SIMULTANEOUS NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS OF A GEODETIC LINE IN COMBINATION WITH SURFACE CALCULATION

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1. Introduction

Calculation of geodetic ellipsoidal co-ordinates , the length of the geodetic line and the azimuths are part of the most important tasks in geometrical geodesy and navigation This is called the direct and inverse geodetic problems.

There is a great deal of algorithms for solving the direct and inverse geodetic problems on a biaxial ellipsoid. The authors of the solutions are made it their to minimise the amount of necessary arithmetic operations. This is the reason why there are different solutions for short, medium and long distances.

From the mathematical point of view one always deals with a system of ordinary differential equations. The equations describe a geodetic line on a biaxial ellipsoid. In the direct problem the initial values are given while in the inverse problem the boundary values are given. When one solves the problem manually it is important to find the method that demands the least amount of arithmetic operations. This situation is changed when using a computer. Then the best solution is to find a general algorithm that can be used in every case and the amount of arithmetic operations is of secondary importance.

2. Solution of the direct problem

in the direct problem the following values are known:

a - major semi-axis of the ellipsoid,

e^2 - eccentricity of the ellipsoid,

B_1, L_1, - geodetic co-ordinates of a point P_1 on the ellipsoid,

B - latitude, L longitude,

s - length of a geodetic line which starts in the point P_1,

A_1 - the geodetic line's azimuth in P_1.

To compute:

B_2, L_2, - geodetic co-ordinates of the end point P_2 of the geodetic line,

A_2 - the geodetic line's azimuth in P_2.

The geodetic line's differential equations are:

$$\frac{dB}{ds}=\frac{\cos A}{M}$$
$$\frac{dL}{ds}=\frac{\sin A}{N\cos B} \qquad (2.1)$$
$$\frac{dA}{ds}=\frac{\sin A\cos B}{N\cos B}$$

where

$$M=\frac{a(1-e^2)^2}{(1-e\ \sin^2 B)^{3/2}}$$

$$N=\frac{a}{(1-e\ \sin^2 B)^{1/2}}$$

The direct problem expressed in mathematical terms:

- solve the system of ordinary differential equations (2.1) with the initial values (B_1, L_1, s, A_1).

This can be solved with some numerical method e.g. Runge-Kutta :

The given system of ordinary differential equations

$$\frac{dY}{ds}=f(Y) \qquad (2.2)$$

with the initial values $Y(s_0)=C$.

In this case:

$$Y(s_0) = \begin{bmatrix} B_1 \\ L_1 \\ A_1 \end{bmatrix}, \quad Y = \begin{bmatrix} B \\ L \\ A \end{bmatrix} \tag{2.3}$$

$$f(Y) = \begin{bmatrix} \dfrac{\cos A}{M} \\ \dfrac{\sin A}{N\cos B} \\ \dfrac{\sin A \sin B}{N\cos B} \end{bmatrix} \tag{2.4}$$

Suppose that $Y_n = Y(s_n)$ is known. Then the approximation Y_{n+1} of $Y(s_n + h)$ is

$$Y_{n+1} = Y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.5}$$

where

$$k_1 = h \cdot f(Y_n) \qquad k_2 = h \cdot f(Y_n + \frac{1}{2}k_1)$$

$$k_3 = h \cdot f(Y_n + \frac{1}{2}k_2) \qquad k_4 = h \cdot f(Y_n + k_{31})$$

The global error is then $O(h^4)$.

3. Solution of the inverse problem

In the inverse problem the following values are known:

a - major semi-axis of the ellipsoid,

e^2 - eccentricity of the ellipsoid,

B_1, L_1, B_2, L_2 - geodetic co-ordinates of two points P_1 and P_2 on the ellipsoid.

To compute:

s - the length the geodetic line between the end points P_1 and P_2.

A_1, A_2 - the geodetic azimuths in the end points P_1 and P_2.

The inverse problem expressed in mathematical terms:

- solve the system of ordinary differential equations (2.1) with the boundary values B_1, L_1, B_2, L_2.

The problem can be solved iterative with some numerical method (e.g. Runge-Kutta).

The iterative process is composed of the following steps.

3.1 Computation of the approximation values of s and A_1.

To get approximations of *s (=s*)* and of A_1 *(=A_1*)* one needs to:

- approximate the ellipsoid with a sphere of the radius R

$$R = \sqrt{M_c N_c}$$

where $$M_c = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\frac{B_1+B_2}{2})^3}}, \quad N_c = \frac{a}{\sqrt{1-e^2\sin^2\frac{B_1+B_2}{2}}}$$

- compute the geocentric latitudes

$$\tan\varphi_1 = (1-e^2)\tan B_1$$

$$\tan\varphi_2 = (1-e^2)\tan B_2$$

- compute s* and A_1* with the help of a spherical triangle (fig. 3.1)

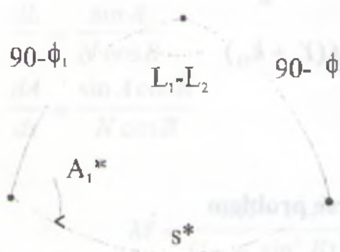

Fig. 3.1 Spherical triangle

$$Q = \arctan\frac{\cos\varphi_2\sin(L - L_1)}{\sin\varphi_2\cos\varphi_1 - \cos\varphi_2\sin\varphi_1\cos(L_2 - L)_1}$$

if $B_2 < B_1$, then $A_1^* = \pi + Q$ otherwise

if $L_2 < L_1$, then $A_1^* = 2\pi + Q$ otherwise $A_1^* = Q$

$$s^* = \arctan\frac{\cos\varphi_2\sin(L_2 - L_1)}{\sin A_2\left[\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2\cos(L_2 - L_1)\right]}$$

if $s^* < 0$, then $s^* := s^* + \pi$

- calculate s^* in meters

$$s^* := s^*\pi$$

3.2 Solution of the direct problem with initial values B_1, L_1, A_1^*, s_1^*

The problem is solved accordingly to chapter 2. The result is written in the form B_2^*, L_2^*, A_2^*.

3.3 Comparison of B_2^* with B_2 and L_2^* with L_2

If $|B_2^* - B_2| < tol$ and $|L_2^* - L_2| < tol$

(where *tol* is the precision with which one wants the result), then the problem is solved. Otherwise improvements are necessary.

3.4 Improvement of s^* and A^*_1

The improvements are written as Δs and ΔA_1.

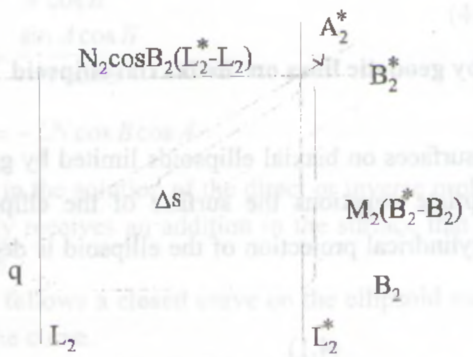

Fig. 3.2 The surrounding of P_2

If the surrounding of P_2 can be considered as a plane, then the following values are received

$$\Delta s = N_2 \cos B_2 \sin A_2^* \,(L_2^* - L_2) + M_2 \cos A_2^* (B_2^* - B_2)$$

$$q = M_2 \sin A_2^* (B_2^* - B_2) - N_2 \cos B_2 (L_2^* - L_2)$$

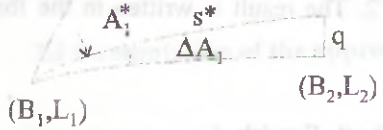

Fig 3.3 Thin perpendicular triangle

From a spherical perpendicular triangle with q being small (Fig.3.3) one receives

$$\Delta A_1 = \frac{N_2 \cos B_2 \cos A_2^*(L_2^* - L_2) - M_2 \sin A_2^*(B_2^* - B_2)}{\sin(s^*/R)\sqrt{M_2 N_2}}$$

3.5 Iterative process

With values of $B_1, L_1, s^* + \Delta s, A_1^* + \Delta A_1$ the calculations in §3.2 are repeated until the inequations in §3.3 are satisfied.

4. Surface on a figure limited by geodetic lines on the biaxial ellipsoid

In some contexts a calculation of surfaces on biaxial ellipsoids limited by geodetic lines are necessary. To receive appropriate equations the surface of the ellipsoid is projected on a cylinder. An equal area cylindrical projection of the ellipsoid is described by:

$$c \cdot dx = MN \cos B dB \qquad dy = c \cdot dL \tag{4.1}$$

from which one receives

$$x = \frac{1}{c} MN \cos B dB \qquad y = c dL \tag{4.2}$$

where c is a arbitrary constant (scaling factor).

According to Green's theorem the surface limited by a closed curve S equals

$$f = \mathrm{S}\oint(xdy - ydx) \quad or$$

$$f = \oint_S xdy \quad or \tag{4.3}$$

$$f = -\oint_S ydx$$

We choose to use the last equation in (4.3).

The addition df to the surface f created by the change ds along the geodetic line s on the ellipsoid is then:

$$\frac{df}{ds} = -y\frac{dx}{ds} \qquad (4.4)$$

From (4.1) one receives $\frac{dx}{ds} = \frac{MN\cos B}{c}\frac{dB}{ds}$ (4.5)

Along the geodetic line: $\frac{dB}{ds} = \frac{\cos A}{M}$ (4.6)

(4.6) put in (4.5) give $\frac{dx}{ds} = \frac{N\cos B\cos A}{c}$ (4.7)

and finally $\frac{df}{ds} = -LN\cos B\cos A$ (4.8)

The system (2.1) of geodetic line's differential equations is expanded with (4.) to

$$\frac{dB}{ds} = \frac{\cos A}{M}$$
$$\frac{dL}{ds} = \frac{\sin A}{N\cos B}$$
$$\frac{dA}{ds} = \frac{\sin A\cos B}{N\cos B} \qquad (4.9)$$
$$\frac{df}{ds} = -LN\cos B\cos A$$

If one in the solution of the direct or inverse problem uses (4.9) instead of (2.1) one automatically receives an addition in the surface that comes from the calculation of the geodetic line.

If one follows a closed curve on the ellipsoid one receives the surface of the figure limited by the curve.

Attention! The sign on the surface depends on whether the curve is being followed according to the clock's motion or not.

5. Testing the method

To test the method in a pc-environment a program has been written. The program uses an algorithm for Runge-Kutta with automatic choice of the integration step (h) so that it will

receive a given a priori precision in the final result [4].

A great deal of useful procedures in the program come from professor G. Dalquist in the department of Numerical Analyses and Computing Science in The Royal Institute of Technology. The program calculates the direct and inverse problem and the surface limited by a closed curve on the ellipsoid. On the screen it draws (in the BL co-ordinate system) lines as it counts and the result is written in a file.

The program has been tested with varying data from [1], [2] and the calculations on the sphere have been made where the results were received manually.

6. Conclusions

The method presented above turns out to be appropriate for short as well as long distances. With one and the same program different, specific, problems can be solved. This can be used in calculation of e.g. only the geodetic lines length. The method also solves the problem of calculating surfaces, limited by geodetic lines on the ellipsoid. There are no restrictions when it comes to the size, and more important, the shape of the surface. This method is superior to the others in the literature, which are limited to calculations of surfaces of a geodetic triangle.

In conclusion one can say that the method solves general problems when it comes to calculations on the biaxial ellipsoid.

References:

[1] P.Vanicek, E.J. Krakiwsky - Geodesy: the concepts, 2nd edition, Elservier Science Publishers B.V. ,1986

[2] W. Szpunar - Postawy geodezji wyzszej, PPWK-Warszawa, 1982

[3] C.F.Baeschlin - Lehrbuch der Geodäsi, Orell Füssli Verlag Zürich, 1948

[4] Dormand, Prince - Solving ordinary differential equation I, Hairer, Norsten Wanner page 433.

ABSTRACT

The purpose of this paper is to present a new method for simultaneous numerical solutions of differential equations of a geodetic line in combination with surface calculation. This method was evaluated as a tool for determination of borders between Sweden and the Union of Soviet Socialist Republics but it has not been published yet. This method is superior to the others in the literature, which are limited to calculations of surfaces of a geodetic triangle. There are no restrictions when it comes to the size, to the distances, and more important, the shape of the surface. With one and the same program different, specific, problems can be solved. This can be used in calculation of e.g. only the geodetic lines length. The method also solves the problem of calculating surfaces, limited by geodetic lines on the ellipsoid. In this paper the whole solution of the problem step by step is presented. The results from test calculations are discussed and prove the theoretical model to be correct. In conclusion one can say that the method solves general problems when it comes to calculations on the biaxial ellipsoid.