ГЕОДЕЗІЯ

EARTH'S RADIAL PROFILES BASED ON LEGENDRE-LAPLACE LAW

A. N. Marchenko

(State University "Lviv Polytechnic")

Abstract. *The famous (and oldest) solution of Clairaut's equation or Legendre-Laplace law of density was used for the parameterization of the hydrostatic/adiabatic Earth's radial density distribution in the two standard forms: continuous and piecewise radial cases. As a result, the set of recursive formulae were found for the solution of direct and inverse problems (on the ground of fundamental geodetic constants and seismic data simultaneously). The mentioned piecewise Legendre-Laplace density is in a good agreement with the PREM-density model.*

Introduction

Traditional consideration of the famous expression of a planet's Newtonian potential

$$V(P) = G\iiint_v \frac{dm}{\ell} = \iiint_v \frac{\rho}{\ell}dv, \qquad (1)$$

leads to well-known conclusion that all problems regarding the determination of a planet (Earth) gravitational potential may be solved if the corresponding volume density function ρ of this body is known. In (1) G=6.673·10^{-8} $cm^2g^{-1}sec^{-2}$ is the gravitational constant; dv is an element of the volume v; dm is an element of mass; ℓ is the distance between the mass element $dm=\rho dv$ and the attracted point P with unit mass. In practice we have another or inverse situation: we want to determine this function ρ (to take into account our poor knowledge about densities and some initial information about planets). First of all we want to determine the planet's radial densities $\rho(\ell)$, for instance, from the solution of inverse gravitational problem with an additional application of the Earth's seismic data and other geophysical data.

Recently, instead of the standard polynomial representation of a piecewise radial density some old hypotheses for density distribution (see, Bullen, 1975) were analyzed especially in view of Clairaut and Williamson-Adams equations (Marchenko, 1999; Marchenko, 2000). The latter leads to the special investigation of the *hydrostatic/adiabatic Earth*. In particular, Darwin's law (1884) of radial density was used for the transformation to the famous *Gaussian distribution* that was called by the *Earth's density normal law* with the treatment of Roche's law as a truncated Taylor series expansion of Gauss' or Legendre-Laplace model (Marchenko, 1999; Marchenko, 2000). The created radial profiles are in a good agreement with the PREM-density model, including gravity distribution. For this reason just Gauss' normal model was applied further for the representation and comparison of planet's radial densities profiles. The goal of this paper is a study in the same way the oldest solution of Clairaut's equation, which is well-known as Legandre-Laplace law of the radial density distribution.

1. Basic expressions for the stratified spherical Earth

As before, we shall assume the figure of the Earth's in form of a sphere with the certain mean radius R=6371 km. In our formulation the planet's radial density will be considered only as a function $\rho(\ell)$ of the current radius ℓ ($0 \le \ell \le R$). Since we use a sphere instead an ellipsoid with the flattening f=0, the gravitational potential V will be treated also as the gravity potential W (Moritz, 1990).

The planet mass M and its mean moment of inertia I will be chosen as observed data. For latter use, we shall rewrite several well-known formulae in the case of a layered sphere of the radius ℓ, restricted planet masses. If a stratification of the Earth leads to its division into m shells, we shall represent a volume radial density by one model within every shell separately

$$\rho_i(\ell), \quad i = 1,2,\ldots m. \qquad (2)$$

with a suitable mathematical representation of the functions (2). Then we shall consider these relationships for a *spherical* stratified Earth. Expression for the *mass* can be written now as

$$M(\ell) = 4\pi\sum_{i=1}^{m-1}\int_{\ell_{i-1}}^{\ell_i}\rho_i(x)x^2dx + \int_{\ell_{m-1}}^{\ell}\rho_m(x)x^2dx, \quad (\ell_0 = 0), \qquad (3)$$

and the *mean moment of inertia* will be represented in the similar way

$$I(\ell)=\frac{8\pi}{3}\left[\sum_{i=1}^{m-1}\int_{\ell_{i-1}}^{\ell_i}\rho_i(x)x^4dx+\int_{\ell_{m-1}}^{\ell}\rho_m(x)x^4dx\right],$$

$$(\ell_0=0). \qquad (4)$$

where dx is the element of a line. The mean density $D(\ell)$ and the gravity inside a planet admit the following standard representations

$$D(\ell)=\frac{3}{4\cdot\pi\cdot\ell^3}M(\ell), \qquad (5)$$

$$g(\ell)=\frac{4\pi G}{3}\ell D(\ell). \qquad (6)$$

In the case of the Earth we shall use the seismic velocities V_p and V_s as well-known function

$$\Phi=\Phi(\ell)=V(\ell)_P^2-\frac{4}{3}V(\ell)_S^2, \qquad (7)$$

by applying their grid values in accordance with the PREM-model (Dziewonski and Anderson, 1981) which reflects the results of seismic radial tomography of the Earth interior.

Assuming that such a piecewise density model should fulfill some standard differential equations, our nearest goal will consist of the computation of the integrals (3), (4) after the choice of an appropriate mathematical expression for the density distribution within each shell. The separation of the spherical planet into convenient shells has to be choice at those spheres, where discontinuities in the parameter Φ or in its derivative can be observed.

2. Auxiliary relationships for Legendre-Laplace model

According to (Bullen, 1975; Moritz, 1990, etc.) there exist three famous solutions of Clairaut's equation for the radial density distribution ρ inside the Earth. First one is *Legandre-Laplace law*, the second one is *Roche's law*, and the third one is *G. Darwin law*. The first law of density distributions will be studied below.

2.1. Continuous radial density distribution

As well-known the radial density ρ may fulfil the so-called Williamson-Adams equation for each shell and it corresponds to the *hydrostatic/adiabatic Earth* under the following assumptions: the Earth is globally in *hydrostatic* equilibrium; the temperature is *adiabatic* in every shell; chemical composition and phase transformations are *homogeneous* in each shell. Now, by applying the observed seismic velocities (7), in view of the gravitational and hydrostatic relationships Williamson-Adams equation can be written finally as

$$\frac{d\ln\rho(\ell)}{d\ell}=-\frac{g(\ell)}{\Phi(\ell)}. \qquad (8)$$

As a result, (8) is a formula to derive the radial density from the seismic velocities data, fulfilled under the assumptions listed above. In order to use (8) we will try further to apply Legendre-Laplace law

$$\rho(\ell)=\rho_0\frac{\sin(\gamma x)}{\gamma x}, \quad (\gamma=\text{const}), \qquad (9)$$

and to express the observed seismic data by the corresponded function of a depth. Here x is the dimensionless "radius-vector" $x=\ell/R$ regarding to R; ρ_o = const and may be considered as the density at the origin. Note that Taylor expansion of (9) leads to Roche's model

$$\rho(x)=\rho_0\left(1-\frac{\gamma^2}{6}x^2\right)=a+bx^2, \qquad (10)$$

if we disregard other higher powers of x.

Next, in view of (9) the straightforward integration of the expressions (3) and (4) admits the next remarkable expressions for the Earth's *mass*

$$M(\ell)=-\frac{4\pi R^2}{\gamma^2}\cdot\ell^2\rho'(\ell), \qquad (11)$$

with the following expression for radial derivative

$$\rho'(\ell)=\frac{d\rho(\ell)}{d\ell}=[\rho_0\cos(\gamma\cdot x)-\rho(\ell)]/\ell, \qquad (12)$$

and for the *mean moment of inertia* in the standard way

$$I(\ell)=\frac{2}{3}M(\ell)\cdot\left[\ell^2-\frac{6R^2}{\gamma^2}\right]+\frac{16\pi R^2}{3\gamma^2}\ell^3\rho(\ell), \qquad (13)$$

and in the dimensionless form

$$I_d(\ell)=\frac{2}{3}\cdot\left[x^2-\frac{6\cdot R^2}{\gamma^2}\right]+\frac{4\rho(\ell)}{\gamma^2 D(\ell)}. \qquad (14)$$

Applying the last formula to a surface of the spherical Earth ($\ell=R$), where the observed density $\rho(R)=\rho_s$ is known, we come to the next expression

$$I_d=I_d(R)=\frac{4}{\gamma^2}\cdot\left[\frac{\rho_s}{D}-1\right]+\frac{2}{3}, \qquad (15)$$

where $D=D(R)$ represents the Earth's mean density (D=5.514 g/cm^3). Thus, if the values D, ρ_s, and I_d are known from observations we come to the solution of a non-linear inverse problem by means of two closed expressions for 2 basic parameters. The first one is some *qualitative characteristic* of the global density distribution in the form of a simple formula for the coefficient γ of Legendre-Laplace model

$$\gamma^2 = 4 \cdot \left[\frac{\rho_s}{D} - 1\right] \Big/ \left[I_d - \frac{2}{3}\right], \quad (16)$$

which equal to γ=2.47541 for the next initial values: D=5.514 g/cm^3, ρ_s=2.67 g/cm^3, and I_d=0.329979. The second one represents the *quantitative parameter* of the model (9):

$$\rho_0 = \frac{\gamma^3 D}{3 \cdot [\sin(\gamma) - \gamma \cdot \cos(\gamma)]}, \quad (17)$$

which can be computed on the ground of the expression (16) and γ=2.47541, D=5.514 g/cm^3. As a result, this parameter is equal to ρ_0=10.873 g/cm^3 in the case of the continuous Legandre-Laplace distribution. The solution (16)-(17) provides finally the exact agreement of ρ_s, the mean density D and the mean moment of inertia I_d, but without such good agreement of the density ρ_0 at the centre of the Earth's masses. This model agrees best of all with the continuous Roche's model that is reflected in Figure 1, were comparison with piecewise PREM-model is done.

Note that the determination of 2 parameters of the continuous Roche's model is based traditionally on the mean density D and the mean moment of inertia I_d only. In fact, in the case of the continuous Legendre-Laplace and Gauss' density model (also with 2 parameters), *all standard 3=2+1 conditions (for surface density ρ_s as well) can be replaced by one relationship (16) for the corresponding qualitative characteristic γ*. The quantitative characteristic ρ_0 now is the non-linear function (17) of the computed γ and the Earth's mean density.

2.2. Piecewise radial density distribution

Next we assume that the Earth's stratification gives in the form of its division into m shells, then we will represent the density distribution by one Legendre-Laplace model within every shell separately

$$\rho_i(\ell) = \delta_i \frac{\sin(\gamma_i \cdot x)}{\gamma_i \cdot x}, \quad i = 1,2,\ldots m. \quad (18)$$

Inserting (18) into (3) - (6) we come to the *recurrence formulae* for a sequential computation of the mass, the mean density and the mean moment of inertia, respectively

$$M_{1,m}(\ell) = M_{1,m-1}(\ell_{m-1}) + M_m(\ell), \quad (\ell_{m-1} \le \ell \le \ell_m), \quad (19)$$

$$D_{1,m}(\ell) = \left(\frac{\ell_{m-1}}{\ell}\right)^3 D_{1,m-1}(\ell_{m-1}) + D_m(\ell), \quad (20)$$

$$I_{1,m}(\ell) = I_{1,m-1}(\ell_{m-1}) + I_m(\ell), \quad (21)$$

where every integral in (3), (4) can be expressed by means of the auxiliary relationships

$$\rho_i'(\ell) = \frac{d\rho_i(\ell)}{d\ell} = [\delta_i \cos(\gamma_i \cdot x) - \rho_i(\ell)]/\ell, \quad (22)$$

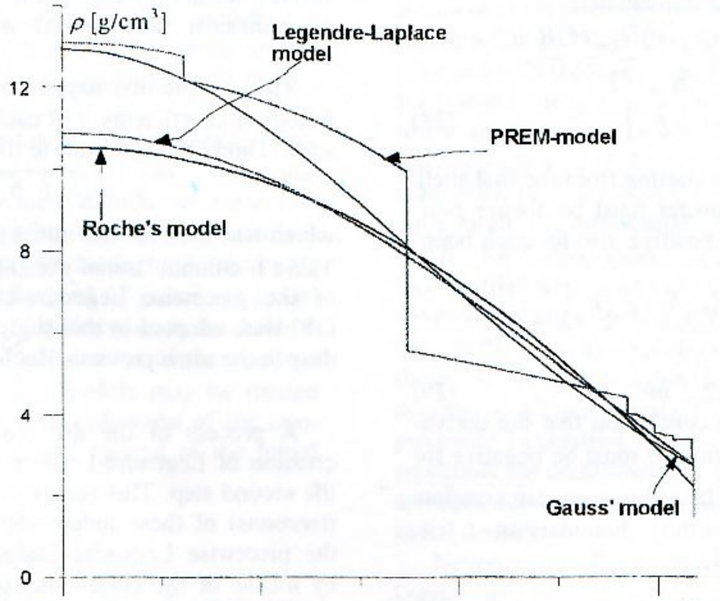

Figure 1. Legendre-Laplace, Roche, and Gauss continuous densities regarding the PREM-model

$$M_i(\ell) = \frac{4\pi R^2}{\gamma_i^2}\left[\ell_{i-1}\rho_i'(\ell_{i-1}) - \ell\rho_i'(\ell)\right],$$

$$M_{1,1}(\ell) = M_1(\ell), \qquad (23)$$

$$D_i(\ell) = \frac{3R^2}{\gamma_i^2 \ell^3}\left[\ell_{i-1}\rho_i'(\ell_{i-1}) - \ell\rho_i'(\ell)\right],$$

$$D_{1,1}(\ell) = D_1(\ell), \qquad (24)$$

$$I_i(\ell) = \frac{4\cdot R^2}{3\gamma_i^2}\left\{4\pi\cdot\left[\ell^3\rho_i(\ell) - \ell_i^3\rho_i(\ell_i)\right]\right.$$

$$\left. - 2\pi\cdot\left[\ell^4\rho_i'(\ell) - \ell_i^4\rho_i'(\ell_i)\right] - M_i(\ell)\right\}$$

$$I_{1,1}(\ell) = I_1(\ell), \qquad (25)$$

starting from the first shell $(0 \le \ell \le \ell_1)$. In these formulae ℓ_j (j=1,2,...m-1) are the fixed radius-vectors, where discontinuities of the radial density are assumed. Formula for the gravity is based on the expressions (6) and (24):

$$g_{1,m}(\ell) = \frac{4\pi G}{3}\ell D_{1,m}(\ell), \quad (\ell_{m-1} \le \ell \le \ell_m). \qquad (26)$$

For the formulae of the seismic parameter Φ and its jumps we shall find

$$\frac{d\ln\rho_i(\ell)}{d\ell} = \frac{\rho_i'(\ell)}{\rho_i(\ell)} = \frac{\gamma_i\cot(\gamma_i x)}{R} - \frac{1}{\ell}, \qquad (27)$$

and in view of Williamson-Adams equation (8) and (27) for such model we get immediately

$$\Phi_{1,m}(\ell) = \frac{4\pi G\rho_m(\ell)D_{1,m}(\ell)R}{3\rho_m'(\ell)},$$

$$(\ell_{m-1} \le \ell \le \ell_m), \qquad (28)$$

the expression for Φ again starting from the first shell. As well-known, this parameter must be always *positive* and the ratio must be positive also for each boundary of two shells, that is

$$\frac{\Phi_{1,i}(\ell_j)}{\Phi_{1,i+1}(\ell_j)} = \frac{\rho_{i+1}'(\ell_j)}{\rho_i'(\ell_j)}\cdot\frac{\rho_i(\ell_j)}{\rho_{i+1}(\ell_j)} > 0,$$

$$i = 1,2,\ldots m. \qquad (29)$$

In particular, we come to conclusion that the derivatives $\rho_i'(\ell_j)$, $\rho_{i+1}'(\ell_j)$ in (29) must be negative for Legendre-Laplace model. In addition we can compute now a seismic jump of Φ at the j - boundary as

$$\Delta\Phi = \Delta\Phi_{i,i+1} = \Phi_i(\ell_j) - \Phi_{i+1}(\ell_j)$$

$$= \frac{4\pi G D_{1,i}(\ell_j)\ell_j}{3}\left[\frac{\rho_{i+1}(\ell_j)}{\rho_{i+1}'(\ell_j)} - \frac{\rho_i(\ell_j)}{\rho_i'(\ell_j)}\right]. \qquad (30)$$

These formulae may be used also as the additional conditions for data processing, because the left-hand side of (29), (30) is known from seismic data. Note also that the density at the origin according to Legendre-Laplace model is depended on the *observed* Φ as

$$\Phi_1(0) = \frac{4\pi G R^2}{\gamma_1^2}\delta_1, \qquad (31)$$

where $\Phi_1(0)$ corresponds to the first piece of the seismic data Φ at the origin. Relationships of the same type for the piecewise Roche's model and for the piecewise Gauss' model can be found in (Marchenko, 1999) and (Marchenko, 2000) respectively.

3. *Piecewise Legendre-Laplace model*

Roche's model (10) can be chosen now within every shell separately as initial iteration for further construction of the piecewise Legendre-Laplace density profile (18), which should be agreed with the whole initial information about the seismic data and astro-geodetic data. Regarding the discontinuities in the seismic velocities we are led to the following separation (Table 1) into shells according to (Marchenko, 1999), where a mathematical description of the Earth's density is based on the piecewise Roche's model with the same separation. This piecewise Roche's model is in a good agreement with the PREM-density, with the exception of the crust shells: on the final step a "geodetic version" of the piecewise Roche's density profile with the surface density ρ_s=2.67 g/cm^3 (included into solution as additional information) was build (Marchenko, 1999).

Thus, on the first step we can get a preliminary solution for coefficients γ_i of each shell by the comparison (9) and (10) that leads to the next formula

$$\gamma_i = \sqrt{-6\cdot b_i/\delta_i}, \qquad (32)$$

which was used for the initial determination of γ_i (see Table 1, column "initial iteration"). The coefficients δ_i of the piecewise Legendre-Laplace density profile (18) were adopted in this study as fixed values according to the same previous Roche's model (Table 1):

$$\delta_i = a_i, \qquad (33)$$

A process of the differential correction for the creation of Legendre-Laplace model was applied on the second step. This consists of the readjustment (by iterations) of these independent pieces of density to the piecewise Legendre-Laplace density distribution by means of the closest approximation of the set of seismic data and other additional information about fundamental geodetic constants.

Table 1. Piecewise Legendre-Laplace density model (m=7)

Shell	δ_i	Initial iteration γ_i	Final iteration γ_i	ℓ_j, km	Density jump
1	13.061	2.02098	2.02334		
2	12.483	2.00252	2.05393	1221.5	0.575
3	6.370	1.55708	1 60817	3480.0	4.429
4	6.058	1.59760	1.68174	5701.0	0.354
5	5.784	1.61810	1.77825	5971.0	0.395
6	6.057	1.69578	1.87310	6151.0	0.084
7	6.622	1.89230	2.11959	6346.6	0.420

In fact, the set of the coefficients δ_i and γ_i yield Legendre-Laplace density (18) denoted in Table 1 as "*initial iteration*". The final version of the piecewise Legendre-Laplace radial density is denoted in Table 1 as "*final iteration*" (see Figure 2). This model was constructed by means of the linearization of necessary equations and the direct approximation of the observed seismic data Φ in accordance with (Dziewonski and Andeson, 1981). Note also that the discussed problem represents a good example of ill-posed problem and in view of stability of the solution only differential correction of γ_i, was applied here without any changes in δ_i,. In practice we get a violation of the requirement of stability. Usually a solution can be obtained by the direct application of some additional information about it. Thus, simultaneous determinations δ_i, γ_i leads in the case of the piecewise Legendre-Laplace model (or in the case of Gauss' model) to a "collapse" of matrix inversion and our choice in this study was the differential correction of the coefficient γ_i only.

As a result, this non-linear inverse problem was based on the expressions (18)-(31) and 3 additional conditions for the Earth's mean density, the mean moment of inertia, and the surface density. The latter was adopted here as in the previous cases for the piecewise Roche's and Gauss' models: ρ_s=2.67 g/cm^3. As a matter of fact, the behaviour of differences in the Table 1 between the initial and final versions of the coefficients γ_i of Legendre-Laplace models may be treated here for every pieces by the application of the nonlinear Legendre-Laplace model instead of the initial linear Roche's model

Conclusions

In addition to the preceding results and conclusions, finally we shall characterize some nice properties of the considered *Legendre-Laplace model*, *Roche's model* (Marchenko, 1999), and *Gauss' (or normal model) radial density distribution* (Marchenko, 2000).

On the one hand, the *continuous* radial density distribution (the Earth as one shell) in the forms of Roche's profile and Legendre-Laplace model have a best agreement, which is reflected in Figure 1. Both these models satisfy to Clairaut's equation. Gauss' radial profile satisfies to Williamson-Adams equation and it describes best of all a *general trend* of the Earth's radial density, which was represented here by the piecewise PREM-model of density.

On the other hand, the piecewise Earth's *Legendre-Laplace radial density distribution* can be tested with the same approach as in (Marchenko, 1999; Marchenko, 2000). For the piecewise Roche's model we have got the appropriate solution on the ground of the golden section technique (step by step) with the readjustment of all pieces by parameters with some additional conditions. Note that Roche's radial distribution connects with the inversion of the *linear* operator (see, for instance (10)). *Gauss'* and *Legendre-Laplace* nonlinear models connect with the corresponding *non-linear* operators and obvious difficulties in their inversion (with high sensitivity to a stability of solution). As a result, the differential correction of one coefficient β_i, (for Gauss' model) and γ_i, (for Legendre-Laplace profile) were used only as a possible way for the solution of such ill-posed problem. So, in view of (10) the coefficients a_i of Roche's model were fixed in these last cases. Note again that simultaneous determinations of all parameters (Abrikosov, 2000) of the piecewise Legendre-Laplace or Gauss' profile require especially the additional information about the desired solution, for example, a general trend of the Earth's radial density.

Nevertheless all investigated piecewise Roche's, Gauss', and Legendre-Laplace profiles are in a good agreement with the standard PREM-model (taking into account the considered assumptions about hydrostatic/adiabatic Earth listed before). Obviously, a final

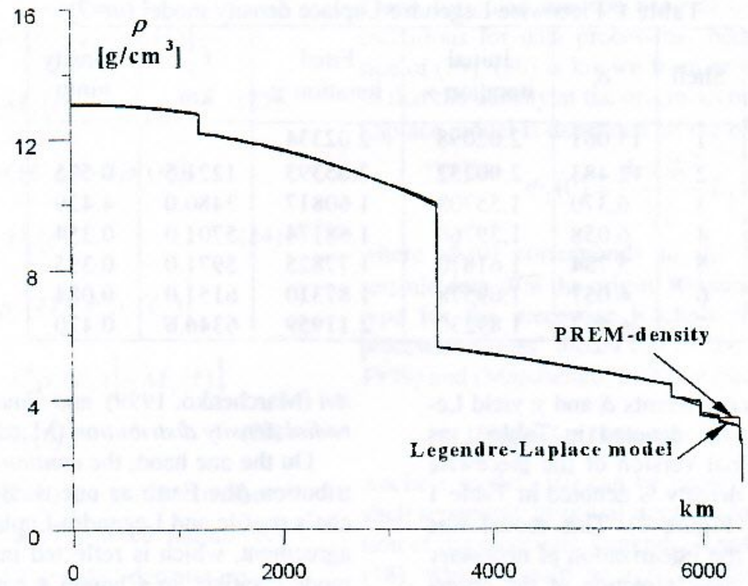

Figure 2. Comparison of the final Legandre-Laplace model with the PREM-density

choice can be done after the above-mentioned study regarding a possible stabilization in the frame of the regularization technique. In this step of investigations of analysed models, Gauss' profile is rather more preferable. On the one hand, this model admits the most appropriate agreement with a general trend of the Earth's radial density. On the other hand, (independently on other models) it leads, as the partial solution of Williamson-Adams equation, to the simplest relationships for initial coefficients on the ground of seismic data.

References

Abrikosov, O.A. (2000) On the stable determination of some Earth's radial density models Geodynamics, n.1(2), 1999, 11-17.

Bullen, K.E. (1975) *The Earth's Density*. Chapman and Hall, London.

Darwin, G. (1884) On the figure of equilibrium of a planet of heterogeneous density. *Proceeding of the Royal Society*, Vol. XXXVI, 158-166.

Dziewonski, A.M. and Anderson, D.L. (1981) Preliminary reference Earth model. *Physics of the Earth and Planetary Interiors*, Vol. 25, 297-356.

Marchenko, A.N. (1999) Simplest solutions of Clairaut's equation and the Earth's density model. In: *"Quo vadis geodesia...?" Festschrift for Erik W. Grafarend on the occasion of his 60th birthday* (F. Krumm and V.S.Schwarze, Eds.). Universität Stuttgart, Technical Reports Department of Geodesy and Geoinformatics, Report Nr.1999.6-2, 303-311.

Marchenko, A.N. (2000) Earth's radial density profiles based on Gauss' and Roche's distributions. *Bolletino di Geodesia e Scienze Affini*, vol. LIX, n.3, 201-220.

Moritz, H. (1990) *The Figure of the Earth. Theoretical Geodesy and Earth's Interior*, Wichmann, Karlsruhe, 1990.

О. Марченко

РАДІАЛЬНИЙ РОЗПОДІЛ ГУСТИНИ ЗЕМЛІ НА ОСНОВІ МОДЕЛІ ЛЕЖАНДРА-ЛАПЛАСА

Резюме

Відоме рішення рівняння Клеро у формі закону Лежандра-Ларласа, використане для параметризації радіального розподілу густини Землі в рамках гідростатично-адіабатичної теорії. Відзначений радіальний розподіл густини розглянутий в двох варіантах: як неперервний та як кусково-неперервний. В результаті були одержані рекурентні формули (для планети із сферичною стратифікацією), які дають розв'язок прямої і оберненої задач на основі сейсмічних і астрономо-геодезичних даних. Отриманий кусково-неперервний радіальний розподіл добре погоджується з моделлю густини PREM.

А.Марченко

РАДИАЛЬНОЕ РАСПРЕДЕЛЕНИЕ ПЛОТНОСТИ ЗЕМЛИ НА ОСНОВЕ МОДЕЛИ ЛЕЖАНДРА-ЛАПЛАСА

Резюме

Известное решение уравнения Клеро в форме закона Лежандра-Ларласа, использовано для параметризации радиального распределения плотности Земли в рамках гидростатически-адиабатическокой теории. Отмеченное радиальное распределение плотности рассмотрено в двух вариантах: как непрерывное, так и ее кусочно-непрерывное распределение. В результате был получен ряд рекуррентных формул (для планеты со сферической стратификацией), доставляющих решение прямой и обратной задач на основе сейсмических и астрономо-геодезических данных. Полученное кусочно-непрерывное радиальное распределение хорошо согласуется с моделью плотности PREM.