MATHEMATICAL MODELS OF A PRIORI ACCURACY ESTIMATION IN PHOTOGRAMMETRY

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Abstract

The prediction mathematical models of the random and systematic errors influence in photogrammetry are given in this paper.

1. Introduction. Initial mathematical model.

Suppose that initial equations are described as

$$A\,U + l + \Theta + \Delta = V, \tag{1}$$

where A is matrix of the coefficients of the correction's equations;

U — vector of most probably values of the unknown parametries,

$(l + \Theta + \Delta)$ — vector of the measured values,

l — vector of the true values of the measurings,

Θ — vector of the systematic errors of the measured values,

Δ — vector of the random errors of the measured values,

V — vector of residual deviations.

Minimising *of* ($V^T V$) gives:

$$U = -(A^T A)^{-1} A^T (l + \Theta + \Delta). \qquad (2)$$

From this expression the distortions of the vector U under the influence of the vectors Δ and Θ are :

$$dU_1 = (A^T A)^{-1} (A^T \Delta), \qquad (3)$$

$$dU_2 = (A^T A)^{-1} (A^T \Theta). \qquad (4)$$

The vectors dU_1 and dU_2 permit to construct models of the accuracy prediction.

2. The model of a priori estimation of the accuracy in the presence of the random errors vector.

It is known, that vector dU_1 and corresponding vector of the residual deviations V_1 are characterised by the covariance matrix K_u and K_v :

$$K_u = \sigma^2 (A^T A)^{-1}, \qquad (5)$$

$$K_v = \sigma^2 (A (A^T A)^{-1} A^T). \qquad (6)$$

In photogrammetry the initial equations (1) represent either approximating functions or linear equations which is congenial by form. In this case we can imagine the covariance matrix (5) as :

$$K_u = \sigma^2 \begin{bmatrix} \lambda_{11} Q_{\xi 11} & \lambda_{12} Q_{\xi 12} & \dots & \lambda_{1m} Q_{\xi 1m} \\ \lambda_{12} Q_{\xi 12} & \lambda_{22} Q_{\xi 22} & \dots & \lambda_{2m} Q_{\xi 2m} \\ \dots & \dots & \dots & \dots \\ \lambda_{1m} Q_{\xi 1m} & \lambda_{2m} Q_{\xi 2m} & \dots & \lambda_{mm} Q_{\xi mm} \end{bmatrix}, \qquad (7)$$

where λ_{ij} are functions of the image size and airsurvey parameters;

$Q_{\xi\ ij}$ are functions only of the points quantity and location.

Form of functions for λ_{ij} and $Q_{\xi IJ}$ depends from form of the initial equations (1).

So, if approximating (power algebraic) polynoms are used as the initial equations[1], matrix (7) is :

$$K_u = \sigma^2 \begin{bmatrix} Q_{\xi 11} & \frac{1}{c} Q_{\xi 12} & \dots & \frac{1}{c^{(k_m + l_m)}} Q_{\xi 1m} \\ \frac{1}{c} Q_{\xi 12} & \frac{1}{c^2} Q_{\xi 22} & \dots & \frac{1}{c^{(k_m + k_2) + (l_m + l_2)}} Q_{\xi 2m} \\ \dots & \dots & \dots & \dots \\ \frac{1}{c^{(k_m + l_m)}} Q_{\xi 1m} & \frac{1}{c^{(k_m + k_2) + (l_m + l_2)}} Q_{\xi 2m} & \dots & \frac{1}{c^{2(k_m + l_m)}} Q_{\xi mm} \end{bmatrix}, \qquad (8)$$

where σ — standard of weight unit,

$Q_{\xi ij}$ are the elements of the matrix

$$Q_\xi = (A_\xi^T \ A_\xi)^{-1}, \qquad (9)$$

here A_ξ — matrix, and it's $i\,j$ - th. element is

$$a_{\xi\,ij} = \xi_i^{k_j} \cdot \eta_i^{l_j}, \qquad \xi_i = x_i / |c|; \qquad \eta_i = y_i / |c| \qquad (10)$$

$i = 1,\dots, n$ (n - quantity of the points, which are used for approximation);

$j = 1,\dots, m$ (m - quantity of approximating polynom terms);

$|c|$ — the modulus of the maximum value of the point abscissa (ordinate) on the approximation area.

After the approximation the vector $d\,U_l$ causes a residual deviations V_{li} which are characterised by the covariance matrix according to (6) and (8)

$$K_V = \sigma^2 (A_\xi \; Q_\xi \; A_\xi^{\;T}). \qquad (11)$$

As approximation is implemented with using of the power algebraic polynoms, the mathematical correlation appears among the vector V_l elements. On these conditions, the matrix Q_ξ which is a covariance matrix (with accuracy up to σ^2) of the residual deviations for separate terms of the approximating polynom for points with co-ordinates $|x|=|y|=c$ has a predominant importance for the prediction.

Then value of a dispersion the residual deviation for this point is

$$D_{vi} = \sigma^2 \sum_{i=1}^{m} \sum_{j=1}^{m} (Q_{\xi ij}) \qquad (12)$$

Using the elements of matrix Q_ξ with various quantity of the control points and with various geometric schemes of the location ones it is possible :

— to determine the dispersion value of the residual deviation with the given precision of a determination of the approximating polynom separate terms.

— to find the optimum scheme of the points location on the images for given kind of the approximating polynom.

If the linear equations of collinearity [2] are uses as the initial equations, (for determination of the elements of image absolute orientation), then covariance matrix (7) is [3] :

$$K_u=\sigma^2\begin{array}{c}\begin{matrix} x_s & y_s & z_s & \alpha & \omega & \kappa\end{matrix}\\ \begin{bmatrix} \frac{H^2}{f^2}Q_{\xi 11} & 0 & 0 & \frac{H}{c^2}Q_{\xi 14} & 0 & 0 \\ 0 & \frac{H^2}{f^2}Q_{\xi 22} & 0 & 0 & \frac{H}{c^2}Q_{\xi 25} & 0 \\ 0 & 0 & \frac{H^2}{c^2}Q_{\xi 33} & 0 & 0 & 0 \\ \frac{H}{c^2}Q_{\xi 41} & 0 & 0 & \frac{f^2}{c^4}Q_{\xi 44} & 0 & 0 \\ 0 & \frac{H}{c^2}Q_{\xi 52} & 0 & 0 & \frac{f^2}{c^4}Q_{\xi 55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{c^2}Q_{\xi 66} \end{bmatrix}\end{array}, \tag{13}$$

where $Q_{\xi ij}$ matrix elements:

$$Q_{\xi}=\sigma^2\begin{bmatrix} \frac{r}{D} & 0 & 0 & -\frac{t}{D} & 0 & 0 \\ 0 & \frac{r}{D} & 0 & 0 & -\frac{t}{D} & 0 \\ 0 & 0 & \frac{1}{\sum(\xi_i^2+\eta_i^2)} & 0 & 0 & 0 \\ -\frac{t}{D} & 0 & 0 & \frac{n}{D} & 0 & 0 \\ 0 & -\frac{t}{D} & 0 & 0 & \frac{n}{D} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sum(\xi_i^2+\eta_i^2)} \end{bmatrix}, \tag{14}$$

here Q_{ξ} was determined by formula (9);

$$r=\frac{f^4}{c^4}\cdot n+\frac{2f^2}{c^2}\cdot\sum\xi_i^2+\sum\xi_i^4+\sum\left(\xi_i^2\cdot\eta_i^2\right), \tag{15.1}$$

$$t = \frac{f^2}{c^2} \cdot n + \sum \xi_i^2 , \qquad (15.2)$$

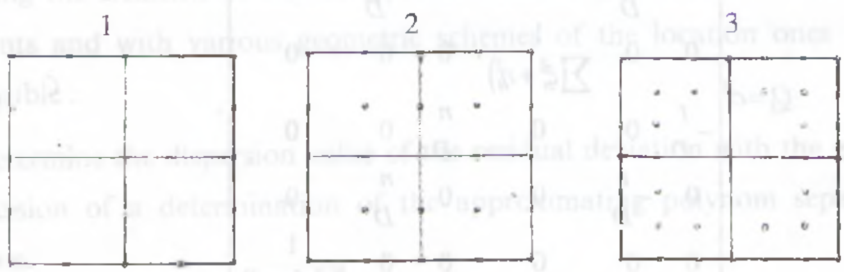
$$D = n\,r - t^2 , \qquad (15.3)$$

Matrixes (13), (14) are written for case of the symmetrical points location.

Matrix (14), by analogy with (9) is covariance matrix (with accuracy up to σ^2) of the resudial deviations *Vc* on the point with co-ordinate x=y=c. Each element of the vector *Vc* is conditioned by error of the certain element of absolute orientation of the image.

For example the numerical values of the matrix Q_x elements for the approximating polynom of 3th power with various schemes of the points location are given lower:

schemes points location



n=16 n=24 n=40

for schema 1

$$Q_\xi=\begin{bmatrix} a_0 & a_1x & a_2y & a_3x^2 & a_4xy & a_5y^2 & a_6x^3 & a_7x^2y & a_8xy^2 & a_9y^3 \\ 0.90 & 0 & 0 & -0.61 & 0 & -0.61 & 0 & 0 & 0 & 0 \\ 0 & 2.57 & 0 & 0 & 0 & 0 & -2.27 & 0 & -0.67 & 0 \\ 0 & 0 & 2.57 & 0 & 0 & 0 & 0 & -0.67 & 0 & -2.27 \\ -0.61 & 0 & 0 & 0.56 & 0 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.17 & 0 & 0 & 0 & 0 & 0 \\ -0.61 & 0 & 0 & 0.33 & 0 & 0.56 & 0 & 0 & 0 & 0 \\ 0 & 2.27 & 0 & 0 & 0 & 0 & 2.21 & 0 & 0.38 & 0 \\ 0 & 0 & -0.67 & 0 & 0 & 0 & 0 & 0.57 & 0 & 0.38 \\ 0 & -0.67 & 0 & 0 & 0 & 0 & 0.38 & 0 & 0.57 & 0 \\ 0 & 0 & -2.27 & 0 & 0 & 0 & 0 & 0.38 & 0 & 2.21 \end{bmatrix},$$

$$D_v=0.9\sigma^2$$

for schema 2

$$Q_\xi=\begin{bmatrix} a_0 & a_1x & a_2y & a_3x^2 & a_4xy & a_5y^2 & a_6x^3 & a_7x^2y & a_8xy^2 & a_9y^3 \\ 0.72 & 0 & 0 & -0.50 & 0 & -0.50 & 0 & 0 & 0 & 0 \\ 0 & 1.93 & 0 & 0 & 0 & 0 & -1.69 & 0 & -0.62 & 0 \\ 0 & 0 & 1.93 & 0 & 0 & 0 & 0 & -0.62 & 0 & -1.69 \\ -0.50 & 0 & 0 & 0.45 & 0 & 0.29 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0 & 0.29 & 0 & 0.45 & 0 & 0 & 0 & 0 \\ 0 & -1.69 & 0 & 0 & 0 & 0 & 1.62 & 0 & 0.40 & 0 \\ 0 & 0 & -0.62 & 0 & 0 & 0 & 0 & 0.48 & 0 & 0.40 \\ 0 & -0.62 & 0 & 0 & 0 & 0 & 0.40 & 0 & 0.48 & 0 \\ 0 & 0 & -1.69 & 0 & 0 & 0 & 0 & 0.40 & 0 & 1.62 \end{bmatrix},$$

$$D_v=0.7\sigma^2$$

for schema 3

$$Q_\xi=\begin{bmatrix} a_0 & a_1x & a_2y & a_3x^2 & a_4xy & a_5y^2 & a_6x^3 & a_7x^2y & a_8xy^2 & a_9y^3 \\ 0.16 & 0 & 0 & -0.12 & 0 & -0.12 & 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0 & 0 & 0 & -0.47 & 0 & -0.17 & 0 \\ 0 & 0 & 0.50 & 0 & 0 & 0 & 0 & -0.17 & 0 & -0.47 \\ -0.12 & 0 & 0 & 0.19 & 0 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ -0.12 & 0 & 0 & 0.05 & 0 & 0.19 & 0 & 0 & 0 & 0 \\ 0 & -0.47 & 0 & 0 & 0 & 0 & 0.57 & 0 & 0.03 & 0 \\ 0 & 0 & -0.17 & 0 & 0 & 0 & 0 & 0.32 & 0 & 0.03 \\ 0 & -0.17 & 0 & 0 & 0 & 0 & 0.03 & 0 & 0.32 & 0 \\ 0 & 0 & -0.47 & 0 & 0 & 0 & 0 & 0.03 & 0 & 0.57 \end{bmatrix}.$$

$$D_v=0.6\sigma^2$$

So this model for the prediction uses matrix Q_ξ instead of covariation matrix K_ν. By this such advantages as simplicity of determination and reliability are reached.

3. The model of the prediction of the systematic errors influence.

Assume, that the elements of the vector Θ of the systematic errors in (4) are described by power algebraic polynom. Then vector Θ is written as:

$$\Theta = A_p b, \tag{16}$$

where A_p is matrix with elements:

$$a_{ih} = x_i^{k_h} \cdot y_i^{l_h} \tag{17}$$

h=1,2,..., p; p - quantity of the polynom terms; b-vector of the polynom coefficients.

Imagine the distortion vector δU_2 as a function of polynom terms for the point with the co-ordinate $x = y = |c|$. Then :

$$\delta U_2 = G_u \cdot d , \tag{18}$$

where $G_u = - (A^T \cdot A)^{-1} \cdot (A^T \cdot A_{p\xi}),$ (19)

$$d = [d_1 \; d_2 \; \dots \; d_h \; \dots \; d_p]^T, \tag{20}$$

$$d_h = b_h \cdot c^{(k_h + l_h)}, \tag{21}$$

$A_{p\xi}$ — the matrix of the coefficient with d_h, where

$$a_{p\xi(ih)} = \xi_i^{k_h} \cdot \eta_i^{l_h}. \tag{22}$$

The residual deviations V_2 caused by vector $d\,U_2$ is

$$V_2 = (A_{p\xi} - G) \cdot d \tag{23}$$

where $G = A_\xi \; (A_\xi^T \cdot A_\xi)^{-1} \cdot (A_\xi^T A_{p\xi}).$ (24)

If the initial equations are approximating polynoms, the elements of the matrix A_ξ are determinate by the formula (10), and vector Θ includes only unconsidered polynom terms.

As the elements of matrix A_ξ and $A_{p\xi}$ are the function of the image points co-ordinates, the elements of matrix G and $(A_{p\xi} - G)$ can be beforehand determined for various points quantity and the schemes of the location ones. And here the matrix $(A_{p\xi} - G)$ elements show what share of the h[th] polynom unconsidered terms (20) remains on the control points after the approximation.

Therefore using comparison of values $(A_{p\xi} - G)$ for various schemes of the points location and various approximating polynom power it is possible to find:

— the optimum scheme of the points location with the approximation for given polynom power;

— the optimum power of the approximating polynom;

— the expected value V_2 (in the shares from d_h) for the given scheme of the points location and for the given polynom power.

The analysis of the structure of the matrix Gu and G (expression (19) and (24)) shows that the basis of this prediction model is a finding of the matrix $G_{u\xi}$, which is:

$$G_{u\xi} = (A_\xi^T \cdot A_\xi)^{-1} \cdot (A_\xi^T A_{p\xi}) \qquad (25)$$

The matrix $G_{u\xi}$ elements give possibility to analyse how does each term of the approximating polynom reduce the unconsidered terms of the higher powers(on point $x = y = c$).

For any points it is

$$G_{u\xi\,(jh)i} = G_{u\xi(jh)} \cdot \xi_i^{k_j} \cdot \eta_i^{h_j} \qquad (26)$$

The numerical values of the matrix $G_{u\xi}$ for the unconsidered terms of the 4-th power are given lower:

for schemes 1,2

$G_{u\xi}=$

	xy^3	x^2y^2	x^3y	x^4	y^4
	0	–0.99	0	–0.05	–0.05
	0	0	0	0	0
	0	0	0	0	0
	0	1.0	0	1.07	–0.08
	0.83	0	0.83	0	0
	0	1.0	0	–0.08	1.07
	0	0	0	0	0
	0	0	0	0	0
	0	0	0	0	0
	0	0	0	0	0
Σ	0.83	1.01	0.83	0.94	0.94

for schema 3

$G_{u\xi}=$

	xy^3	x^2y^2	x^3y	x^4	y^4
	0	–0.37	0	–0.20	–0.20
	0	0	0	0	0
	0	0	0	0	0
	0	0.58	0	1.13	0.07
	0.78	0	0.78	0	0
	0	0.58	0	0.07	1.13
	0	0	0	0	0
	0	0	0	0	0
	0	0	0	0	0
	0	0	0	0	0
Σ	0.78	0.79	0.78	1.00	1.00

If the initial equation (1) are formed from known photogrammetric equations, then the matrix A_ξ in (24) is a transformed matrix A under the condition that x, y are determined by formula (10) and all elements of the matrix A_ξ are the result of the simple transformations do not depend on aerosurvey parameters and image size [3].

For task of the determination of the elements of image absolute orientation the matrix $G_{u\xi}$ elements (formula 25)will show, what share of the systematic errors dx (dy) on the points with co-ordinate $x=y=c$ is absorb by each element of the image absolute orientation . According to formulas (25), (14), (15.1-15.3) the analytical expression for matrix $G_{u\xi}$ elements in the form of functions of the values t, r ,n are obtain. Numerical values of the matrix $G_{u\xi x}$ and $G_{u\xi y}$ which characterise the influence of the systematic errors of the abscissa and ordinate of the image points are given below for schema 2 of the point location.

$$
G_{\xi x} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \Sigma \end{array}
\begin{array}{cccccccccc}
b_0 & b_1x & b_2y & b_3xy & b_4x^2 & b_5y^2 & b_6xy^2 & b_7x^2y & b_8x^3 & b_9y^3 \\
1 & 0 & 0 & 0 & -0.08 & 0.56 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.92 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0.42 & 0 \\
0 & 0 & 0 & 0 & 0.40 & -0.02 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.60 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0 & 0 & 0 & 0 & -0.25 & 0 & -0.42 \\
1 & 0.5 & -0.5 & -0.32 & 0.32 & 0.54 & 0.25 & -0.25 & 0.42 & -0.42
\end{array}
\begin{array}{c} \\ X_S \\ Y_S \\ Z_S \\ \alpha \\ \omega \\ \kappa \\ \\ \end{array}
$$

$$
G_{\xi y} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \Sigma \end{array}
\begin{array}{cccccccccc}
b_0 & b_1x & b_2y & b_3xy & b_4x^2 & b_5y^2 & b_6xy^2 & b_7x^2y & b_8x^3 & b_9y^3 \\
0 & 0 & 0 & -0.92 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0.56 & -0.08 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0.42 \\
0 & 0 & 0 & 0.60 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.02 & 0.40 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0 & 0 & -0.25 & 0 & -0.42 & 0 \\
1 & -0.5 & 0.5 & -0.32 & 0.54 & 0.32 & -0.25 & -0.25 & -0.42 & 0.42
\end{array}
\begin{array}{c} \\ X_S \\ Y_S \\ Z_S \\ \alpha \\ \omega \\ \kappa \\ \\ \end{array}
$$

So the developed approach to a priori accuracy estimation solves to some extent the problem of the prediction of the influence of the random and systematic errors.

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