ON THE COMPUTATION OF THE REGULARIZATION PARAMETER

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***Summary**. Two possibilities are considered for the determination of the regularization parameter on the basis of the misclosure principle.*

Introduction

In the paper (Abrikosov, 1999) three different principles were applied for the determination of the regularization parameter in the variational problem of data processing. The equations of observations without a systematic part (Moritz, 1980) were considered as

$$\mathbf{l} = \mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{l}$ is the observation vector, $\mathbf{s}$ is the signal vector, which is characterized by the covariance matrix $\mathbf{C}_{ss}$, and $\mathbf{n}$ is the noise vector, which is characterized by the covariance matrix $\mathbf{C}_{nn}$. By using the standard variational principle (Moritz, 1980):

$$\Phi_\gamma = \mathbf{n}^T\mathbf{C}_{nn}^{-1}\mathbf{n} + \gamma \mathbf{s}^T\mathbf{C}_{ss}^{-1}\mathbf{s} = \min, \qquad (2)$$

with the positive weighting coefficient γ:

$$0 < \gamma < \infty \qquad (3)$$

the estimations for signal and noise were obtained in the following form:

$$\hat{\mathbf{s}}(\gamma) = \mathbf{C}_{ss}(\mathbf{C}_{ss} + \gamma\mathbf{C}_{nn})^{-1}\mathbf{l}, \qquad (4)$$

$$\hat{\mathbf{n}}(\gamma) = \gamma\mathbf{C}_{nn}(\mathbf{C}_{ss} + \gamma\mathbf{C}_{nn})^{-1}\mathbf{l}. \qquad (5)$$

By applying the next principle

$$\left\|\mathbf{C}_{nn} - \hat{\mathbf{C}}_{nn}(\gamma)\right\| = \left\|\Delta\mathbf{C}_{nn}(\gamma)\right\| = \min, \qquad (6)$$

where $\hat{\mathbf{C}}_{nn}(\gamma)$ is the a posteriori covariance matrix of the estimation (5), two possible values for the regularization parameter were obtained. These are

$$\gamma = 1, \qquad (7)$$

which corresponds to traditional least squares collocation solution of the system (1), and

$$\gamma = 1 + \sqrt{1 + \frac{(\mathbf{C}_{ss}, \mathbf{C}_{nn})}{\left\|\mathbf{C}_{nn}\right\|^2}}. \qquad (8)$$

which realizes the solution of (1) under the misclosure principle.

In the present paper our goal consists of consideration of some additional possibilities for the determination of the regularization parameter γ on the basis of the misclosure principle.

Basic relationships

By introducing the notation

$$\mathbf{Q}(\gamma) = \mathbf{C}_{ss} + \gamma\mathbf{C}_{nn}, \qquad (9)$$

we can rewrite the estimations (4) and (5) as

$$\hat{\mathbf{s}}(\gamma) = \mathbf{C}_{ss}\mathbf{Q}^{-1}(\gamma)\mathbf{l}, \qquad (10)$$

$$\hat{\mathbf{n}}(\gamma) = \gamma\mathbf{C}_{nn}\mathbf{Q}^{-1}(\gamma)\mathbf{l}. \qquad (11)$$

The covariance matrixes of these estimations are

$$\hat{\mathbf{C}}_{ss}(\gamma) = \mathbf{C}_{ss}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(1)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{ss}, \qquad (12)$$

$$\hat{\mathbf{C}}_{nn}(\gamma) = \gamma^2\mathbf{C}_{nn}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(1)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{nn}, \qquad (13)$$

and the covariance matrixes of errors are

$$\mathbf{E}_{ss}(\gamma) = \mathbf{C}_{ss} - 2\mathbf{C}_{ss}\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{ss} + \mathbf{C}_{ss}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(1)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{ss}, \qquad (14)$$

$$\mathbf{E}_{nn}(\gamma) = \mathbf{C}_{nn} - 2\gamma\mathbf{C}_{nn}\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{nn} + \gamma^2\mathbf{C}_{nn}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(1)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{nn}. \qquad (15)$$

Supposing that all matrixes in the right hand sides of above expressions have a full rank, we can transform (14) and (15) to

$$\mathbf{E}_{ss}(\gamma) = \mathbf{E}_{nn}(\gamma) = \mathbf{C}_{ss}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(\gamma^2)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{nn}, \qquad (16)$$

Regularization parameter from the matrix $\hat{\mathbf{C}}_{nn}(\gamma)$

Let us suppose that a value of the regularization parameter fulfills the equality

$$\hat{\mathbf{C}}_{nn}(\gamma) = \mathbf{C}_{nn}. \qquad (17)$$

By substitution (12) into (17) we get

$$\gamma^2\mathbf{C}_{nn}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(1)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{nn} = \mathbf{C}_{nn}, \qquad (18)$$

and after obvious transformations we come to the equation

$$\gamma(\gamma - 2)\mathbf{C}_{nn} = \mathbf{C}_{ss}. \qquad (19)$$

This expressions is nothing else but the system of linear equations regarding the unknown

$$x = \gamma(\gamma - 2). \qquad (20)$$

Classical least squares solution of this system leads to the value

$$x = \frac{\mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{nn})}{\mathrm{trace}(\mathbf{C}_{nn}\mathbf{C}_{nn})}, \qquad (21)$$

and, in view of inequalities (3), we get immediately

$$\gamma = 1 + \sqrt{1 + \frac{\mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{nn})}{\mathrm{trace}(\mathbf{C}_{nn}\mathbf{C}_{nn})}}. \qquad (22)$$

As we can see, this expression coicides exactly with the expression obtained in (Abrikosov, 1999) on the basis of the misclosure principle.

Regularization parameter from the matrix $\mathbf{E}_{ss}(\gamma)$

Now let us suppose that a value of the regularization parameter fulfills the equality

$$\hat{\mathbf{E}}_{ss}(\gamma) = \mathbf{C}_{nn}. \qquad (23)$$

By substitution (16) into (23) we get

$$\mathbf{C}_{ss}\mathbf{Q}^{-1}(\gamma)\mathbf{Q}(\gamma^2)\mathbf{Q}^{-1}(\gamma)\mathbf{C}_{nn} = \mathbf{C}_{nn}, \quad (24)$$

and after obvious transformations we come to the equation

$$\gamma(\gamma-2)\mathbf{C}_{ss} = \gamma^2\mathbf{C}_{nn}. \quad (25)$$

Excluding the trivial root $\gamma=0$, we can rewrite (25) in the form

$$(\gamma-2)\mathbf{C}_{ss} = \gamma\mathbf{C}_{nn}, \quad (26)$$

and treate it as a system of linear equations

$$\mathbf{C}_{ss}x = \mathbf{C}_{nn} \quad (27)$$

regarding the unknown

$$x = \frac{\gamma-2}{\gamma} \quad (28)$$

Classical least squares solution of the system (27) leads to the value

$$x = \frac{\mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{nn})}{\mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{ss})}, \quad (29)$$

and we can see that in the considered case

$$\gamma = \frac{2\,\mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{ss})}{\mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{ss}) - \mathrm{trace}(\mathbf{C}_{ss}\mathbf{C}_{nn})}. \quad (30)$$

Discussion

Thus, we have obtained two expressions for the computation of the regularization parameter in accordance with misclocure prenciple. The value (22) minimizes the Euclidean norm of the difference between a priori and a posteriori covariance matrix of the noise vector (Abrikosov, 1999), whereas the value (30) minimizes the Euclidean norm of the difference between a priori covarianbce matrix of the noise and a posteriori covariance matrix of errors of the signal vector estimation. In other words, the value (30) leads to the agreement of the accuracy estimation of the signal with the magnitude of errors in initial data. As a result, this value also may be treated as a possible realization of the misclosure principle.

For better understanding of the behaviours of the discussed values of the regularization parameter, it have a sense to consider the simplest case

$$\mathbf{C}_{ss} = c\mathbf{I}, \quad (31)$$

$$\mathbf{C}_{nn} = d\mathbf{I}, \quad (32)$$

where **I** is a unit matrix, c is the variance of the signal, and d is the variance of the measured data errors. With (31), (32) the expressions (22) and (30) may be rewritten, respectively, as

$$\gamma = 1 + \sqrt{1+z}, \quad (33)$$

$$\gamma = \frac{2z}{z-1}, \quad (34)$$

where

$$z = \frac{c}{d} \quad (35)$$

In other words, both values are the functions of the relation c/d only in this simplest case. As we can see, the function (33) increases with the increasing of the relation c/d and has only singularity at infinity, whereas the function (34) decreases with the increasing of this relation, has the singularity for $d=c$, and takes negative values if d>c. The discussed functions are shown on the Figure 1.

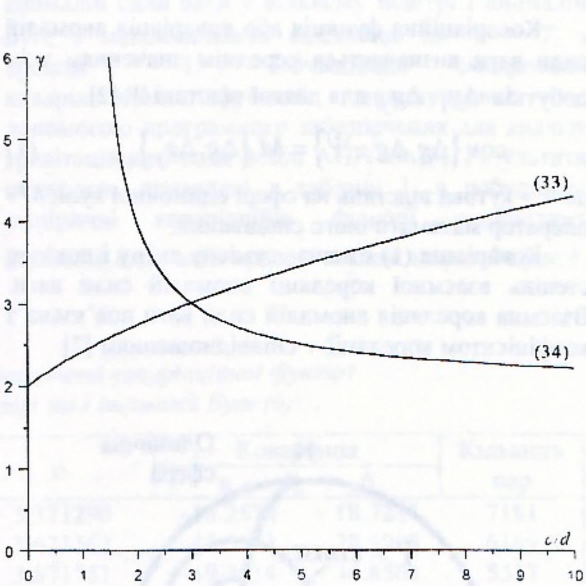

Fig. 1. Propagation of the functions (33) and (34)

References

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ПРО ОБЧИСЛЕННЯ ПАРАМЕТРУ РЕГУЛЯРИЗАЦІЇ

Резюме

Розглянуті дві можливості визначення параметру регуляризації на основі принципу нев'язки.

О.А. Абрикосов

О ВЫЧИСЛЕНИИ ПАРАМЕТРА РЕГУЛЯРИЗАЦИИ

Резюме

Рассмотрены две возможности определения параметра регуляризации на основе принципа невязки.