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CIRCUIT MATHEMATICAL MODELS OF ELECTRIC DEVICES

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Abstract. On the basis of the main laws of electro–magnetic circuit theory we propose general approach to creating four concepts of circuit mathematical models of electrical devices. It is shown that by the same assumptions their discrete analogues are different. Owing to this idea, the real possibility to present differential equatons of industrial electrical devices (static and electromechanic ones) and the systems formed by them in the normalized Cauchy form is shown, which is very important for the analysis of the long-term processes.

Key words: electromagnetic circuit theory, four mathematical models of electrical devices.

1. Introduction

Three types of mathematical models of electric devices can be recognized: circuit, circuit-field and field ones [1]. In this article only the circuit model is described. The mathematical model is created using the equations of the device, based on some assumptions. Therefore, an erroneous conclusion can be drawn, that all mathematical models built on the same assumptions are equivalent, if they are made up according to strict mathematical rules. However, it is necessary to remember that a mathematical model is completed by creating its discrete analogue. And exactly this analogue determines the amount of calculations and their precision, as well as competitiveness of the model. Therefore, the concepts of creating mathematical models possess an important place in mathematical modeling. There are four of them according to [1]. Each of them is built on the basis of the theory of electromagnetic circuits.

Electric, magnetic and electromagnetic circuits are distinguished. The electromagnetic circuit contains the features of two previous which are considered to be its sub-circuits. The analysis of electric circuits is related to the determination of the integral values of an electric field, namely, currents and electric voltages. The analysis of magnetic circuits is related to the determination of the integral values of a magnetic field – magnetic fluxes and magnetic voltages. In electro-magnetic circuits the first and second values are equally unknown. So, from this point of view electrical devices are first and foremost electromagnetic circuits [2].

2. Formulation of the technical problem

The electric sub-circuits of electrical devices in most cases consist of separate electric circuits, made by magnetic windings and their electric voltages. When necessary, it is possible to impose any structural connectivity on these windings on the basis of the Kirchhoff law. The differential equations of the separate electric circuits of the sub-circuit are the equations of real inductance coils

$$d\Psi_{ik}/dt = u_{ik} - r_{ik}, \quad i = 1,2,...,m_k;\ k = 1,2,...,n, \qquad (1)$$

where Ψ_{ik}, u_{ik}, i_{ik} are linkage flux, electric voltage and current of the *ik*-th winding; r_{ik} is its resistance. Subscripts *i* and *k* show its belonging to the *i*-th winding from the set m_k and the *k*-th magnetic branch from the *n,* thus m_k is the number of windings of the *k*- th magnetic branch.

Linkage flux Ψ_{ik} conditionally will give as sum

$$\Psi_{ik} = L_{sik} i_{ik} + w_{ik}\Phi_k, \qquad (2)$$

where Φ_k is the magnetic flux of the *k*-th branch; w_{ik} is the number of winding turns; L_{sik} is dissipation inductance.

The first element in right part (2) represents dissipation linkage flux, and the second is the main linkage flux of a winding. Magnetic force lines of dissipation flux partly or fully flow through air. Therefore, it is possible to accept with sufficient accuracy that L_{sik} = const, otherwise the field models should be used.

Solving (2) in relation to a current, we obtain

$$i_{ik} = \alpha_{ik}(\Psi_{ik} - w_{ik}\Phi_k), \qquad (3)$$

where $\alpha_{ik}=1/L_{sik}$ is reverse dissipation inductance.

Unknown magnetic streams in (3) are found from the structural equations of the magnetic sub-cirquit

$$\sum_{k=1}^{n}\Phi_k = 0; \quad \sum_{k=1}^{n}\sum_{i=1}^{n_k} V_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{m_k} F_{ik}, \qquad (4)$$

where n_k is the number of magnetic elements of k -th magnetic branch, and from the equations of elements

$$V_{ik} = \rho'_{ik}(\Phi_k)\Phi_k; \ F_{ik} = w_{ik} i_{ik}, \qquad (5)$$

where V_{ik}, F_{ik} are magnetic voltage and magnetomotive force (MMF) of ik -th element of a branch, and

$$\rho'_{ik}(\Phi_k) = V_{ik}(\Phi_k)/\Phi_k \qquad (6)$$

is static magnetic resistance that is obtained by the magnetization curve $V_{ik}(\Phi_k)$ of a magnetic conductor. Resistance of a nonmagnetic gap is $\rho'_{ik} = \text{const}$.

Expressions (1), (3)–(6) form the complete system of fundamental algebraic differential equations describing the electromagnetic circuit of the electrical device. On the basis of fundamental equations the analysis can not be performed. They only provide the foundation for building $\mathrm{N}-, \Psi-, \mathrm{A}-, L$ – mathematical models.

Principle of construction of all mentioned above mathematical models at the level of theory of electric machines was offered by us in [3]. In this work, the extension of the method on the level of general theory of electromagnetic circuits was carred out, that extends possibilities of the method of construction of mathematical models of any implementation.

3. N-model

This model is the oldest (it was developed in precomputer times) and nearest to the fundamental equations of electromagnetic circuits. Here we only describe the method of solving the nonlinear algebraic equations of the magnetic sub-circuit.

Inserting (5) in contour equations (4), we obtain

$$\sum_{k=1}^{n}\Phi_k = 0; \quad \sum_{k=1}^{n}\sum_{i=1}^{n_k}\rho'_{ik}(\Phi_k)\Phi_k = \sum_{k=1}^{n}\sum_{i=1}^{m_k} w_{ik} i_{ik}. \qquad (7)$$

Expression (3) gives an opportunity to calculate the unknown MMFs

$$F_{ik} = w_{ik}\alpha_{ik}(\Psi_{ik} - w_{ik}\Phi_k). \qquad (8)$$

Inserting (8) in (7), together with node equations (4), we obtain the equations of magnetic fluxes

$$\sum_{k=1}^{n}\Phi_k = 0; \sum_{k=1}^{n}\left(\sum_{i=1}^{n_k}\rho'_{ik}(\Phi_k) + \sum_{i=1}^{m_k}\alpha_{ik}w_{ik}^2\right)\Phi_k = \sum_{k=1}^{n}\sum_{i=1}^{m_k}\alpha_{ik}w_{ik}\Psi_{ik}. \qquad (9)$$

If necessary, the order of the system (9) can be reduced, passing to the equations of contour magnetic fluxes, node magnetic voltage and others.

Algebraic differential equations (1), (3) (9) form the mathematical N-model of the electrical device. It is the best for the investigation of processes by the implicit methods of integration in case of stiff differential equations.

4. Ψ-model

The prevailing majority of practical cases is not connected with stiffness of differential equations. The necessity of creating iteration cycles at every point of time for solving nonlinear equations (9) in a previous model is a too laborious procedure related to the problem of convergence. Therefore, it is a reason for replacing these equations with differential ones.

Differentiating (9) with respect to time and inserting (1), we obtain

$$\sum_{k=1}^{n}\frac{d\Phi_k}{dt} = 0; \sum_{k=1}^{n}\left(\sum_{i=1}^{n_k}\rho''_{ik}(\Phi_k) + \sum_{i=1}^{m_k}\alpha_{ik}w_{ik}^2\right)\frac{d\Phi_k}{dt} = \sum_{k=1}^{n}\sum_{i=1}^{m_k}\alpha_{ik}w_{ik}(u_{ik} - r_{ik}i_{ik}), \qquad (10)$$

where $\rho''_{ik}(\Phi_k)$ is differential magnetic resistance

$$\rho''_{ik}(\Phi_k) = dV_{ik}(\Phi_k)/d\Phi_k, \qquad (11)$$

which can be found by the same curve $V_{ik}(\Phi_k)$ that the static one (6).

Differential equations (1), (10) together with independent expression (3) form the mathematical Ψ-model of the electrical device. It is the best for co-operation in circuit-field models. Its differential equations can always be presented in the normalized Cauchy form [1], avoiding the same laborious procedure at every point of time in integration numerical methods

$$\frac{d\Phi_k}{dt} = f_k(\Phi_1, \Phi_2, ..., \Phi_n), \quad k = 1, 2, ..., n. \qquad (12)$$

5. A-model

The A-model is the improved variant of the Ψ-model. The linkage flux is here eliminated, being of no practical interest.

Differentiating (3) with respect to time and inserting (1), (12) into the resulting equation, we finally obtain

$$\frac{di_{ik}}{dt} = \alpha_{ik}\left((u_{ik} - r_{ik}i_{ik}) - w_{ik}f_k(\Phi_1, \Phi_2, ..., \Phi_n)\right), \quad i = 1, 2, ..., m_k; \ k = 1, 2, ..., n. \qquad (13)$$

Differential equalizations (12), (13) form the mathe–matical A-model of the electrical device. It is the best for the analysis of transitional and steady-state processes in the widest range of electrical devices. Its differential equations can also be noted in the normalized Cauchy form [1].

6. *L*-model

The *L*-model is based on the concepts of inductances of electric circuits, which are the constituent part of the theory of electric circuits. The name of the model originates from the symbol *L* used for inductances in the circuitry. The *L*-model was first introduced by P. Sylvester in 1965 [4]. On the base of the electric circuit theory he proposed a formula for calculating the coefficients of self- and mutual induction between two moving electric circuits in physical coordinates. This area is truly developed by R. Filts [5]. But in the area of coordinate transformations of differential equation theory the formula proposed by P. Sylvester suits only for orthogonal circuits. We managed to obtain a relevant formula for arbitrarily oriented contours [1]. It gave us the possibility to work freely on the theory of differential equations. And on the base of the theory of elektromagnetic circuits creating the *L*-models is simplified considerably, as it is shown below.

Inserting (2) in (1) and differentiating the result with respect to time, we obtain

$$L_{sik}\frac{di_{ik}}{dt}+w_{ik}\frac{d\Phi_k}{dt}=u_{ik}-r_{ik},\ i=1,2,...,m_k;\ k=1,2,...,n. \quad (14)$$

By the rules of analysis of electric circuits, the derivative $d\Phi_k/dt$ must be eliminated from (14), as Φ is an integral value that characterizes magnetic or electromagnetic circuits. Just that realization of this idea results in the *L*-model. For this purpose we differentiate (7) with respect to time

$$\sum_{k=1}^{n}\frac{d\Phi_k}{dt}=0;\ \sum_{k=1}^{n}\sum_{i=1}^{n_k}\rho''_{ik}(\Phi_k)\frac{d\Phi_k}{dt}=\sum_{k=1}^{n}\sum_{i=1}^{m_k}w_{ik}\frac{di_{ik}}{dt}, \quad (15)$$

where $\rho''_{ik}(\Phi_k)$ is differential magnetic resistance (11).

Solving (15) in relation to the derivatives of fluxes and inserting the result in (14), we obtain

$$(L_{si}+L_i)\frac{di_i}{dt}+\sum_{i=1,i\neq k}^{m}L_{ik}\frac{di_k}{dt}=u_i-r_ii_i,\ i,k=1,2,...,m, \quad (16)$$

where L_i, L_{ik} are differential inductances

$$L_i=w_i^2/\rho''_i(\Phi_i);\quad L_{ik}=w_iw_k/\rho''_i(\Phi_i), \quad (17)$$

and $m=\sum_k m_k$ is the general number of excited windings.

The coefficients L_i and L_{ik} are the functions of fluxes, so equations (12) and (16) should be integrated together. They form the needed *L*-model.

As usual, in the theory of electric circuits it is very difficult to solve equations (16). It is evident that in the theory of electromagnetic circuits this problem does not exist.

7. Comparative analysis of the proposed models

Let us generalize the results obtained.

The equation of the *N* -model in general case is possible to be written as follows

$$d\Psi/dt=U-RI;\ \Phi=\Phi(\Psi);\ I=\Lambda(\Psi-W\Phi), \quad (18)$$

where Ψ, *U*, *I* are columns of linkage fluxes, electric voltages, currents; Φ is the column of main magnetic fluxes; *R* is the matrix of winding resistances; Λ is the matrix of inverse dissipation inductances; *W* is the matrix of the number of winding turns.

The equations of the Ψ -model differ from (18) that the equations of the relation between the main fluxes and linkage fluxes are replaced by the equations of the relation between their increments

$$d\Psi/dt=U-RI;\ d\Phi/dt=Gd\Psi/dt;\ I=\Lambda(\Psi-W\Phi), \quad (19)$$

where *G* is the matrix of the relations between the increments of main fluxes and linkage fluxes.

The equations of the A-model are

$$dI/dt=\mathrm{A}(U-RI);\ d\Phi/dt=G(U-RI), \quad (20)$$

where A is the matrix of inverse inductances.

The equations of the *L*-model can be noted as follows

$$LdI/dt=(U-RI);\ d\Phi/dt=G(U-RI), \quad (21)$$

where *L* is the matrix of inductances.

The analysis of the last three mathematical models intended for the explicit methods of numerical integration (19)–(21) results in such statements:

1. All three mathematical models are obtained through the same assumptions, so they are equivalent from the point of view of mathematical analysis, however from the point of view of adequacy to the numerical methods they are absolutely different.

2. The A-model is the most perfect one, since its differential equation for the most common devices can always be written in the normalized Cauchy's form [1] that enables using their discrete analogues. It is the only mathematical model which does not require the subtraction of two neighbour values. This advantage increases the accuracy of calculations and makes possible the investigations of long-term processes.

3. The Ψ-model is the easiest way to create cirquit-field and field mathematical models.

4. The traditional *L*-model is the most unusable one. The discrete analogue of differential equations is found here through the inverse matrix of inductances $\mathrm{A}=L^{-1}$. However, it is not advisable to decline the *L*-models completely. They can be applied in the tasks of analysis of the simple systems by the method of loop currents. It should be mentioned, that this method is used rather seldom, because real systems contain many independent circuits and the limited number of independent nodes. Therefore, in such a case the method of node voltages is more preferable.

5. The *N*-model is the most suitable for the implisit methods of numerical integration and it is as good as the A-model in these cases.

It should be noted that mixed models may be built where some circuits are created according to the Ψ–, A – or *L*-models. The construction of such models is obvious – it is realized by the rules of the construction of «pure» models.

When a device contains movable electric circuits, the equations of the considered mathematical models are complemented by the equations of motion [1].

7. Examples of application

For evident comparison let us show all four models of the simplest device at the accepted denotations of a double-wound transformer [2].

1. N -model:

$$d\Psi_i / dt = u_i - r_i i; \; i_i = \alpha_i(\Psi_i - w_i\Phi), \;\; i = 1,2;$$
$$\left(\rho'(\Phi) + \sum_{i=1}^{2} w_i^2\alpha_i\right)\Phi = \sum_{i=1}^{2} w_i\alpha_i\Psi_i. \quad (22)$$

2. Ψ-model:

$$d\Psi_i / dt = u_i - r_i i_i; \; i_i = \alpha_i(\Psi_i - w_i\Phi), \;\; i = 1,2;$$
$$\frac{d\Phi}{dt} = \frac{1}{\rho''(\Phi) + \sum_{i=1}^{2} w_i^2\alpha_i}\sum_{i=1}^{2} w_i\alpha_i(u_i - r_i i_i). \quad (23)$$

3. A-model:

$$\frac{di_i}{dt} = \alpha_i(1 - \alpha_i A_i)(u_i - r_i i_i) - \\ - \sum_{k=1,k\neq i}^{2} \alpha_i\alpha_k A_{ik}(u_k - r_k i_k), \; i = 1,2, \quad (24)$$

where

$$A_i = \frac{w_i^2}{\rho''(\Phi) + \sum_{i=1}^{2} w_i^2\alpha_i}; \;\; A_{ik} = \frac{w_i w_k}{\rho''(\Phi) + \sum_{i=1}^{2} w_i^2\alpha_i}. \quad (25)$$

4. L-model:

$$(L_{si} + L_i)\frac{di_i}{dt} + \sum_{k=1,k\neq i}^{2} L_{ik}\frac{di_k}{dt} = u_i - r_i i_i, \; i = 1,2, \quad (26)$$

where L_i, L_{ik} are differential inductances:

$$L_i = w_i^2 / \rho''(\Phi); \;\; L_{ik} = w_i w_k / \rho''(\Phi). \quad (27)$$

We should notice that differential equations (24) and (26) are integrated jointly with differential equation of fluxes (23). It is possible not to use one differential equation, for example, the first one, finding the current according to algebraic equation (7)

$$i_1 = (\rho'(\Phi)\Phi - w_2 i_2) / w_1. \quad (28)$$

8. Conclusion

1. Each of the offered four types of circuit mathematical models has its purpose:

– N -model is used in the algorithms of the integration of the differential equations of state by implicit methods;

– A-model is used in the algorithms of the integration of differential equations by explicit methods, and also for the analysis of the complex systems in node coordinates, e.g. the equation of multipolar elements;

– L-model is intended for the analysis of the complex systems in contour coordinates, e.g. the equation of multipolar elements;

– Ψ-model is intended for the cooperation in building of circuit-field models building.

2. The A-model is the most suitable one for the analysis of the long-term processes.

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КОЛОВІ МАТЕМАТИЧНІ МОДЕЛІ ЕЛЕКТРОТЕХНІЧНИХ ПРИСТРОЇВ

Василь Чабан

На підставі основних положень теорії електро–магнетних кіл, пропоновано загальний підхід до по–будови чотирьох видів колових математичних моделей електротехнічних пристроїв, кожна з яких має своє призначення. Показано, що при однакових допущеннях їхні дискретні аналоги суттєво відрізняються. На цій підставі вперше отримано реальну можливість представити диференціальні рівняння промислових електротехнічних пристроїв (статичних і електромеханічних) та систем, що вони утворюють, у нормальній формі Коші, що дуже важливо для аналізу тривалих перехідних процесів.

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