ГЕОДЕЗІЯ

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ESTIMATION OF THE POTENTIAL GRAVITATIONAL ENERGY OF THE EARTH BASED ON REFERENCE DENSITY MODELS

The estimation of the Earth's gravitational potential energy *E* based on the given density distribution is considered. The global density model was selected as combined solution of the 3D continuous distribution and reference radial piecewise profile with basic density jumps as sampled for the PREM density. This model preserves the external gravitational potential from zero to second degree/order, the dynamical ellipticity, the planet's flattening, and basic radial density-jumps. The rigorous error propagation of adopted density parameterization was derived to restrict a possible solution domain. Comparison of lateral density anomalies with estimated accuracy of density leads to values of the same order in uncertainties and density heterogeneities. As a result, radial-only density models were chosen for the computation of the potential energy *E*. *E*-estimates were based on the expression $E = -(W_{\min} + \Delta W)$ derived from the conventional relationship for *E* through the Green's identity. The first component $W_{\min}$ expresses some minimum amount of the work *W* and the second component ΔW represents a deviation from $W_{\min}$ treated via Dirichlet's integral on the internal potential. Relationships for the internal potential and *E*, including error propagation were developed for continuous and piecewise densities. Determination of *E* provides the inequality with two limits for *E*-values corresponding to different density models. The upper limit E_H agrees with the homogeneous distribution. The minimum amount E_{Gauss} corresponds to Gauss' continuous radial density. All *E*-estimates were obtained for the spherical Earth since the ellipsoidal reduction gives two orders smaller quantity than the accuracy σ_E = ±0.0025×10^{39} ergs of the energy *E*. Thus, we get a perfect agreement between E_{Gauss} = –2.5073×10^{39} ergs, $E =$ –2.4910×10^{39} ergs derived from the piecewise Roche's density, E_{PREM} = –2.4884×10^{39} ergs based on the PREM model, and *E*-values from simplest models separated into core and mantle only. Distributions of the internal potential and its first and second derivatives were derived for piecewise and continuous density models. Influence of the secular variation in the zonal coefficient $\overline{C}_{20}$ on global density changes is discussed using the adopted 3D continuous density model as restricted solution of the three-dimensional Cartesian moments problem inside the ellipsoid of revolution.

Key words: Gravitational potential energy, Internal potential, Density distribution, Error propagation.

Introduction

Determination of the 3D density distribution δ inside the Earth's interior Σ from given external potential data requires a solution of the inverse problem of the gravitational potential. This problem has no unique solution and treats traditionally as improperly posed problem due to a violation of conditions of solvability. One of suitable solutions follows from Mescheryakov's [5] theorem: "if the numerical value of the Earth's gravitational potential energy *E* and the density on the Earth's surface σ are given prior, this problem transforms to a properly posed problem in the Tikhonov sense" [9] with representation of δ through the three-dimensional Cartesian moments of the density of a gravitating body. The gravitational potential energy *E* taken with the sign (–) represents the quadratic functional $W(= -E)$ of δ [27] and therefore can be applied for a stable solution of the discussed inverse problem. This functional *W* represents the work of gravitation required to transport the masses, having the total Earth's mass *M*, from a state of infinite diffusion to their actual condition inside the planet. A remarkable expression for the work *W* gives Dirichlet's integral $D(V,V)$ on the gravitational potential *V* being extended throughout all space ([29], [19], [1]). To make the concept of potential energy useful, in addition to standard definition of *E* we need appropriate explicit relationships for the energy *E* and accurate numerical *E*-value based on the known Earth's reference models of density.

One of possible approaches leads to the search of the stationary value *E* or the so-called Gauss' problem ([14], [1]). Gauss proved in his famous memoir [14] that $W = -E$ has some minimal value $W_{\min}$. Thomson and Tait [29] wrote, in particular: "The manner in which Gauss independently proved Green's theorems is more immediately and easily interpretable in terms of energy". Therefore, Gauss' results can be treated as one of deep and central topics of the potential theory having the direct connection with Dirichlet's problem and Green's function. The minimum amount of W is $W_{\min} = M \cdot V_0 / 2$, if all masses are concentrated on the boundary σ considered as a level surface where the gravitational potential $V_0 =$ **const** and M is the Earth's mass [1]. According to Moritz [27] a remarkable summary of Gauss' problem for the homogeneous sphere reads: "minimum and maximum potential energy correspond to physically (for the Earth) meaningless cases: a surface distribution and a mass point. The 'true' Earth lies somewhere in between".

It is obvious that the potential energy E can be estimated from the known density δ and the internal gravitational potential. However only some E-estimates are found in the literature usually for the homogeneous planet and a body differentiated into several homogeneous shells ([27], [11], [12], [28] with references). The simplest continuous Legendre-Laplace law, Roche's law (as solutions of the Clairaut's equation), Bullard's model, and Gaussian (normal) distribution together with the piecewise Roche's profile with 7 basic shells as sampled for the PREM density [13] were applied in [23] for several estimations of the potential energy E that lead to the inequality with minimum limit corresponding to Gauss' radial profile.

In contrast to the last paper this study aims to derive according to Maxwell [26] other kind of expression for the gravitational potential energy $E = -(W_{\min} + \Delta W)$ based on the first Green's identity. This representation allows a simple estimation of $W_{\min}$ and important treatment of the deviation ΔW from this minimal amount $W_{\min}$ as Dirichlet's integral on the internal potential V_i generated by an adopted density distribution.

The Earth's mass and three principal moments of inertia represent initial information for unique and exact solution of the restricted 3D Cartesian moments problem [5], providing in this way the global density δ inside the ellipsoidal planet and the gravitational potential energy E. This model includes the reference piecewise radial profile with density jumps from discontinuities in seismic velocities as sampled for PREM. Such combined model of global piecewise density was adopted to preserve the external gravitational potential from zero to second degree/order, the dynamical ellipticity H_D, the planet's flattening f, and radial jumps of density. Components of the Earth's tensor of inertia are derived from the consistent set of the five 2nd degree harmonic coefficients $\overline{C}_{2m}$, $\overline{S}_{2m}$ and H_D ([24], [22]) and used for the computation of the density δ. It has to be pointed out, that accuracy of this global density should be derived especially from error propagation to restrict the possible solution domain in such a way that a reasonable solution may be selected either from spatial or radial density. Accuracy estimation of the gravitational potential energy is also applied additionally to restrict the solution either inside the ellipsoidal Earth or spherical planet.

Therefore, this study focuses on (a) the determination of the 3D global density distribution from the Earth's fundamental parameters including error propagation; (b) the background of adopted restrictions of possible solutions domain taking into account accuracy estimation; (c) the derivation of formulae for the internal potential and the gravitational potential energy $E = -(W_{\min} + \Delta W)$; (d) the estimation of the potential energy $E = E_{\text{PREM}}$ based on the most widely used PREM density model. Distributions of the internal potential and its first and second derivatives are also given for the Earth's continuous and piecewise density models. The influence of the secular non-tidal drift in the second-degree zonal coefficient $\overline{C}_{20}$ on density changes is discussed based on the adopted 3D global density model.

Basic relationships for the Earth's global density distribution

As a preparation, consider according to Mescheryakov [7] the mathematical model of the 3D global density distribution of the Earth having a shape of the ellipsoid of revolution with the flattening f and the semimajor axis a:

$$\delta = \tilde{\delta} - \sum_{j=1}^{k} h_j \cdot \theta(\rho - \rho_j)\,. \qquad (1)$$

In the expression above $\tilde{\delta}$ represents the exact (restricted by the order 2) solution of the three-dimensional Cartesian moments problem for the continuous mass density distribution, h_j is the j-density jump at the relative boundary $\rho_j = r_j / R$ (R=6371 km is the mean Earth' radius),

$$\theta(\rho - \rho_j) = \begin{cases} 0 \;\Rightarrow\; \rho - \rho_j < 0 \\ 1 \;\Rightarrow\; \rho - \rho_j > 0 \end{cases}, \qquad (2)$$

is the Heaviside's function, and $\rho = r / R$ is the relative distance. The relationship for $\tilde{\delta}$ as the mentioned solution of the Cartesian moments problem reads

$$\tilde{\delta}(\rho,\vartheta,\lambda) = K + F_1 + \rho^2 (K_1 \sin^2\vartheta \cos^2\lambda + \\ + K_2 \sin^2\vartheta \sin^2\lambda + K_3 \cos^2\vartheta + F_2)\,, \qquad (3)$$

$$K = \frac{5}{4}\delta_m [5 I_{000} - 7(I_{200} + I_{020} + I_{002}/\chi^2)]\,, \qquad (4)$$

$$K_1 = \frac{35}{4}\delta_m(3I_{200} + I_{020} + I_{002}/\chi^2 - I_{000}), \quad (5)$$

$$K_2 = \frac{35}{4}\delta_m(I_{200} + 3I_{020} + I_{002}/\chi^2 - I_{000}), \quad (6)$$

$$K_3 = \frac{35}{4}\delta_m(I_{200} + I_{020} + 3I_{002}/\chi^2 - I_{000}), \quad (7)$$

where $\chi = 1 - f$ is expressed via the flattening f of the ellipsoidal Earth assuming a homothetic stratification f=const; δ_m is the mean density; ρ ($0 \le \rho \le 1$) is the relative distance from the origin of the coordinate system to a current point, ϑ and λ are the polar distance and longitude of this point. Another two parameters from Eq. (3) are the functions of k given density jumps

$$F_1 = \frac{5}{4}\sum_{j=1}^{k} h_j[5(1-\rho_j^3) - \frac{21}{5}(1-\rho_j^5)], \quad (8a)$$

$$F_2 = \frac{35}{4}\sum_{j=1}^{k} h_j\rho_j^3(1-\rho_j^2). \quad (8b)$$

Eq. (3) is given in the geocentric coordinate system of the principal axes of inertia ($\overline{A}$, $\overline{B}$, $\overline{C}$) and agreed with the Earth's mass M and all components of the Earth's tensor of inertia to preserve in this way the external gravitational potential from zero to second degree/order, the dynamical ellipticity H_D, and the planet's flattening f. Mechanical parameters in Eqs. (4–7) are expressed through the dimensionless Cartesian moments of the density of a gravitating body (see definition in Grafarend et al. [15]) restricted here by the order n=p+q+r=2:

$$I_{pqr}(\delta) = \frac{1}{Ma^n}\int_\tau \delta x^p y^q z^r d\tau, \; (p+q+r=n) \quad (9)$$

which for n=2 can be computed by means of the Earth's mass and the dimensionless principal moments of inertia A, B, and C normalized by the factor $1/Ma^2$:

$$I_{000} = 1, \qquad I_{200} = \frac{B+C-A}{2}, \qquad I_{020} = \frac{A-B+C}{2}, \quad I_{002} = \frac{A+B-C}{2}, \quad (10)$$

where A, B, and C can be expressed via the 2nd-degree harmonic coefficients $\overline{A}_{20}$, $\overline{A}_{22}$ given in the principal axes system and the dynamical ellipticity H_D:

$$\left.\begin{aligned} A &= \sqrt{5}\overline{A}_{20} - \frac{\sqrt{15}\overline{A}_{22}}{3} - \frac{\sqrt{5}\overline{A}_{20}}{H_D}, \\ B &= \sqrt{5}\overline{A}_{20} + \frac{\sqrt{15}\overline{A}_{22}}{3} - \frac{\sqrt{5}\overline{A}_{20}}{H_D}, \\ C &= -\frac{\sqrt{5}\overline{A}_{20}}{H_D}. \end{aligned}\right\} \quad (11)$$

Eqs. (10) lead to the following relationship for the Trace(**I**) of the tensor of inertia **I**:

$$\mathrm{Trace}(\mathbf{I}) = (A + B + C) \quad (12a)$$

$$\mathrm{Trace}(\mathbf{I}) = 2(I_{200} + I_{020} + I_{002}). \quad (12b)$$

Thus, in the above formulae x, y, z are the Cartesian coordinates of an internal point; $d\tau$ is the volume element of the ellipsoid of revolution; $\overline{A}_{20}, \overline{A}_{22}$ are the fully normalized (non-zero) harmonic coefficients adopted here as Stokes constants in the Earth's principal axes system $O\overline{A}\overline{B}\overline{C}$.

The corresponding radial density distribution or the well-known Roche's model is the average of Eq. (1) over the surface of ellipsoid ρ=const (see Eq. (23b)):

$$\left.\begin{aligned} \delta(\rho) &= \tilde{\delta}(\rho) + F_1 + \rho^2 F_2 - \sum_{i=1}^{k} h_j\theta(\rho - \rho_j), \\ \tilde{\delta}(\rho) &= K + \rho^2 D, \end{aligned}\right\} \quad (13)$$

$$D = \frac{35}{12}\delta_m\left[5\left(I_{200} + I_{020} + I_{002}/\chi^2\right) - 3I_{000}\right], \quad (14)$$

which is agreed with the Earth's mass and the mean moment of inertia I_m because the parameters D and K can be expressed via M and I_m.

It has to be noted, that Eq. (1) and Eq. (13) lead to significant differences about the origin between such global density distributions and well-known density models. To avoid these differences we will use one modification of the considered approach given by Mescheryakov [6], which is based on the additional information about piecewise radial density profile such as PREM [13] including density jumps.

If some piecewise reference radial density model $\delta(\rho)_R$ is given, it is easy to verify that this modification for the global density model $\delta(\rho, \vartheta, \lambda)$ ($0 \le \rho = r/R \le 1$, R=6371km) can be written in the following manner

$$\delta(\rho, \vartheta, \lambda) = \left(\delta(\rho)_R + \left[\tilde{\delta}(\rho, \vartheta, \lambda) - \delta(\rho)_R\right]\right) \quad (15a)$$

$$\delta(\rho, \vartheta, \lambda) = \delta(\rho)_R + \Delta\delta(\rho, \vartheta, \lambda), \quad (15b)$$

$$\Delta\delta(\rho, \vartheta, \lambda) = \Delta K + \rho^2(\Delta K_1 \sin^2\vartheta\cos^2\lambda + \Delta K_2 \sin^2\vartheta\sin^2 + \Delta K_3\cos^2\vartheta), \quad (16)$$

where

$$\Delta K = \frac{5}{4}\delta_m[5\Delta I_{000} - 7(\Delta I_{200} + \Delta I_{020} + \Delta I_{002}/\chi^2)], \quad (17)$$

$$\Delta K_1 = \frac{35}{4}\delta_m(3\Delta I_{200} + \Delta I_{020} + \Delta I_{002}/\chi^2 - \Delta I_{000}), \quad (18)$$

$$\Delta K_2 = \frac{35}{4}\delta_m(\Delta I_{200} + 3\Delta I_{020} + \Delta I_{002}/\chi^2 - \Delta I_{000}), \quad (19)$$

$$\Delta K_3 = \frac{35}{4}\delta_m(\Delta I_{200} + \Delta I_{020} + 3\Delta I_{002}/\chi^2 - \Delta I_{000}), \quad (20)$$

were derived by subtracting from Eqs. (4 – 7) the corresponding Cartesian moments I_{000}^R, I_{200}^R,

I_{020}^R, and I_{002}^R of the reference density $\delta(\rho)_R$:

$$\Delta I_{000} = I_{000} - I_{000}^R, \quad \Delta I_{200} = I_{200} - I_{200}^R, \quad \Delta I_{020} = I_{020} - I_{020}^R, \quad \Delta I_{002} = I_{002} - I_{002}^R \quad (21)$$

The reference model $\delta(\rho)_R$ includes individual information about density jumps, the mean density δ_m^R, and the mean moment of inertia I_m^R, which have been selected preliminary for the construction of the radial profile $\delta(\rho)_R$. In contrast to Mescheryakov's solution [6] the Cartesian moments I_{000}^R, I_{200}^R, I_{020}^R, and I_{002}^R were adopted here for one common set of the conventional constants δ_m and I_m of the model (15) and density jumps entering into $\delta(\rho)_R$ [23]:

$$\left.\begin{aligned} I_{000}^R &= \frac{\delta_m^R}{\delta_m},\ I_{200}^R = I_{020}^R = \frac{3 I_m^R \delta_m^R}{2\delta_m(\chi^2+2)}, \\ \delta_m^R &= 3\int_0^1 \delta(\rho)_R \rho^2 d\rho,\ I_{002}^R = \frac{3\cdot\chi^2 I_m^R \delta_m^R}{2\cdot\delta_m(\chi^2+2)}, \\ I_m^R &= \frac{2(\chi^2+2)}{3\delta_m^R}\int_0^1 \delta(\rho)_R \rho^4 d\rho. \end{aligned}\right\} \quad (22)$$

This radial density $\delta(\rho)_R$ is also treated within the ellipsoid of revolution if we use according to Moritz [27] the following formula for the radius vector r_e by applying the first order theory (disregarding f^2 and other higher powers of f):

$$r_e = R\left[1 - \frac{2}{3} f \cdot P_2(\cos\vartheta)\right], \quad (23a)$$

where $P_2(\cos\vartheta)$ is the 2nd-degree Legendre polynomial. Eq. (23a) results from the average of r_e over the unite sphere that gives the mean Earth's radius R=6371 km. It has to be pointed out, that all basic formulas [Eq. (1) to Eq. (22)] are valid for a homothetic stratification (geometrically similar) when f=const inside the ellipsoidal Earth ([7], [6]). Hence, if the set of the internal ellipsoidal surfaces $\tilde{r}_e$ is labeled by the associated mean radius r of a sphere we get

$$\tilde{r}_e = r\left[1 - \frac{2}{3} f \cdot P_2(\cos\vartheta)\right] \Rightarrow$$

$$\Rightarrow \rho = \frac{r}{R} = \frac{\tilde{r}_e}{r_e}, \quad (0 \le \rho \le 1) \quad (23b)$$

By averaging $\delta(\rho,\vartheta,\lambda)$ over the ellipsoid surface we define the piecewise $\delta(\rho)$ function inside the ellipsoid of revolution as

$$\left.\begin{aligned} \delta(\rho) &= \delta(\rho)_R + [\Delta K + \rho^2 \Delta D], \\ \Delta D &= \frac{35}{12}\delta_m\left[5\left(\Delta I_{200} + \Delta I_{020} + \Delta I_{002}/\chi^2\right) - 3\Delta I_{000}\right], \end{aligned}\right\} \quad (24)$$

with the treatment of the reference density $\delta(\rho)_R$ also within the ellipsoidal Earth. Since the dimensionless radius ρ is constant for each $\tilde{r}_e$ the radial densities $\delta(\rho)_R$ and $\delta(\rho)$ are also constant by Eq. (24) at the ellipsoidal surface (23b).

Error propagation and lateral density heterogeneities

To prepare error propagation from starting values to the Earth 3D global density distribution we should keep in mind that information about accuracy of the adopted mean density δ_m^R, the mean moment of inertia I_m^R, and density jumps in various piecewise radial profiles $\delta(\rho)_R$ (such as PREM) are not found in literature or were not easily accessible to the authors. For this reason we will treat the reference density model $\delta(\rho)_R$ as exact constituent or "normal density" and come therefore only to the accuracy estimation of the density distribution $\tilde{\delta}(\rho,\vartheta,\lambda)$ [Eq. (3)] since $\tilde{\delta}(\rho,\vartheta,\lambda)$ is involved in Eq. (16) by Eq. (15) in the implicit form. Thus, the variance-covariance matrix of the principal moments of inertia, accuracy of the mean density σ_{δ_m} and accuracy of the flattening σ_f were chosen as initial data.

Given as initial information is the vector **a** containing the degree 2 harmonic coefficients $\overline{A}_{20}, \overline{A}_{22}$ in the principal axes system and the dynamical flattening H_D,

$$\mathbf{a} = \left[\overline{A}_{20}, \overline{A}_{22}, H_D\right]^{\mathrm{T}}, \quad (25)$$

(the symbol T denotes transposition) and the (3×3) variance-covariance matrix $\mathbf{C}_{\mathrm{aa}}$ of the parameters (25). Starting from the formulae of Eqs. (3) to (14) the necessary matrices of partial derivatives and variance-covariance matrices of the corresponding parameters are obtained by applying the error propagation rule.

Thus, defining the vector

$$\mathbf{J} = [A, B, C]^{\mathrm{T}}, \quad (26)$$

by differentiating Eq. (26) in view of Eqs. (11) we get the (3×3) matrix of partial derivatives of the vector $\mathbf{J}$ with respect to the vector **a**:

$$\frac{\partial \mathbf{J}}{\partial \mathbf{a}} = \begin{bmatrix} \sqrt{5}(1 - 1/H_D) & -\sqrt{15}/3 & -C/H_D \\ \sqrt{5}(1 - 1/H_D) & \sqrt{15}/3 & -C/H_D \\ -\sqrt{5}/H_D & 0 & -C/H_D \end{bmatrix}. \quad (27)$$

Hence Eq. (27) allows to apply the error propagation rule for the computations of the variance-covariance matrix $\mathbf{C}_{\mathrm{JJ}}$ of the principal moments of inertia A, B, and C from the variance-covariance matrix $\mathbf{C}_{\mathrm{aa}}$:

$$\mathbf{C}_{\mathrm{JJ}} = \left(\frac{\partial \mathbf{J}}{\partial \mathbf{a}}\right)\mathbf{C}_{\mathrm{aa}}\left(\frac{\partial \mathbf{J}}{\partial \mathbf{a}}\right)^{\mathrm{T}}. \qquad (28)$$

Then the accuracy σ_{I_m} of the mean moment of inertia reads

$$\sigma_{I_m} = \sqrt{\left(\frac{\partial I_m}{\partial \mathbf{J}}\right)\mathbf{C}_{\mathrm{JJ}}\left(\frac{\partial I_m}{\partial \mathbf{J}}\right)^{\mathrm{T}}}, \quad \frac{\partial I_m}{\partial \mathbf{J}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \qquad (29)$$

In order to create the covariance matrix $\mathbf{C}_{\mathrm{I_2I_2}}$ of the normalized Cartesian moments I_{200}, I_{020}, and I_{002} [Eq. (10)] we define the new vector

$$\mathbf{I}_2 = \begin{bmatrix} I_{200} & I_{020} & I_{002} \end{bmatrix}^{\mathrm{T}}, \qquad (30)$$

and, taking into account Eq. (10), we find the (3×3)-matrix of partial derivatives

$$\frac{\partial \mathbf{I}_2}{\partial \mathbf{J}} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad (31)$$

so that the variance-covariance matrix $\mathbf{C}_{\mathrm{I_2I_2}}$ becomes

$$\mathbf{C}_{\mathrm{I_2I_2}} = \left(\frac{\partial \mathbf{I_2}}{\partial \mathbf{J}}\right)\mathbf{C}_{\mathrm{JJ}}\left(\frac{\partial \mathbf{I_2}}{\partial \mathbf{J}}\right)^{\mathrm{T}}. \qquad (32)$$

Then, the determination of the variance-covariance matrix $\mathbf{C}_{\mathrm{I_2I_2}}$ provides a simple possibility to estimate accuracy of 1D and 3D density distributions. For the radial density $\tilde{\delta}(\rho)$ [Eq. (13)] initially we determine the matrix of partial derivatives of the vector $\mathbf{K}_{1D} = [K, D]^{\mathrm{T}}$ with respect to the vector $\mathbf{p} = [I_{200}, I_{020}, I_{002}, \delta_m, f]^{\mathrm{T}}$:

$$\frac{\partial \mathbf{K}_{1D}}{\partial \mathbf{p}} = \frac{35\delta_m}{4}\begin{pmatrix} -1 & -1 & -\frac{1}{\chi^2} & \frac{4K}{35\delta_m^2} & \frac{2I_{002}}{\chi^2} \\ \frac{5}{3} & \frac{5}{3} & \frac{5}{3\chi^2} & \frac{4D}{35\delta_m^2} & \frac{10I_{002}}{3\chi^2} \end{pmatrix}, \qquad (33)$$

the variance-covariance matrix $\mathbf{C}_{K_{1D}K_{1D}}$:

$$\mathbf{C}_{K_{1D}K_{1D}} = \left(\frac{\partial \mathbf{K_{1D}}}{\partial \mathbf{p}}\right)\mathbf{C}_{\mathrm{pp}}\left(\frac{\partial \mathbf{K_{1D}}}{\partial \mathbf{p}}\right)^{\mathrm{T}}, \qquad (34)$$

by involving additionally to $\mathbf{C}_{\mathrm{I_2I_2}}$ accuracy estimates of the mean density σ_{δ_m} and the flattening σ_f in the variance-covariance matrix $\mathbf{C}_{\mathrm{pp}}$:

$$\mathbf{C}_{\mathrm{pp}} = \begin{bmatrix} \mathbf{C}_{\mathrm{I_2I_2}} & 0 & 0 \\ 0 & \sigma_{\delta_m}^2 & 0 \\ 0 & 0 & \sigma_f^2 \end{bmatrix}. \qquad (35)$$

As a result, by differentiating Eq. (13) for $\tilde{\delta}(\rho)$ with respect to the elements of $\mathbf{K}_{1D} = [K, D]^{\mathrm{T}}$ we get accuracy of the radial density profile

$$\left.\begin{aligned} \sigma_{\tilde{\delta}(\rho)} &= \sqrt{\left(\frac{\partial \tilde{\delta}(\rho)}{\partial \mathbf{K}_{1D}}\right)\mathbf{C}_{K_{1D}K_{1D}}\left(\frac{\partial \tilde{\delta}(\rho)}{\partial \mathbf{K}_{1D}}\right)^{\mathrm{T}}}, \\ \frac{\partial \tilde{\delta}(\rho)}{\partial \mathbf{K}_{1D}} &= [1, \quad \rho^2], \end{aligned}\right\}. \qquad (36)$$

Accuracy $\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)}$ of the 3D density distribution [Eq. (3)] can be derived in a similar manner after determination of the matrix $\frac{\partial \mathbf{K}_{3D}}{\partial \mathbf{p}}$ of partial derivatives of the vector $\mathbf{K}_{3D} = [K, K_1, K_2, K_3]^{\mathrm{T}}$ with respect to the vector $\mathbf{p} = [I_{200}, I_{020}, I_{002}, \delta_m, f]^{\mathrm{T}}$:

$$\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)} = \sqrt{\left(\frac{\partial \tilde{\delta}(\rho,\vartheta,\lambda)}{\partial \mathbf{K}_{3D}}\right)\mathbf{C}_{K_{3D}K_{3D}}\left(\frac{\partial \tilde{\delta}(\rho,\vartheta,\lambda)}{\partial \mathbf{K}_{3D}}\right)^{\mathrm{T}}}, \qquad (37)$$

where

$$\frac{\partial \tilde{\delta}(\rho,\vartheta,\lambda)}{\partial \mathbf{K}_{3D}} = \left[1, (\rho\sin\vartheta\cos\lambda)^2, (\rho\sin\vartheta\sin\lambda)^2, (\rho\cos\vartheta)^2\right], \qquad (38)$$

$$\mathbf{C}_{K_{3D}K_{3D}} = \frac{\partial \mathbf{K}_{3D}}{\partial \mathbf{p}}\mathbf{C}_{pp}\left(\frac{\partial \mathbf{K}_{3D}}{\partial \mathbf{p}}\right)^{\mathrm{T}}, \qquad (39)$$

$$\frac{\partial \mathbf{K}_{3D}}{\partial \mathbf{p}} = \frac{35\delta_m}{4\chi^3}\begin{pmatrix} -\chi^3 & -\chi^3 & -\chi & (4\chi^3 K)/(35\delta_m^2) & 2I_{002} \\ 3\chi^3 & \chi^3 & \chi & (4\chi^3 K_1)/(35\delta_m^2) & 2I_{002} \\ \chi^3 & 3\chi^3 & \chi & (4\chi^3 K_2)/(35\delta_m^2) & 2I_{002} \\ \chi^3 & \chi^3 & 3\chi^3 & (4\chi^3 K_3)/(35\delta_m^2) & 6I_{002} \end{pmatrix}. \qquad (40)$$

Table 1 illustrates adopted initial parameters and their accuracy. Accuracy estimation σ_{δ_m} of the mean density δ_m requires additional remarks because δ_m represents a scale factor of the considered theory [Eq. (3) to Eq. (40)]. If GM = (398600.4418±0.0008)×10^9 m^{3}s^{-2} and the gravitational constant G = (6.673±0.010)×10^{-11} m^{3}kg^{-1}s^{-2} suggested by the IERS Conventions 2003 [25] are selected, we get $\sigma_{\delta_m} = 0.08$ g/cm^3. According to the IAG recommendations for G and GM in Table 1 another values of the mean density δ_m = (5.51483±0.0026) g/cm^3 and $\sigma_{\delta_m} = 0.0026$ g/cm^3 were estimated.

Thus, the global density distribution and accuracy at different depths were based on the flattening f, the principal moments of inertia A, B, and C from Table 1, and the value $\delta_m = (5.51483 \pm 0.0026)$ g/cm^3. The principal moments of inertia (given here in the zero frequency tide system) are results from the adjustment [22] of the 2nd-degree harmonic coefficients of 6 gravity field models (EGM96, GGM01S, GGM02C, EIGEN-CHAMP03S, EIGEN-GRACE02S, EIGEN-GL04S1) and 7 values H_D of the dynamical ellipticity all transformed to the common value of precession constant at epoch J2000.

Table 1.
Initial parameters and their accuracy

Reference	Adopted parameters
Groten [16]	$G=(6.67259 \pm 0.0003) \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2}
Groten [16]	$GM=(398600.4418 \pm 0.0008) \cdot 10^9$ m^{3}s^{-2}
Marchenko [22]	$A=0.32961274 \pm 0.0000005$
Marchenko [22]	$B=0.32962001 \pm 0.0000005$
Marchenko [22]	$C=0.33069901 \pm 0.0000005$
Marchenko and Schwintzer [24]	$1/f=298.25650 \pm 0.00001$

Table 2 lists necessary parameters for the determination of the 3D and 1D density distributions and their accuracy based on these principal moments of inertia and variance-covariance matrix taken from [22].

Finally, the density distribution [Eqs. (15 – 16)] and accuracy [Eq. (37)] at different depths were found from the consistent set of the Earth's fundamental parameters under the conditions to conserve the Earth's mass (δ_m), f, and all principal moments (A, B, C) of inertia. The reference radial density profile $\delta(\rho)_R$ in Eq. (15) was selected in the form of the simple piecewise Roche's law separated into seven basic shells [21], which is slightly different from the PREM-density.

Table 2.
Cartesian moments and other parameters of 3D and 1D density distributions

Para-meter	Value	Para-meter	Value
I_{200}	0.16535314 ±0.00000025	K	10.5298 ±0.0005 g/cm^3
I_{020}	0.16534588 ±0.00000025	K_1	-8.3587 ±0.0004 g/cm^3
I_{002}	0. 16426687 ±0.00000025	K_2	-8.3594 ±0.0004 g/cm^3
D	-8.3583 ±0.0004 g/cm^3	K_3	-8.3567 ±0.0007 g/cm^3

Therefore, with $\delta(\rho)_R$ known as exact constituent, the accuracy estimation $\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)}$ [Eq. (37)] of the 3D continuous global density distribution $\tilde{\delta}(\rho,\vartheta,\lambda)$ [Eq. (3)] (based only on the Earth's mechanical parameters) and the lateral density heterogeneities $\Delta\delta(\rho,\vartheta,\lambda)$ [Eq. (15)] are straightforward.

Comparison of these lateral density anomalies $\Delta\delta(\rho,\vartheta,\lambda)$ (Fig. 1) with the accuracy $\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)}$ at the same depths (Fig. 2) of the continuous constituent leads generally to values of the same order in uncertainties and density heterogeneities taken for various depths. Since discussed uncertainties are increasing when radius ρ is decreasing to zero (origin) [3] we will use below only radial density models for the determination of the gravitational potential energy E.

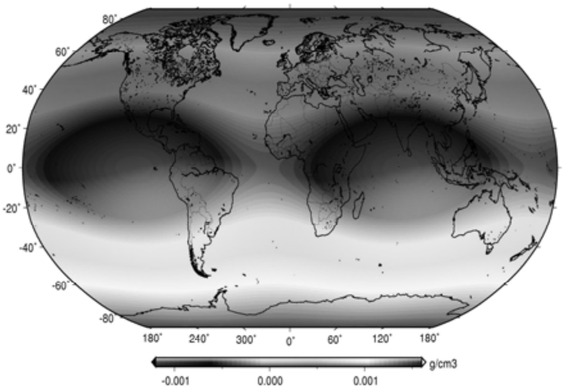

Fig. 1. Density anomalies [g/cm^3] $\Delta\delta(\rho,\vartheta,\lambda) = \delta(\rho,\vartheta,\lambda) - \delta(\rho)_R$ [Eq. (16)] at the mantle/crust boundary (r=6346.6 km)

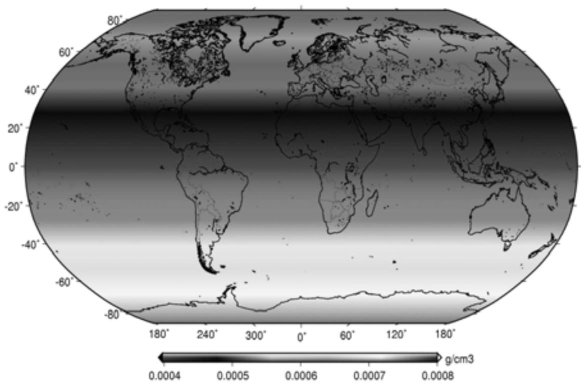

Fig. 2. Accuracy [g/cm^3] $\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)}$ [Eq. (37)] of the continuous constituent of 3D density distribution at the mantle/crust boundary (r=6346.6 km)

Basic relationships for the gravitational potential energy

The computation of the Earth's gravitational potential energy is based on the following conventional expression ([14], [29], [19], [1], [30], [27]):

$$E = -W = -\frac{1}{2}\int_{\tau} V_i \cdot \delta \cdot d\tau, \quad (41)$$

where $\delta = \delta(r, \vartheta, \lambda)$ is the planet's volume density, V_i is the Earth's internal gravitational potential, τ is the planet's volume enclosed by the surface σ, and W is the work of gravitation required to transport the masses M from a state of infinite diffusion to their actual condition inside the Earth.

However we prefer to use another treatment of Eq. (41) in terms of the internal potential only that leads to the equivalent and useful relationship for the energy E. Since our basic model of density [Eq. (15)] includes density jumps and represents some piecewise bounded function, let us suppose that $\delta \in L_2(\Sigma)$ is defined on the Hilbert space $L_2(\Sigma)$ of square-integrable functions inside the Earth's interior Σ. In this case the internal potential V_i has generalized second derivatives and satisfies Poisson's equation in almost all points of Σ [8], [18]:

$$\nabla^2 V_i = -4\pi G\delta, \quad \left(\delta = -\frac{\nabla^2 V_i}{4\pi G}\right). \quad (42)$$

Substitution of Eq. (42) into Eq. (41) by means of the first Green's identity applied with Maxwell [26] to Eq. (41) gives

$$W = -\frac{1}{8\pi G}\int_{\tau} V_i \cdot \nabla^2 V_i d\tau =$$

$$= -\frac{1}{8\pi G}\left[\int_{\sigma} V_i \frac{\partial V_i}{\partial n} d\sigma - \int_{\tau} D(V_i, V_i) d\tau\right], \quad (43)$$

$$\int_{\tau} D(V_i, V_i) d\tau = \int_{\tau}\left[\left(\frac{\partial V_i}{\partial x}\right)^2 + \left(\frac{\partial V_i}{\partial y}\right)^2 + \left(\frac{\partial V_i}{\partial z}\right)^2\right] d\tau =$$

$$= \int_{\tau} \left|\operatorname{grad} V_i\right|^2 d\tau. \quad (44)$$

Green's transformation [Eq. (43)] is valid if we suppose that the function V_i and its first derivatives are continuous or even piecewise [2]. Eq. (44) denotes always-nonnegative Dirichlet's integral. Note that initial Eq. (41) may also be transformed via Green's identity as [19]:

$$4\pi \int_{\tau} V_i \cdot \delta \cdot d\tau = \int_{\varpi} D(V_i, V_i) d\tau, \quad (45)$$

where Dirichlet's integral is extended throughout all space ϖ. The interpretation of Dirichlet's integral in terms of the gravitational potential energy follows from Eq. (45).

Let us now assume that the boundary σ represents a level surface where the gravitational potential $V_i = V_0 = \text{const}$. By this Eq. (43) gives

$$W = -\frac{1}{8\pi G}\left[V_0 \int_{\sigma} \frac{\partial V_i}{\partial n} d\sigma - \int_{\tau} D(V_i, V_i) d\tau\right] \quad (46a)$$

$$W = \frac{V_0 M}{2} + \frac{1}{8\pi G}\int_{\tau} D(V_i, V_i) d\tau \quad (46b)$$

as a consequence of Gauss' theorem [19], [17] applied to the first integral in the brackets of Eq. (46):

$$-4\pi GM = \int_{\sigma} \frac{\partial V_i}{\partial n} d\sigma. \quad (47)$$

According to Dirichlet's principle [19], [27] the work $W = -E$ has some minimal value $W_{\min}$ if all masses are concentrated on the level surface σ when the gravitational potential $V_0 = \text{const}$ and the interior is empty. In this case the internal potential $V_i = V_0 = \text{const}$ represents the harmonic function inside the surface σ and leads to zero Dirichlet's integral in Eq. (46). Thus, the minimum amount of W becomes

$$W_{\min} = M \cdot V_0 / 2, \quad (48)$$

and represents the solution of the variational Gauss' problem [14], [1]. Substitution of Eq. (48) into Eq. (46) leads to the following basic formulae

$$E = -(W_{\min} + \Delta W),$$

$$\Delta W = \frac{1}{8\pi G}\int_{\tau} D(V_i, V_i) d\tau. \quad (49)$$

Thus, other kind of expression for E, given under the assumption that the boundary σ is a level surface, provides a simple estimation of $W_{\min}$ and remarkable treatment of the deviation ΔW from this minimal amount $W_{\min}$ as non-zero Dirichlet's integral when all masses are distributed inside τ according to an adopted density law.

Now we consider the application of Eq. (49) to the spherically symmetric density distribution $\delta = \delta(r)$ within the spherical Earth. In this case the gravitational potential $V_0 = \text{const}$ will coincide with the potential of a point mass and the potential $V_0 = GM / R$ of the surface distribution $\delta = \delta(R) = \text{const}$. By this Eq. (48) becomes

$$W_{\min} = GM^2 / 2R, \quad (50)$$

usually related to a homogeneous planet [1]. The second term ΔW in Eq. (49) transforms to

$$\Delta W = \frac{4\pi}{8\pi G}\int_0^R \left(\frac{dV_i(r')}{dr'}\right)^2 r'^2 dr' =$$

$$= \frac{1}{2G}\int_0^R \left(\frac{GM(r')}{r'^2}\right)^2 r'^2 dr' = \frac{1}{2G}\int_0^R U^2 dr', \quad (51)$$

where according to Moritz [27] the gravity (gravitational attraction) $g(r)$ inside a stratified spherical Earth is expressed through the part of the Earth's mass $M(r)$:

$$g(r) = -\frac{dV_i(r)}{dr} = \frac{GM(r)}{r^2}, \quad (52)$$

$$M(r) = 4\pi\int_0^r \delta(r')\cdot r'^2 dr'. \quad (53)$$

Therefore, the gravitational potential energy of the spherically symmetric density distribution $\delta = \delta(r)$ becomes

$$E = -\left(\frac{GM^2}{2R} + \frac{1}{2G}\int_0^R \left(\frac{GM(r')}{r'}\right)^2 dr'\right) \quad (54a)$$

$$E = -\left(\frac{GM^2}{2R} + \frac{1}{2G}\|U\|^2_{L_2[0,R]}\right). \quad (54b)$$

The first term within brackets of Eq. (54) represents the minimal work of gravitation required to transport masses, having the total Earth's mass *M*, from a state of infinite diffusion onto the spherical planet with the radius *R*. Obviously, the mass $M(r)$ given by Eq. (53) represents the part of mass of the spherical Earth restricted by the radius *r*.

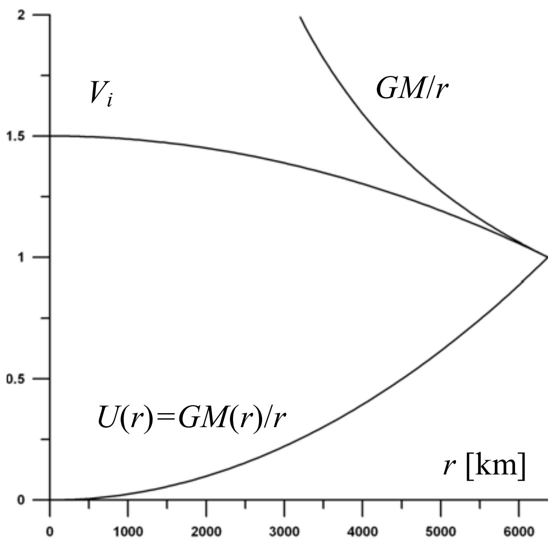

Fig. 3. Normalized values of the point potential GM/r, the internal potential V_i, and the function $V(r) = GM(r)/r$ given for the homogeneous spherical Earth

In view of Eq. (49) or Eq. (51) the integral in the second term is bounded and can be treated through the norm $\|U\|_{L_2[0,R]}$ of the simple function $U(r) = GM(r)/r$ in the Hilbert space L_2 of square-integrable functions on the segment [0, *R*]. Fig. 3 shows normalized values of different functions computed for the homogeneous spherical Earth having the same value on the Earth's sphere. Fig.3 illustrates also zero value of $U(r) = GM(r)/r$ when $r = 0$ and singularity of the point potential GM/r at the origin.

If the function $\delta = \delta(r)$ corresponds to the piecewise density of the layered Earth, Eq. (54) can be transformed by the partial integration to the following relationship

$$E = \sum_{j=1}^{k} E_j, \quad (j=1,2,\dots k), \quad (55)$$

where E_j expresses the contribution of the energy of the *j*-shell in the total value *E*.

E-estimates for simplest radial density models

As a preparation, consider additionally to the homogeneous Earth the determination of *E* for the following radial-only continuous density profiles: the law of Legendre-Laplace, the law of Roche (as solutions of the Clairaut's equation), the Bullard's model, and the Gaussian (normal) distribution (Fig. 4). Therefore, in order to determine the gravitational potential energy *E* we use density laws from Table 3 initially for the spherical Earth. The parameters of the simplest density models (Fig. 4) listed in Table 3 were derived from the solution of the inverse problem based on the well-known conditions to keep the Earth's mean density δ_m, the mean moment of inertia I_m, and the density δ_s on the Earth' surface [27]. Only first two conditions are applied to the determination of the continuous Roche's model. Thus, we get the following expressions for the parameters of the law of Legendre-Laplace [20]:

$$\left.\begin{aligned} \gamma^2 &= 4\cdot\left[\frac{\delta_s}{\delta_m} - 1\right] \Big/ \left[I_m - \frac{2}{3}\right], \\ \delta_0 &= \frac{\gamma^3\delta_m}{3\cdot[\sin\gamma - \gamma\cos\gamma]}, \end{aligned}\right\} \quad (56)$$

the law of Roche

$$\left.\begin{aligned} \delta_0 = a &= \frac{5}{8}\delta_m[10 - 21 I_m], \\ b &= \frac{5}{3}[\delta_m - \delta_0], \\ \delta_s &= \frac{5}{4}\delta_m[7 I_m - 2], \end{aligned}\right\} \quad (57)$$

the Bullard's model

$$\left.\begin{aligned} \delta_0 = a &= \frac{5}{32}\left[12\delta_s - 7\delta_m(27I_m - 10)\right],\\ b &= \frac{35}{16}\left[\delta_m(45I_m - 14) - 4\delta_s\right],\\ c &= \frac{63}{32}\left[4\delta_s - 5\delta_m(7I_m - 2)\right], \end{aligned}\right\} \quad (58)$$

and the Gauss' model [21]:

$$\left.\begin{aligned} \beta^2 &= \frac{1}{I_m}\left[1 - \frac{\delta_s}{\delta_m}\right],\\ \delta_0 &= \frac{4\cdot\beta^3\delta_m\cdot\exp(\beta^2)}{3\cdot\left(\sqrt{\pi}\cdot\exp(\beta^2)\cdot\operatorname{erf}(\beta) - 2\beta\right)}, \end{aligned}\right\} \quad (59)$$

where δ_0 is the density at the origin; erf(x) is the integral of the Gaussian distribution from 0 to x. Fig. 4 illustrates the radial continuous density profiles (Legendre-Laplace law, Roche's law, Bullard's model, and Gauss' model) now obtained by means of Eqs. (56–59) through adopted in Table 3 set of parameters based on the mean moment of inertia I_m= 0.3299773±0.0000005 taken from Table 1, the mean density δ_m = 5.51483±0.0026 g/cm^3, and the density on the Earth surface δ_s = 2.67 g/cm^3.

Table 3.

Expressions for different radial density models ($\rho = r / R$)

Model	Expression	Values of parameters
Homogeneous planet	$\delta(r) = \delta_m = \text{const}$	δ_m = 5.51483 g/cm^3
Legendre-Laplace law	$\delta(r) = \delta_0 \sin(\gamma\rho)/\gamma\rho$	δ_0=10.993 g/cm^3, γ=2.4929
Roche's law	$\delta(r) = a + b\rho^2$	a =10.583, b=–8.447 [g/cm^3]
Bullard's model	$\delta(r) = a + b\rho^2 + c\rho^4$	a =11.585, b=–13.121, c= 4.206 [g/cm^3]
Gauss' model (Marchenko, 2000)	$\delta(r) = \delta_0 \exp(-\beta^2\rho^2)$	δ_0=13.097 g/cm^3, β=1.26126

Table 4.

Internal potential V_i (spherical Earth) for different radial density models

Model	Expression for the internal gravitational potential
Homogeneous planet	$V_i(r) = \dfrac{2\pi G\delta_m}{3}(3R^2 - r^2)$
Legendre-Laplace law	$V_i(r) = \dfrac{4\pi G\delta_0 R^2}{\gamma^2}\left(\dfrac{R\sin(\gamma r/R)}{\gamma r} - \cos\gamma\right)$
Roche's law	$V_i(r) = -\dfrac{\pi G}{15R^2}\left[10aR^2(r^2 - 3R^2) + 3b(r^4 - 5R^4)\right]$
Bullard model	$V_i(r) = -\dfrac{4\pi G}{420R^4}\left[(70aR^4(r^2 - 3R^2) + 21bR^2(r^4 - 5R^4) + 10c(r^6 - 7R^6)\right]$
Gauss' model	$V_i(r) = \dfrac{\pi G\delta_0 R^2}{\beta^2}\left(\dfrac{R\sqrt{\pi}\operatorname{erf}(\beta r/R)}{\beta r} - 2\exp(-\beta^2)\right)$

Taking into account the expressions above, the relationships for the internal potential V_i corresponding to the mentioned set of density laws were derived (Table 4). In contrast to the previous paper [23] we prefer Eq. (54) for the computation of the potential energy E of the spherical Earth for different radial density models.

Thus, explicit relationships (Table 5) for the estimation of the gravitational potential energy E of the spherical Earth were obtained via two components $W_{\min}$ and ΔW treated as some minimal work and Dirichlet's integral, respectively. With the above-mentioned δ_m, I_m, δ_s, and values of parameters from Table 3 given for the law of Legendre-Laplace, the law of Roche, the Bullard's model, and the Gaussian distribution numerically we get estimations of the energy E given in Table 6, which includes E-values given by Mescheryakov [4] and Rubincam [28] for further comparisons.

Thus, there are the following limits for all computed E:

$$E_{\text{Gauss}} \le E_{\text{Earth}} \le E_{\text{H}} < -W_{\min}. \quad (60)$$

The amount $-W_{\min}$ corresponds to the surface distribution of the total Earth's mass M. The upper limit E_{H} agrees with the homogeneous Earth. The difference between $-W_{\min} = -GM^2/2$ and

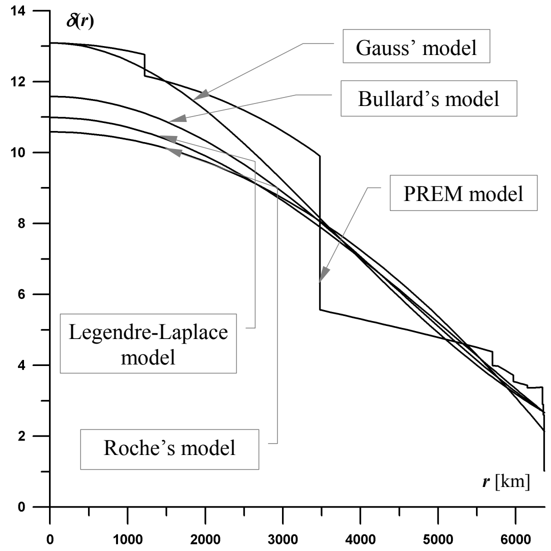

Fig. 4. Legendre-Laplace, Roche, Bullard, and Gauss continuous densities compared with the PREM-density model $\delta(\rho)$ [g/cm^3]

Table 6.
Estimations of the gravitational potential energy E (spherical Earth)

Model	W_{min}, ergs×10^{-39}	ΔW, ergs×10^{-39}	E, ergs×10^{-39}
Mescheryakov [4]	–	–	–2.34
Rubincam [28]	–	–	–2.45
Homogeneous Earth	1.8687	0.3737	–2.2425
Legendre-Laplace model	1.8687	0.5965	–2.4652
Roche's model	1.8687	0.6101	–2.4788
Bullard's model	1.8687	0.6132	–2.4820
Gauss' model	1.8687	0.6386	–2.5073

$E_H = -3GM^2/5R$ has the well-known value $GM^2/10R$. The minimum amount E_{Gauss} corresponds to the Gauss' model. The inequality (60) has the following explanation. The first term of the Taylor series expansion of the potential energy

Table 5.
Two components W_{min} and ΔW from Eq. (54) for the estimation of the potential energy $E = -(W_{min} + \Delta W)$ (spherical Earth)

Model	$W_{min} = GM^2/2R$	$\Delta W = \lVert V \rVert^2_{L_2(0,R)}/2G$
Homogeneous planet	$(8/9)\pi^2 G\delta_m^2 R^5$	$(8/45)\pi^2 G\delta_m^2 R^5$
Legendre-Laplace law	$\frac{8\pi^2 G\delta_0^2 R^5}{\gamma^6}\left[(\gamma^2-1)\cos^2\gamma - -2\gamma\sin\gamma\cos\gamma + 1\right]$	$\frac{4\pi^2 G\delta_0^2 R^5}{\gamma^6}\left[\gamma\sin\gamma\cos\gamma - 2\sin^2\gamma + \gamma^2\right]$
Roche's law	$\frac{8\pi^2 GR^5}{225}\left[5a+3b\right]^2$	$\frac{16\pi^2 GR^5}{1575}\left[35a^2+30ab+7b^2\right]$
Bullard model	$\frac{8\pi^2 GR^5}{11025}\left[35a+21b+15c\right]^2$	$\frac{8\pi^2 GR^5}{4729725}\left[105105a^2+90090ab+50050ac + 21021b^2+24570bc+7425c^2)\right]$
Gauss' model	$\frac{\pi^2 G\delta_0^2 R^5\exp(-2\beta^2)}{2\beta^6}\left[2\beta - -\sqrt{\pi}\exp(\beta^2)\mathrm{erf}(\beta)\right]^2$	$\frac{\sqrt{\pi^5} G\delta_0^2 R^5}{2\beta^6}\left[\sqrt{2}\beta\mathrm{erf}(\sqrt{2}\beta) - \sqrt{\pi}\mathrm{erf}(\beta)^2\right]$

E_{Gauss}, corresponding to the Gauss' model from Table 5, represents the gravitational potential energy of the homogeneous Earth E_H. Generally speaking every expression from Table 5 includes the main term, which is equal to E_H. But the sum of other terms with E_H leads on the whole to a smaller E than the value E_H (Table 6). It has to be pointed out, that the energy E derived by Mescheryakov [4] as $2E = -V_m M$ was based on the known Earth's mass M and the mean-value theorem after preliminary computation of the mean value V_m of the internal potential V_i inserted as $V_i = V_m$ in Eq. (41). The estimation of E given by Rubincam [28] was found for the spherical Earth differentiated into homogeneous core and homogeneous mantle with one jump at the core-mantle boundary. To verify the inequality (60) we will apply a similar approach to the above-discussed profiles using the direct approximation of the PREM density by these four simplest piecewise two shells with the same basic jump at the models separated into core/mantle boundary. Fig. 5 illustrates results of such approximations, which are characterized by r.m.s. deviations from the PREM density based in each case on the additional conditions to keep δ_m, I_m, and δ_s. Despite the best value of r.m.s. for the Bullard's model we prefer to use below also a simplest law of Roche because of a smaller number of the parameters a_j and b_j (j=1,2,…k) introduced for each shell.

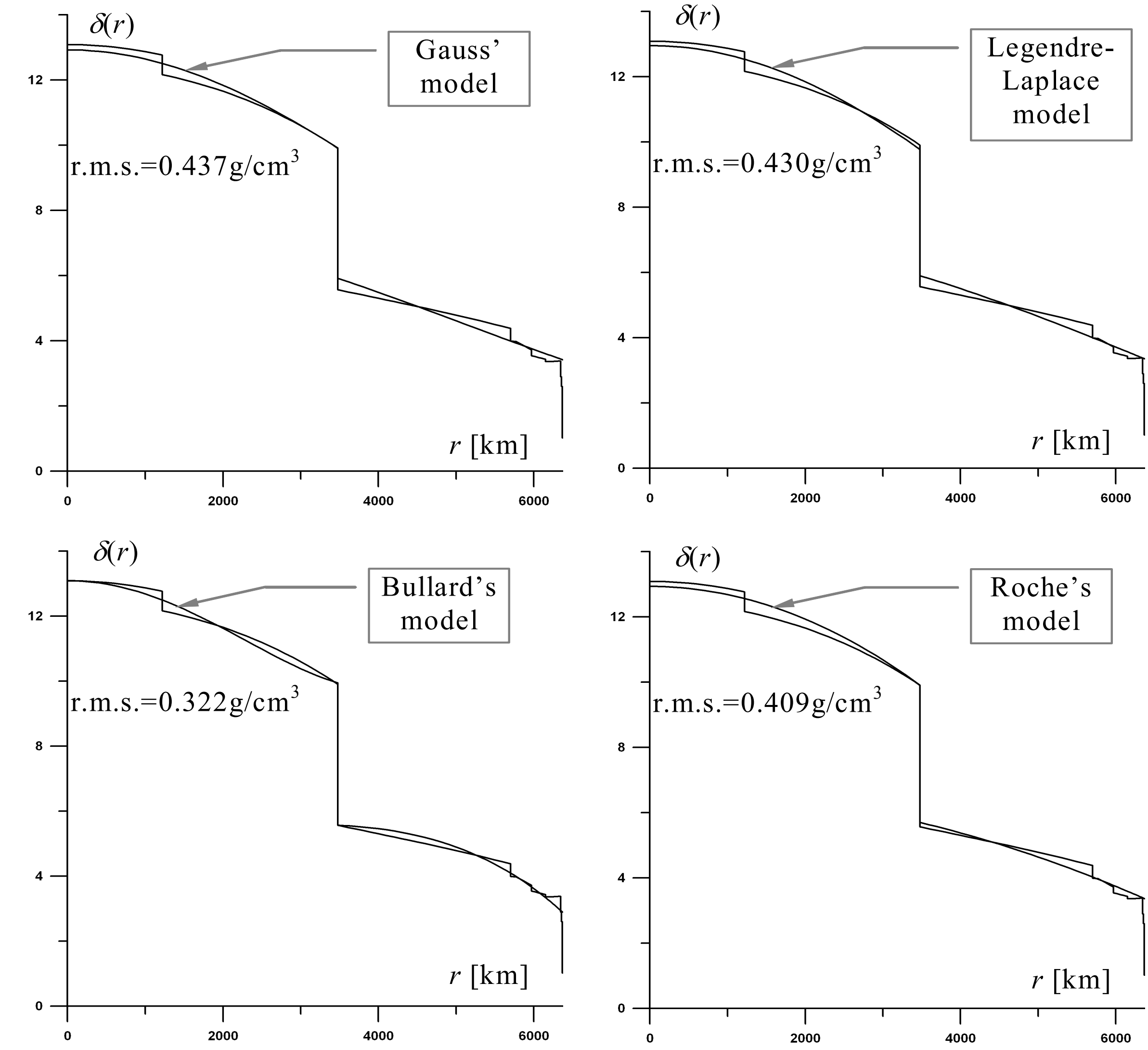

Fig. 5. Results of the direct approximation of the PREM density by some piecewise models with two shells taking into account one basic jump at the core/mantle boundary

Table 7.
Estimations of E for the spherically symmetric Earth with basic jump at the core/mantle boundary

Model	Value E
Homogeneous Earth (2 shells)	-2.4401×10^{39} ergs
Rubincam [28] (2 shells)	$-2.45\ \ \times10^{39}$ ergs
Legendre-Laplace model (2 shells)	-2.4944×10^{39} ergs
Roche's model (2 shells)	-2.4938×10^{39} ergs
Bullard's model (2 shells)	-2.4907×10^{39} ergs
Gauss' model (2 shells)	-2.4940×10^{39} ergs

The comparison of E-estimates from Table 6 (continuous radial density) and Table 7 (spherical Earth with one jump, k=2) gives better-quality agreement between the values E when the basic jump of density at the core/mantle boundary is taking into consideration. E-estimates given in Table 7 fulfill again to the inequality (60) with two limits E_{Gauss} and E_{H} from Table 6. In the case of these piecewise radial models with one jump of density at the core/mantle boundary (Table 7) all values of E are very close to the minimum amount of E_{Gauss}. That is why the accuracy estimation $\sigma_E = \sigma^E_{\text{Gauss}}$ for E_{Gauss} was derived by error propagation under the assumption that σ_E depends only on accuracy of the mean density δ_m and the mean moment of inertia I_m given above.

Numerically we get E_{Gauss} = (-2.5073±0.0025)× ×10^{39} ergs. Therefore, if a spherical Earth differentiates into present-day core and mantle we get in view of the estimated accuracy σ_E = ±0.0025)×10^{39} ergs a perfect accordance between *E*-values corresponded to the layered Legendre-Laplace, Roche, Bullard, and Gauss models with 2 shells. This quantity σ_E is certainly larger than *E*-estimate contained in the 2nd-degree harmonics [28] and for this reason we will use again radial-only piecewise density models for the determination of the Earth's potential energy *E*.

Gravitational potential energy based on piecewise density models

The internal potential V_i inside the ellipsoid of revolution with the radial density $\delta(r = \rho \cdot R)$ was adopted according to Moritz [27], p.41:

$$V_i = \frac{4\pi G}{\widetilde{r}_e}\int_0^r \delta(r')r'^2 dr' + 4\pi G\int_r^R \delta(r')r'dr' - \frac{8\pi G}{15\widetilde{r}_e^3}P_2(\cos\vartheta)\int_0^r \delta(r')d(fr'^5) - \frac{8\pi G\widetilde{r}_e^2}{15}P_2(\cos\vartheta)\int_r^R \delta(r')\frac{df}{d\widetilde{r}_e}dr' \quad (61)$$

where the radius vector $\widetilde{r}_e$ of the ellipsoid and the radius *r* of the associated mean sphere are connected by Eq. (23b). Then in Eq. (61) we express $\widetilde{r}_e$ by Eq. (23b) and get

$$\frac{1}{\widetilde{r}_e} = \frac{1}{r}\left[1 + \frac{2}{3}f \cdot P_2(\cos\vartheta)\right]. \quad (62)$$

Substitution of Eq. (62) into Eq. (61) gives for the homothetic stratification *f*=const:

$$V_i = V_i^{sphere} + \Delta V_i^{ell} = \frac{4\pi G}{r}\int_0^r \delta(r')r'^2 dr' + 4\pi G\int_r^R \delta(r')r'dr' + \Delta V_i^{ell}, \quad (63)$$

$$\Delta V_i^{ell} = \frac{8\pi G \cdot f}{3r}P_2(\cos\vartheta) \times \left(\int_0^r \delta(r')r'^2 dr' - \int_0^r \delta(r')r'^4 dr'\right), \quad (64)$$

the internal potential of the heterogeneous ellipsoidal Earth [Eq. (63)] in the form of the internal potential of the heterogeneous spherical planet V_i^{sphere} reduced to V_i by the ellipsoidal reduction ΔV_i^{ell} [Eq. (64)]. Eqs. (63–64) allow the direct computation of the gravitational potential energy *E* in the obvious form

$$E = E_{sphere} + \Delta E_{ell}, \quad (65)$$

if inserted into Eq. (41) or Eq. (54). Nevertheless, taking into consideration the flattening f we will estimate the corresponding ellipsoidal reduction ΔE_{ell} beforehand. Since all *E*-values of the piecewise radial models with one density-jump (Table 7) are very close to the lower limit E_{Gauss} in Eq. (60) it is enough to estimate ΔE_{ell} by applying again the Gauss' continuous model $\delta(\rho) = \delta_0 \exp(-\beta^2\rho^2)$ inside the ellipsoid with the homothetic stratification ($f = \text{const}$). With adopted parameters numerically we get $\Delta E_{ell} = \Delta E_{ell}^{\text{Gauss}} \approx 0.000045 \times 10^{39}$ ergs two orders smaller value than accuracy σ_E = ±0.0025)×10^{39} ergs. Hence, it is sufficient to accept the reduction $\Delta V_i^{ell} = 0$ in Eq. (63) for the internal potential V_i.

Now we can consider a general polynomial representation of piecewise density within the spherical planet

$$\delta_j(r) = \sum_{i=0}^{n_j} a_j^i \rho^i = \sum_{i=0}^{n_j} A_j^i r^i, \; \rho = r/R, \quad (66)$$

where n_j is the maximal degree of polynomial (66) in the *j* shell; *r* is the current radius within the *j* shell; the initial coefficients a_j [g/cm^3] are given for each *j* shell separately with the artificial zero shell $a_0 = 0$, $r_0 = 0$, which is involved for the generalization of basic formulae; $A_j^i = a_j^i \cdot R^i$ are the coefficients of the polynomial approximation in relation to *r*; *R* is the Earth's mean radius.

For further computation of $W_{\min}$ and ΔW through Eq. (54) we need a similar to Eq. (66) representation of the function $M(r)$ [Eq. (53)] via polynomials. When the radius r is considered within the *j* shell ($r_{j-1} < r < r_j$) substitution of Eq. (66) into Eq. (53) for $M(r)$, representing the part of the Earth's mass bounded by the radius r, after simple algebraic manipulations gives

$$M(r) = \sum_{i=0}^{j-1}\Delta M_i - M_j(r_{j-1}) + M_j(r) = \sum_{m=0}^{n_j+3} B_j^m r^m, \quad (67)$$

where

$$M_j(r) = 4\pi\int_0^r \delta_j(r')r'^2 dr' = 4\pi\sum_{i=0}^{n_j}\frac{A_j^i}{(i+3)}r^{i+3}, \quad (68)$$

$$\Delta M_j = 4\pi\int_0^{r_j}\delta_j(r')r'^2 dr' - 4\pi\int_0^{r_{j-1}}\delta_j(r')r'^2 dr' = M_j(r_j) - M_j(r_{j-1}), \quad (69)$$

with the coefficients B_j^m:

$$B_j^m = \frac{4\pi}{m} A_j^{m-3}, \quad (3 \le m \le n_j + 3),$$
$$B_j^0 = \sum_{i=0}^{j-1} \Delta M_i - M_j(r_{j-1}), \qquad (70)$$
$$B_1^0 = B_j^1 = B_j^2 = 0,$$

and the squared mass $M^2(r)$ according to Eq. (67):

$$M^2(r) = \sum_{i=0}^{n_j+3} \sum_{m=0}^{n_j+3} B_j^i B_j^m r^{i+m} . \qquad (71)$$

All this is substituted into Eq. (54) with the final result for $W_{\min}$ and ΔW :

$$W_{\min} = \frac{GM^2(R)}{2R}, \ \Delta W = \sum_{j=1}^{k} w_j , \qquad (72)$$

$$w_j = \frac{G}{2} \int_{r_{j-1}}^{r_j} \left(\frac{1}{r'^2} \sum_{i=0}^{n_j+3} \sum_{m=0}^{n_j+3} B_j^i B_j^m r'^{(i+m)} \right) dr' =$$
$$= \frac{G}{2} \sum_{i=0}^{n_j+3} \sum_{m=0}^{n_j+3} \frac{B_j^i B_j^m}{(i+m-1)} (r_j^{i+m-1} - r_{j-1}^{i+m-1}), \qquad (73)$$

keeping in mind that the radius of the artificial zero shell is $r_0 = 0$ and if the denominator $(i+m-1)$ is equal to zero we ignore such term in Eq. (73) because the corresponding coefficients $B_j^i = B_j^m =$ $= 0$ by Eq. (70).

Gravitational potential energy based on the piecewise Roche's density model

Taking into account a good agreement of the piecewise Roche-density model with the PREM density (Fig. 6), we will apply this radial density profile consisting from 7 shells and representing by polynomials of identical even powers within every shell [21] as initial information in the following form

$$\delta_j(r) = a_j + b_j \left(\frac{r}{R} \right)^2 = a_j + c_j r^2 ,$$
$$(j = 0,1,2,\ldots k), \ a_0 = b_0 = c_0 = 0 , \qquad (74)$$

where adopted k=7; a_j , a_j , and $c_j = b_j / R^2$ are the known coefficients of the model (74) given for each shell (Table 7) with the artificial zero shell $a_0 = b_0 = c_0 =$ 0, $r_0 = 0$. Note also that r.m.s. deviation between these models (Fig. 6) has the value 0.06 g/cm^3 for the core-mantle area and increases only to 0.24 g/cm^3 for the core-mantle-crust.

With $\Delta V_i^{ell} = 0$, =0, and a current point lied within the j-shell at the distance r , the substitution of Eq. (74) into Eq. (53) provides the following

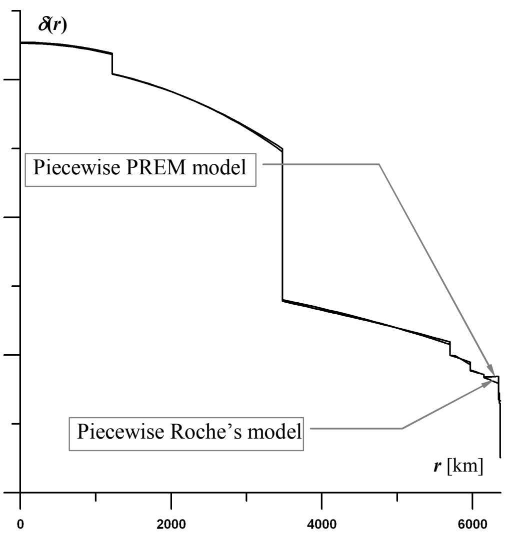

Fig. 6. Piecewise Roche-density model with 7 shells compared with the PREM-density model $\delta(\rho)$ [g/cm^3]

expression for the mass

$$M(r) = \sum_{i=0}^{j-1} \Delta M_i - M_j(r_{j-1}) + \frac{4\pi}{3} a_j r^3 + \frac{4\pi}{5} c_j r^5 =$$
$$= B_j^0 + B_j^3 r^3 + B_j^5 r^5 , \qquad (75)$$

abbreviating

$$B_j^3 = \frac{4\pi}{3} a_j, \quad B_j^5 = \frac{4\pi}{5} c_j, \quad B_j^0 =$$
$$= \left(\sum_{i=0}^{j-1} B_i^3 \left(r_i^3 - r_{i-1}^3 \right) + B_i^5 \left(r_i^5 - r_{i-1}^5 \right) \right) - B_j^3 r_{j-1}^3 - B_j^5 r_{j-1}^5 . \qquad (76)$$

Thus, according to Eqs. (75–76) in the case of the piecewise Roche's density (74) the function $M(r)$ can be represented by polynomials of identical odd powers within every shell. By this, after some algebraic manipulations with Eq. (75) inserted into Eq. (54) we get a simple possibility of the determination of the energy E. The result is

$$W_{\min} = \frac{GM^2(R)}{2R}, \ \Delta W = \sum_{j=1}^{k} w_j , \qquad (77)$$

$$w_j = \frac{G}{2} \left\{ \left(B_j^0 \right)^2 \left(\frac{1}{r_{j-1}} - \frac{1}{r_j} \right) + B_j^0 B_j^3 \left(r_j^2 - r_{j-1}^2 \right) + \right.$$
$$+ \frac{B_j^0 B_j^3}{5} \left(r_j^4 - r_{j-1}^4 \right) + \frac{\left(B_j^3 \right)^2}{5} \left(r_j^5 - r_{j-1}^5 \right) + \qquad (78)$$
$$\left. + \frac{2 B_j^0 B_j^3}{7} \left(r_j^7 - r_{j-1}^7 \right) + \frac{\left(B_j^5 \right)^2}{9} \left(r_j^9 - r_{j-1}^9 \right) \right\}$$

where w_j expresses the contribution of the j-shell in the total value ΔW . With adopted piecewise Roche's density model we get the estimation of $E = -(W_{\min} + \Delta W)$ [$W_{\min} = 1.8681 \times 10^{39}$ ergs, $\Delta W = 0.6269 \times 10^{39}$ ergs] given in Table 8 also via the contributions E_j [Eq. (55)] of each shell. The

Table 8.
Estimation of the gravitational potential energy E derived from the piecewise Roche's density model separated into 7 basic shells [21]

Shell	a_j, g/cm^3	b_j, g/cm^3	r_j, km	Contribution E_j of each shell, ergs	E_j, %
1 (Inner core)	13.061	-8.891	1221.5	-0.0541×10^{39}	2.17
2 (Outer core)	12.483	-8.343	3480.0	-0.9159×10^{39}	36.77
3 (Lower mantle)	6.370	-2.574	5701.0	-1.1625×10^{39}	46.67
4 (Upper mantle 1)	6.058	-2.577	5971.0	-0.1527×10^{39}	6.13
5 (Upper mantle 2)	5.784	-2.524	6151.0	-0.0954×10^{39}	3.83
6 (Upper mantle 3)	6.057	-2.903	6346.6	-0.0998×10^{39}	4.01
7 (Crust)	6.622	-3.952		-0.0104×10^{39}	0.42
Total gravitational potential energy: -2.4910×10^{39} ergs					

quantity $E = -2.4910\times10^{39}$ ergs agrees with E-estimates from Table 7 based on the radial models with one jump of density at the core/mantle boundary and fulfills to the inequality (60) at the vicinity of the minimum $E_{\text{Gauss}} = (-2.5073\pm\pm0.0025)\times10^{39}$ ergs. Taking into account the estimated above accuracy $\sigma_E = \pm0.0025\times10^{39}$ ergs we get a remarkable accordance between $E = -2.4910\times10^{39}$ ergs derived from the piecewise Roche's density with 7 basic shells as sampled for PREM and the values E given by the simplest piecewise Legendre-Laplace, Roche, Bullard, and Gauss models with 2 shells. Note that all E-estimates from Table 8 coincide exactly with the results based on the direct application of Eq. (41) and Eq. (55) [23].

Gravitational potential energy based on the PREM density model

Starting from 1981 PREM piecewise radial profile [13] represents the most widely used Earth's density model and, therefore, one of suitable densities for the estimation of the gravitational potential energy E. In this case we must consider a piecewise polynomial representation of the general kind [Eq. (66)] and final Eqs. (72–73) for the computation of the gravitational potential energy E.

Table 9 contains estimations of the total potential energy $E = -(W_{\min} + \Delta W)$ [$W_{\min} = 1.8685\times10^{39}$ ergs, $\Delta W = -0.6199\times10^{39}$ ergs] and the contributions E_j [Eq. (55)] of each shell. The gravitational potential energy $E_{\text{PREM}} = -2.4884\times\times10^{39}$ ergs agrees well with E-estimates from Table 7 and Table 8 based on different piecewise radial models and fulfills to the inequality (60). In view of the estimated above accuracy $\sigma_E = \pm0.0025\times10^{39}$ ergs we get again remarkable agreement between $E_{\text{PREM}} = -2.4884\times10^{39}$ ergs derived from the piecewise PREM density and all E-values given by the piecewise Roche's density with 7 shells and all simplest piecewise models with 2 shells which are corresponded to the spherically symmetric Earth differentiated into core and mantle only. Gauss' continuous model gives the lower limit $E_{\text{Gauss}} = (-2.5073\pm0.0025)\times10^{39}$ ergs of E for all considered density distributions including PREM model.

It has to be pointed out that we get different values of minimal work $W_{\min} = 1.8687\times10^{39}$ ergs based on all continuous profiles given in Table 6, $W_{\min} = 1.8681\times10^{39}$ ergs derived from the piecewise Roche's density, and $W_{\min} = 1.8687\times\times10^{39}$ ergs corresponded to the piecewise PREM density. Since every considered density profile includes individual information about the Earth's mass and the value G Eq. (50) leads to the conclusion that this fact simply reflects different values GM adopted for the construction of these models.

Internal potential and other related parameters of the Earth's reference models

Since the transformation of Eq. (41) into Eq. (43) was made under the assumption that the internal potential V_i has continuous or piecewise first derivatives we will analyze additionally the function $V_i(r)$ and its derivatives for piecewise (PREM) and continuous (Gauss) radial models of the Earth's density $\delta(r)$. According to Moritz (1990) the internal potential V_i of the stratified spherical planet represents the function of the current radius r:

$$V_i(r) = \frac{4\pi G}{r}\int_0^r \delta(r')r'^2 dr' + 4\pi G\int_r^R \delta(r')r'dr' = V_1(r) + V_2(r) \quad (79)$$

where the first term on the right-hand side in view of Eq. (53) corresponds to the external potential of the sphere bounded by the radius r:

Table 9.
Estimation of the gravitational potential energy E derived from the PREM density models according to [13]

Shell	r_j, km	Contribution E_j of each shell, ergs	E_j, %
1 (Inner core)	1221.5	-0.0542×10^{39}	2.18
2 (Outer core)	3480.0	-0.9128×10^{39}	36.68
3 (Lower mantle 1)	3630.0	-0.0598×10^{39}	2.40
4 (Lower mantle 2)	5600.0	-1.0390×10^{39}	41.75
5 (Lower mantle 3)	5701.0	-0.0623×10^{39}	2.50
6 (Upper mantle 1, Transition zone)	5771.0	-0.0397×10^{39}	1.60
7 (Upper mantle 2, Transition zone)	5971.0	-0.1126×10^{39}	4.52
8 (Upper mantle 3, Transition zone)	6151.0	-0.0951×10^{39}	3.82
9 (Upper mantle 4, LVZ)	6291.0	-0.0734×10^{39}	2.95
10 (Upper mantle 5, LID)	6346.6	-0.0297×10^{39}	1.20
11 (Crust 1)	6356.0	-0.0043×10^{39}	0.17
12 (Crust 2)	6368.0	-0.0050×10^{39}	0.20
13 (Ocean)		-0.0005×10^{39}	0.02
Total potential energy:		-2.4884×10^{39}	100.00

$$V_1(r) = \frac{4\pi G}{r}\int_0^r \delta(r')r'^2 dr' = \frac{GM(r)}{r}. \quad (80)$$

In the second term on the right-hand side we have

$$V_2(r) = \int_r^R 4\pi G\delta(r')r'dr'. \quad (81)$$

Let us now consider the element dm of mass of the ring (with infinitesimal thickness) oriented according to Fig. 7 along any chosen parallel with polar distance ϑ. It is evident that $r\cdot\sin(\vartheta)$ is the radius of this ring, $dh = r\cdot d\vartheta$ is the meridian height, and dr is radial thickness. Thus, the element dm becomes

$$dm = 2\pi\cdot\delta(r)\cdot r\cdot\sin(\vartheta)\cdot r\cdot d\vartheta dr \quad (82a)$$

$$= 2\pi\cdot\delta(r)\cdot r\cdot\sin(\vartheta)\cdot dhdr, \quad (82b)$$

where r is the radius of the considered sphere. Note also that the integration over longitude was already carried out in Eqs. (82a–82b).

Then if the element of mass is given by Eq.

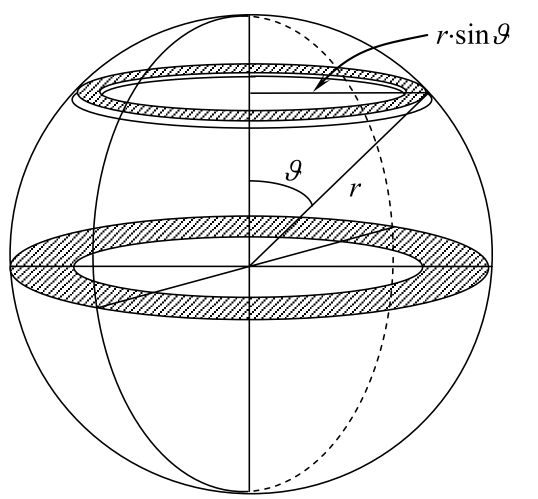

Fig. 7. To the interpretation of Eq. (79)

(82a) after computation of the mass by Eq. (53) we come to Eq. (80) for the external potential $V_1(r)$. Let us now the element of mass is written for the equatorial plane ($\sin(\vartheta) = 1$) by Eq. (82b):

$$dm = 2\pi\delta(r)r \cdot dh \cdot dr \,. \qquad (83)$$

By introducing the surface density

$$\mu(r) = \delta(r) \cdot dh \,, \qquad (84)$$

we get

$$dm = 2\pi\mu(r) \cdot r \cdot dr \,. \qquad (85)$$

This relationship can be interpreted as an element of the mass of equatorial ring with the surface density $\mu(r)$ when the volume density $\delta(r)$ is condensed or compressed into a surface density $\mu(r)$. In terms of Eqs. (84–85) we get

$$m = \int_r^R dm = 2\pi\int_r^R \delta(r')r'dhdr' = 2\pi\int_r^R \mu(r')r'dr' \,, \qquad (86)$$

the mass of the equatorial ring bounded by the radiuses *r* and *R*. Therefore, the comparison of Eq. (86) and Eq. (81) allows the following suggestion: by assuming the volume density $\delta(r)$ numerically equal to the surface density $\mu(r)$ when $dh = 1$ we come to a treatment of the potential V_2 using the mass of this equatorial ring (Fig. 7):

$$V_2(r) = 4\pi G\int_r^R \delta(r')r'dr' = 2Gm \,. \qquad (87)$$

Eq. (80) and Eq. (87) can serve as a basis for computing the internal potential V_i of the stratified spherical Earth. Supposing now the planet separated into *k* shells and making some elementary transformation with Eq. (79), Eq. (80), and Eq. (87) we get

$$V_i(r) = \frac{1}{r}G\left(M_j(r) - M_j(r_{j-1}) + \sum_{i=1}^{j-1}\Delta M_i \right) + $$
$$+ 2G\left(m_j(r_j) - m_j(r) + \sum_{i=j+1}^{k}\Delta m_i \right). \qquad (88)$$

The parameters above admit the following interpretation: $M_j(r)$ is the part of mass of the spherical Earth which is bounded by the radius *r* and possessed by the density distribution according to the *j* shell; ΔM_j is the mass of the *j* spherical shell restricted by the radiuses r_j and r_{j-1}; $m_j(r)$ is the part of mass of the equatorial disk of infinitesimal thickness (Fig. 7) which is bounded by the radius *r* and corresponded to the density distribution of *j* shell; Δm_j is the mass of the *j* equatorial ring of infinitesimal thickness restricted by the radiuses r_j and r_{j-1}.

Now with the piecewise polynomial representation of general kind [Eq. (66)] after substitution of Eq. (66) into Eq. (79) and associated algebraic manipulations the expression for the internal potential corresponding to the density (66) reduces to Eq. (88); $M_j(r)$ and ΔM_j are given by Eq. (68) and Eq. (69), respectively; the relationships for $m_j(r)$ and Δm_j are

$$m_j(r) = 2\pi\int_0^r \delta(r')r'dr' = 2\pi\sum_{i=0}^{n_j}\frac{A_j^i}{(i+2)}r^{i+2} \,, \qquad (89)$$

$$\Delta m_j = m_j(r_j) - m_j(r_{j-1}) \,. \qquad (90)$$

As a result, with the density distribution and internal potential given by Eq. (66) and Eq. (88), respectively, the computation of the gravity *g* and $\frac{dg}{dr}$ are straightforward

$$g(r) = -\frac{dV_i(r)}{dr} = \frac{GM(r)}{r^2} \,, \qquad (91)$$

$$\frac{dg(r)}{dr} = -\frac{d^2V_i(r)}{d^2r} =$$
$$= -\left(\nabla^2 V_i(r) + \frac{2g(r)}{r} \right) = 4\pi G\delta(r)r - \frac{2g(r)}{r} \,. \qquad (92)$$

Because the density $\delta(r)$ is bounded and piecewise function the gravity $g(r)$ [Eq. (91)] represents continuous function (Fig. 9) overall on the segment [0, *R*] by Eq. (53) for $M(r)$. Eq. (92) is valid in almost all points of [0, *R*] excluding a finite number of points of discontinuity. These points have the same position as density jumps where the functions $\delta(r)$ and $\frac{dg}{dr}$ have two limits: limit from the left and limit from the right at the vicinity of each point of discontinuity.

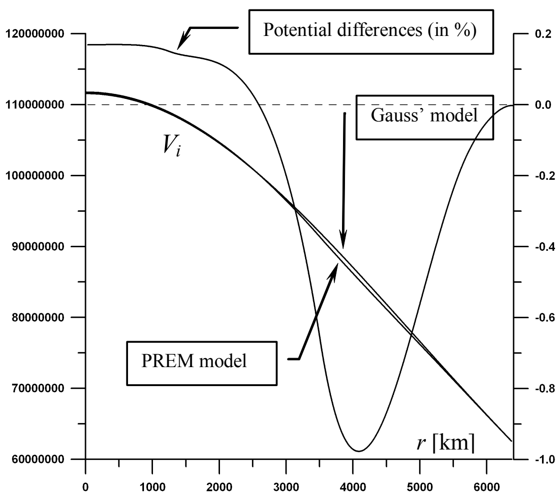

Fig. 8. Distribution of the internal potential V_i [m/s] in accordance with PREM and Gauss' radial models. Differences between PREM and Gauss' internal potentials are shown in percents [%] in relation to $V_i(r)$

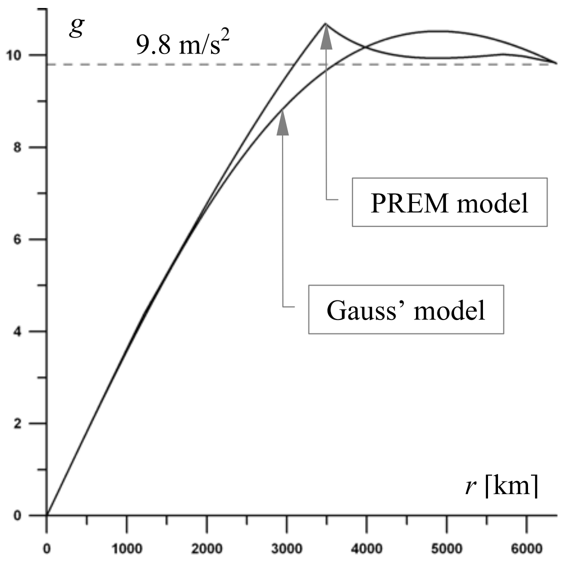

Fig. 9. Distribution of the gravity g [m/s^2] in accordance with PREM and Gauss' radial models

Both functions $g(r)$ and $\frac{dg}{dr}$ have exact limits from the right at the point r=0.

Thus, Fig. 8, Fig. 9 and Fig. 10 illustrate distribution of the internal potential $V_i(r)$ [Eq. (88)], the first derivative of V_i taken with sign (–) as gravity $g(r)$ (gravitational attraction) [Eq. (91)] and the second derivative of V_i taken with sign (–) as $\frac{dg}{dr}$ [Eq. (92)], respectively, in accordance with the PREM and Gauss' radial density models.

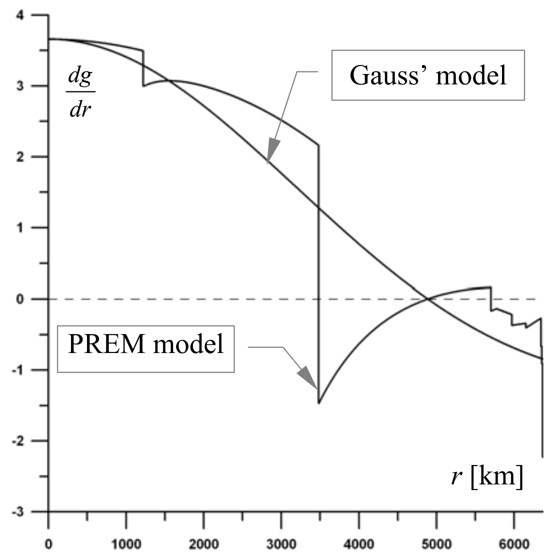

Fig. 10. Distribution of the function $\frac{dg}{dr}\times 10^6$ [1/s^2] according to PREM and Gauss' radial models.

Note that all these three functions are continuous only in the case of the Gauss' continuous density distribution.

The function $\frac{dg}{dr}$ given for the PREM model has evident discontinuities at the depths of density jumps (Fig. 10), which are corresponded to discontinuities in seismic velocities. The function $g(r)$ has no discontinuities in view of Eq. (91). Distribution of the gravity $g(r)$, having as stated by Saigey's theorem [10] some maximum inside the Earth, in the case of the PREM piecewise profile shows greatest value at the core/mantle boundary, local maximum in the upper mantle and minimum in the lower mantle. Note that this minimum of the PREM gravity coincides exactly with the position of global maximum according to Gauss' gravity distribution. The internal potential $V_i(r)$ represents for both models continuous function (Fig. 9) with maximum at the origin and minimum on the Earth's surface. Maximal deviation between these two internal potentials (in relation to $V_i(r)$) generated by piecewise and continuous densities consists value smaller than 1 %, which is shown in Fig. 9 in percents.

Conclusions

The global density $\delta(\rho,\vartheta,\lambda)$ inside the Earth having a shape of the ellipsoid of revolution was selected as combined model of the 3D continuous density $\tilde{\delta}(\rho,\vartheta,\lambda)$, given by the restricted solution of the three-dimensional Cartesian moments problem, and the reference radial piecewise density $\delta(\rho)_R$ with basic density jumps as sampled for the PREM density. This model conserves the Earth's mass, the flattening f, all principal moments (A, B, C) of inertia, and density jumps from discontinuities in seismic velocities. The corresponding 1D Roche's radial density is also treated within the ellipsoid using the conditions to preserve the Earth's mass, the mean moment of inertia, the flattening f, and density jumps. With $\delta(\rho)_R$ chosen as exact constituent, the accuracy $\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)}$ of the 3D continuous global density was derived at different depths from error propagation based on the consistent set of the Earth's mechanical parameters. Comparison of the lateral density anomalies with the accuracy $\sigma_{\tilde{\delta}(\rho,\vartheta,\lambda)}$ at the same depths leads generally to values of the same order in uncertainties and density heterogeneities.

That is why only radial density models were adopted for the determination of the Earth's gravitational potential energy. All E-estimates were based on the following relationship $E=-(W_{\min}+\Delta W)$ [Eq. (49)] derived from the transformation of the conventional expression for E through the first Green's identity. The first component of Eq. (49) $W_{\min}$ expresses some minimum amount of the work W and the second component ΔW represents a certain deviation from $W_{\min}$ treated via Dirichlet's integral on the internal potential V_i [Eq. (43), Eq. (49) and Eq. (54)]. Relationships for both components of $E=-(W_{\min}+\Delta W)$ were derived in the following cases: 1) the continuous radial density laws of Legendre-Laplace, Roche, Bullard, and Gauss; 2) the same radial models with one jump of density at the core/mantle boundary; 3) the piecewise Roche's profile; 4) the piecewise PREM model. The estimation of E according to different continuous density radial laws leads to the following result [Eq. (60)]: there are two limits for all computed E. First one agrees with the homogeneous distribution. Second one corresponds to the Gauss' radial density model.

All determinations of the potential energy E were made for the spherical Earth since the computation of the ellipsoidal reduction ΔE_{ell} gives two orders smaller quantity than the estimated accuracy $\sigma_E = \pm 0.0025\times 10^{39}$ ergs of E. Taking into account this accuracy estimation we get a perfect agreement between $E_{\text{Gauss}} = (-2.5073\pm 0.0025)\times$ $\times 10^{39}$ ergs, $E = -2.4910\times 10^{39}$ ergs derived from the piecewise Roche's density, the gravitational potential energy $E = -2.4884\times 10^{39}$ ergs based on the PREM density model, and the values E given by the simplest piecewise Legendre-Laplace, Roche, Bullard, and Gauss models all corresponded to the spherically symmetric Earth differentiated into core and mantle only. Thus, accuracy of the 3D Earth's global density distribution and accuracy of the gravitational potential energy restrict the possible solution domain in such a way that a sufficient solution was derived from the piecewise radial density model taken only for the spherical Earth. Distributions of the internal potential, the gravity g, and $\frac{dg}{dr}$ were found for piecewise and continuous radial densities inside the spherical Earth's.

Finally we should note that the secular variation $\dot{A}_{20} \cong \dot{C}_{20} = 1.1628\times 10^{-11}$ yr^{-1} in the degree 2 zonal coefficient produces the change $dC = -\frac{2\sqrt{5}\dot{A}_{20}}{3}\times \times (t-t_0)$ in the polar moment C of inertia [31], [24]. By this Eqs. (4-7) give changes in the following parameters

$$\left.\begin{aligned} dK &= \frac{35dC\delta_m(1-\chi^2)}{4\chi^2}, \\ dD &= -\frac{175dC\delta_m(1-\chi^2)}{12\chi^2}, \end{aligned}\right\} \quad (93a)$$

$$\left.\begin{aligned} dK_1 = dK_2 &= \frac{35dC\delta_m(2\chi^2-1)}{4\chi^2}, \\ dK_3 &= -\frac{35dC\delta_m(3-\chi^2)}{4\chi^2}, \end{aligned}\right\} \quad (93b)$$

of the density distribution inside the ellipsoidal Earth [Eqs. (1-3)]. But spherically symmetric distribution is not sufficient: usual Roche's law given by Eq. (13) is not responsible for this variation because $f = 0$ and $\chi = 1$ in Eqs. (93a). Therefore, within the chosen model approach the ellipsoidal 1D [Eq. (13), Eq. (24)] and 3D [Eq. (15)] models provide the time-dependence in the global density distribution and can be used for the estimation of corresponding changes in the Earth's interior.

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ОЦІНКА ПОТЕНЦІАЛЬНОЇ ГРАВІТАЦІЙНОЇ ЕНЕРГІЇ ЗЕМЛІ НА ОСНОВІ РЕФЕРЕНЦНИХ МОДЕЛЕЙ РОЗПОДІЛІВ ГУСТИНИ

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У статті розглядаються питання, присвячені оцінці гравітаційної потенціальної енергії E Землі на основі заданих глобальних розподілів густини. Глобальна модель густини обчислювалась як комбінація тривимірного неперервного розподілу та референцного радіального розподілу з основними стрибками густини, як і у моделі PREM. Даний глобальний розподіл однозначно відтворює зовнішнє гравітаційне поле Землі до другого порядку і степеня включно, є узгодженим зі значеннями геометричного та динамічного стиску планети, а також з основними радіальними стрибками густини. Чисельні дослідження показали, що значення латеральних аномалій густини та їх точність є величинами одного і того ж порядку, внаслідок чого для оцінки гравітаційної потенціальної енергії E використовувались лише радіальні моделі розподілу густини. Всі оцінки енергії виконувались з використанням формули $E = -(W_{\min} + \Delta W)$, отриманої з загальноприйнятого співвідношення для E через тотожність Гріна.

Перший доданок цієї формули $W_{\min}$ виражає мінімальну роботу *W*, а другий ΔW – відхилення від $W_{\min}$, яке трактується через інтеграл Діріхле для внутрішнього потенціалу. В роботі запропоновано співвідношення для обчислення внутрішнього потенціалу та *E*, а також вирази для оцінки точності неперервних та радіально-кускових розподілів густини. Обчислено границі, в межах яких може приймати значення *E*. Верхня межа E_H відповідає однорідній Землі. Нижня межа E_{Gauss} відповідає радіальному розподілу густини за законом Гаусса. Всі оцінювальні значення гравітаційної потенціальної енергії були отримані для сферичної Землі, оскільки еліпсоїдальна поправка дає значення на два порядки менше, ніж точність визначення самої енергії $\sigma_E = \pm 0{,}0025\times10^{39}$ ergs. Таким чином було отримано добре узгодження між трьома оцінювальними значеннями енергії $E_{Gauss} = -2{,}5073\times10^{39}$ ergs (енергія моделі Гаусса), $E = -2{,}4910\times10^{39}$ ergs (енергія кусково-неперервної моделі Роша) та $E_{PREM} = -2{,}4884\times10^{39}$ ergs (енергія моделі PREM). Подібна узгодженість спостерігалась і для оцінювальних значень енергії найпростіших моделей густини, які складались з двох шарів – кора і мантія. В статті наведено розподіли внутрішнього потенціалу та його першої та другої похідних для неперервних та кусково-неперевних моделей густини. З використанням тривимірних моделей густини аналізується вплив вікової варіації зонального коефіцієнта $\bar{C}_{20}$ на глобальні зміни густини.

Ключові слова: гравітаційна потенціальна енергія, внутрішній потенціал, розподіл густини, оцінка точності.

ОЦЕНКА ПОТЕНЦИАЛЬНОЙ ГРАВИТАЦИОННОЙ ЭНЕРГИИ ЗЕМЛИ НА ОСНОВЕ РЕФРЕНЦНЫХ МОДЕЛЕЙ РАСПРЕДЕЛЕНИЙ ПЛОТНОСТИ

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В статье рассматриваются вопросы, посвященные оценке гравитационной энергии *E* Земли на основе заданных глобальных распределений плотности. Глобальная модель плотности представляла собой комбинацию трехмерного непрерывного распределения и референтного радиального распределения с главными скачками плотности, как у модели PREM. Данное глобальное распределение согласовано з внешним гравитационным полем Земли до второго порядка и степени, со значениями геометрического и динамического сжатий планеты, а также с главными радиальными скачками плотности. Численные исследования показывают, что значения латеральных аномалий плотности одного и того же порядка, что и их точность, вследствие чего для оценок использовались только радиальные модели распределения плотности. Для всех оценок энергии использовалась формула $E = -(W_{\min} + \Delta W)$, полученная из общепринятого соотношения для *E* через тождество Грина. Первое слагаемое формулы представляет собою минимальную работу $W_{\min}$, а второе ΔW – отклонение от $W_{\min}$, которое интерпретируется как интеграл Дирихле для внутреннего потенциала. В статье предложены соотношения для расчета внутреннего потенциала и *E*, а также выражения для оценки точности непрерывных и радиально-кусочных распределений плотности. Определены границы, в пределах которых может принимать значения *E*. Верхняя граница E_H соответствует однородной Земле. Нижняя граница E_{Gauss} соответствует радиальному распределению плотности Гаусса. Все оценочные значения гравитационной потенциальной энергии были получены для сферической Земли, поскольку эллипсоидальная поправка на два порядка меньше, чем точность определения энергии $\sigma_E = \pm 0{,}0025\times10^{39}$ ergs. Таким образом получено согласование между тремя оценочными значениями энергии: $E_{Gauss} = -2{,}5073\times10^{39}$ ergs (энергия модели Гаусса), $E = -2{,}4910\times10^{39}$ ergs (энергия кусочно-непрерывной модели Роша) и $E_{PREM} = -2{,}4884\times10^{39}$ ergs (энергия модели PREM). Подобное согласование наблюдалось и для оценочных значений энергии наипростейших моделей плотности, которые состояли из двух слоёв – коры и мантии. В статье приведены распределения внутреннего потенциала и его первой и второй производных для непрерывных и кусочно-непрерывных моделей плотности. С использованием трехмерных моделей плотности анализируется влияния вековой вариации зонального коэффициента $\bar{C}_{20}$ на глобальные изменения плотности.

Ключевые слова: гравитационная потенциальная энергия, внутренний потенциал, распределение плотности, оценка точности.

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