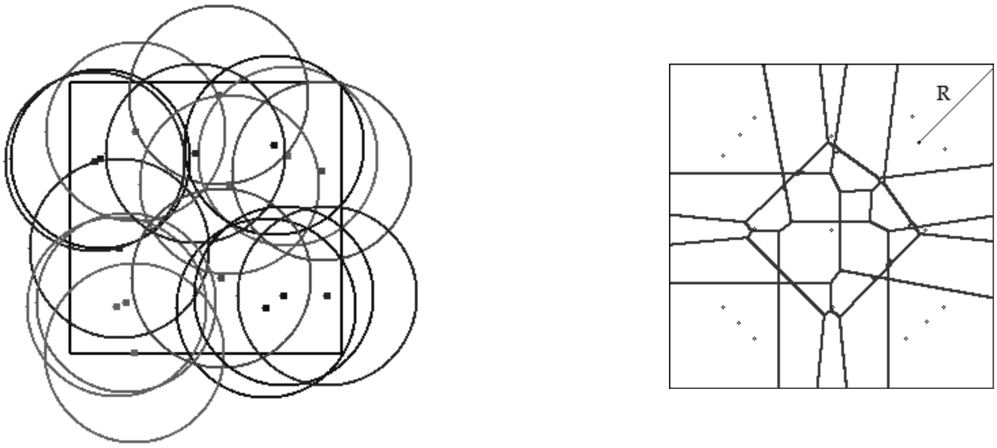
Notated, that solving task (6) is realized using Shor's r-algorithm [12]. Given the fact that this algorithm provides only local minimum non-smooth function search, but task (6) – many-extremal having different start approximations of centers coordinates, there is possible get different local solutions of optimal multiplex-partitioning tasks, and in accordance the radius of many-multiple covering by balls, which centers are the solutions of task (6).



Pic. 1 Two-multiple covering and duplex partitioning of square with N=11 fixed centers



Pic. 2 Two-multiple minimal covering and optimal duplex partitioning of square with placing N=15 centers



Pic. 3 Three-multiple minimal covering and optimal threeplex partitioning of square with placing N=17 centers

Table 1. Minimal radius of two-multiple covering of unitary square

N	Value R(τ^*), get		
	in [2]	using algorithm 2 from [6]	in process solving duplex partitioning task
11	0.31280	0.3164	0.34042
13	0.29106	0.29700	0.29614
15	0.26650	0.27807	0.28771
17	0.25372	0.26570	0,27308
19	0.22766	0.23685	0.23529
21	0.21601	0.22500	0.23306

The minimization of radius of many-multiple covering of bounded set as criteria of optimality in task multiplex set partitioning

Much computational experiments solving multiplex-partitioning tasks with notated above optimality criteria and solving many-multiple ball covering tasks led to the following conclusions. In case fixed centers many-multiple covering radius value get by solving (1) and task **A1-**k are equal. This is false when in multiplex-partitioning task centers are must locate. Are appearing questions: which is must be the target functional of optimal multiplex set partitioning task with centers locating or which is must be partitioned set and metrics on it for situation described above will be true?

Will propose such modify of quality criteria of multiplex-partitioning tasks in **A2-**k **:**

$$F_R(\bar{\omega},\tau^N) = \max_{l=\overline{1,L}} \sup_{x\in\Omega_{\sigma_l}} \max_{i\in\sigma_l}\left(c(x,\ \tau_i)/w_i + a_i\right)\rho(x).$$

It's easy to see, if in $F_R(\bar{\omega},\tau^N)$ choose next parameters: $c(x,\tau_i)$ – the Eucledian metrics, $a_i = 0,\ w_i = 1,\ i = \overline{1,N};\ \ \rho(x) = 1\ \forall x \in \Omega$, then task **A2-**$k$ with such criteria of multiplex-partitioning become equal for task (2) of search set covering by stated number of minimum-radius balls. Thus, it's possible to get another constructive mathematical formalization of optimal many-multiple ball covering problem and it varios generalizations.

Conclusions

So, on examples of solving continuous many-multiple ball set covering problems, and problems of search proper multiple partitioning of these sets, there is shown possiblility to define covering radius in process of solving continuous linear problems of multiplex set partitioning. For multiplex set partitioning problem with centers locating there is proposed the quality covering criteria, which let get such centers placing, that is equal with centers of **minimum**-radius balls covering k –multiple stated set.

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