ON THE STABLE DETERMINATION OF SOME EARTH'S RADIAL DENSITY MODELS

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Abstract. *The regularization algorithm was developed on the basis of such fit of the normal operator, which is closed to a system of linear equations with scalar or unit matrix. The application of the famous theorem on the spectral expansion of normal matrixes led to introducing of the simplest matrix norm (connected with the traditional Euclidean norm) and allowed form special condition, which provides the determination of the regularization parameter. Proposed approach yields the regularization parameter, which is responsible only for an accuracy of an initial operator and therefore, provides a stable inversion. Numerical testing of the algorithm was performed for the construction of piecewise density models of the Earth based on Roche's, Gauss', and Legendre-Laplace laws.*

Introduction

As well-known, results of seismic radial tomography of the Earth's interior give important data for a determination of radial density distribution. One of the basic equations for such a determination is the famous Williamson-Adams equation (Bullen, 1975):

$$\frac{d\ln\rho(\ell)}{d\ell}=-\frac{4\pi G}{\ell^2\Phi(\ell)}\int_0^{\ell}\rho(t)t^2dt\,, \qquad (1)$$

where $\rho(\ell)$ is the density on the distance ℓ from the Earth's centre, $\Phi(\ell)$ is the combination of the seismic wave velocities V_p and V_s on the same distance

$$\Phi(\ell)=V_p^2(\ell)-\frac{4}{3}V_s^2(\ell) \qquad (2)$$

Equation (1) with known values $\Phi(\ell)$ was used alone up to depth 670 km for construction of the PREM model (Dziewonski and Anderson, 1981) where the Earth's density was represented by piecewise polynomials (in general up to degree 3):

$$\rho(\ell_{i-1}<\ell<\ell_i)=\rho_i(\ell)=\sum_{j=0}^{n_i}a_{ij}\left(\frac{\ell}{R}\right)^j. \qquad (3)$$

Here R is the Earth's mean radius, ℓ_i is the geocentric distance of i-th discontinuity of the Earth's density ($\ell_0=0$, $\ell_{max}=R$). Usage of some additional data (such as density jump across the inner-outer core boundary, density at the base of the mantle, density below the Mohorovichich discontinuity etc.) yielded more or less stable inversion of $\Phi(\ell)$. As a result, the appropriate solution was found for the coefficients a_{ij} within 13 shells separately.

Grid values of the function (2) in accordance with the PREM were used (Marchenko, 1999) for the construction of piecewise radial density distribution based on Roche's model:

$$\rho(\ell_{i-1}<\ell<\ell_i)=\rho_i(\ell)=a_i-b_i^2\left(\frac{\ell}{R}\right)^2. \qquad (4)$$

Coefficients a_i, b_i were determined for 7 main shells by means of the golden section technique. Stable solution was obtained by applying the additional conditions for primary geodetic constants and ratios $\lim_{\varepsilon\to 0}\Phi(\ell_i-\varepsilon)/\Phi(\ell_i+\varepsilon)$.

Recently piecewise radial density distributions based on Gauss' and Legendre-Laplace models

$$\rho(\ell_{i-1}<\ell<\ell_i)=\rho_i(\ell)=a_i\exp\left(-b_i^2\frac{\ell^2}{R^2}\right), \qquad (5)$$

$$\rho(\ell_{i-1}<\ell<\ell_i)=\rho_i(\ell)=a_i\frac{\sin\left(b_i\frac{\ell}{R}\right)}{b_i\frac{\ell}{R}}, \qquad (6)$$

were constructed (Marchenko, 2000a, 2000b) for the same 7 shells. Coefficients a_i for all discussed three models are coincided that can be easily seen from Taylor expansions of the functions (5) and (6)). Then only the coefficients b_i were determined from seismic data for the models (5), (6) and the coefficients a_i were adopted in accordance with the piecewise Roche's distribution. In such a formulation, stable solutions were obtained even without any additional information. However, two additional conditions for the Earth's mass and mean moment of inertia were applied for the agreement with these global characteristics of the planet.

Any attempt to determine either all coefficients a_i and b_i or even coefficients a_i only for any model (4) – (6) without applying other additional conditions led to the destruction of solution due to its instability

(Marchenko, 2000b). For this reason, the goals of this paper are the development and application of the special technique for a stable determination of parameters of density distribution from seismic data.

1. Basic equations

Below we will suppose that the Earth's interior is stratified by M shells and in each shell the density is represented by some continuous function with n parameters

$$\rho_i(\ell) = \rho_i(\ell; p_{i1}, p_{i2}, \cdots, p_{in}). \qquad (7)$$

Rewriting the equation (1) as

$$\Phi(\ell_{i-1} < \ell < \ell_i) = \Phi_i(\ell) = -\frac{4\pi G}{\ell^2} \cdot \frac{\rho_i(\ell)}{\rho_i'(\ell)} \times \left(\sum_{k=1}^{i-1} \int_{\ell_{k-1}}^{\ell_k} \rho_k(t) t^2 dt + \int_{\ell_{i-1}}^{\ell} \rho_i(t) t^2 dt \right), \qquad (8)$$

where

$$\rho_i'(\ell) = \frac{d\rho_i(\ell)}{d\ell}, \qquad (9)$$

we can see that any $\Phi_i(\ell)$ depends on $i \cdot n$ parameters and after traditional Taylor linearization we come to a system of linear equations with the quasi triangular block matrix

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{0}_{12} & \cdots & \mathbf{0}_{1M} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{0}_{2M} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{A}_{M1} & \mathbf{A}_{M2} & \cdots & \mathbf{A}_{MM} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_M \end{pmatrix} = \mathbf{L} + \mathbf{V}, \qquad (10)$$

where $\mathbf{A}_{ij}$ is $(m_i \times n)$ matrix, which consists of the derivatives $\partial\Phi_i / \partial p_{jk}$ (k=1,2..n), $\mathbf{0}_{ij}$ is the zero $(m_i \times n)$ matrix. m_i is the number of given ("measured") values of $\Phi_i(\ell)$ in i-th shell, $\mathbf{X}_i$ is $(n \times 1)$ vector of corrections to parameters of the model (7) for the current i-th shell, $\mathbf{L}$ is the vector of "observation minus calculation", $\mathbf{V}$ is the residual vector.

Determination of parameters of the piecewise density distribution (7) requires solution of the system of *non-linear* equations (8) by iterations. On each one, corrections to parameters should be obtained as a solution of *linear system* (10). Obviously, such a solution must be stable in order to provide a convergence of such iterative process.

2. Application of regularization technique

It is well-known that most general approach to derive a stable solution of a system of linear equations

$$\mathbf{AX} = \mathbf{L} + \mathbf{V} \qquad (11)$$

is Tikhonov's regularization (Tikhonov and Arsenin, 1986), which based on minimization of the next smoothing functional

$$F_\gamma = \mathbf{V}^{\mathrm{T}}\mathbf{C}_{nn}^{-1}\mathbf{V} + \gamma\mathbf{X}_\gamma^{\mathrm{T}}\mathbf{\Omega}\mathbf{X}_\gamma, \qquad (12)$$

where $\mathbf{X}_\gamma$ is the stable estimation of the vector $\mathbf{X}$, $\mathbf{C}_{nn}^{-1}$ is the covariance matrix of measured data, $\mathbf{\Omega}$ is the positive defined symmetric matrix called by a stabilizer, $\gamma \geq 0$ is the regularization parameter, the superscript T denotes the transposition. For a given γ vector $\mathbf{X}_\gamma$ is the solution of the system

$$(\mathbf{A}^{\mathrm{T}}\mathbf{C}_{nn}^{-1}\mathbf{A} + \gamma\mathbf{\Omega})\mathbf{X}_\gamma = \mathbf{A}^{\mathrm{T}}\mathbf{C}_{nn}^{-1}\mathbf{L}. \qquad (13)$$

$\mathbf{X}_0$ can be based (for $\gamma = 0$) on the usual least squares principle.

In the case of a given stabilizer $\mathbf{\Omega}$ the main problem consists of the computation of an appropriate value γ. In accordance with the general approach (Tikhonov and Arsenin, 1986), such value γ_{opt} must be agreed with the accuracy of measured data and corresponding operator, which is represented here by the normal matrix

$$\mathbf{N}_0 = \mathbf{A}^{\mathrm{T}}\mathbf{C}_{nn}^{-1}\mathbf{A}. \qquad (14)$$

Standard determination of γ_{opt} requires an iterative process, which starts from the initial value $\gamma = 0$ and may lead to essential difficulties in a case of ill-conditioned matrix $\mathbf{N}_0$. For this reason, we shall try to use another approach based in particular on some spectral properties of the normal matrix

$$\mathbf{N}_\gamma = \mathbf{A}^{\mathrm{T}}\mathbf{C}_{nn}^{-1}\mathbf{A} + \gamma\mathbf{\Omega}. \qquad (15)$$

3. Some important properties of the normal matrix

Taking into account that both matrixes $\mathbf{\Omega}$ and $\mathbf{N}_\gamma$ $(\gamma > 0)$ are positive defined, we may apply well-known theorem on the matrix spectral expansion (Horn and Johnson, 1986) and transform the system (13) to the system with diagonal matrix

$$(\mathbf{\Lambda} + \gamma\mathbf{I})\mathbf{Y}_\gamma = \mathbf{U}. \qquad (16)$$

Omitting some elementary details of these spectral expansions and corresponding transformations, note only that $\mathbf{I}$ is unit ($n \times n$) matrix, n is the order of system (13), Λ is the diagonal matrix

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_n),$$
$$\lambda_1 > 0, \quad \lambda_i \geq 0, \quad i = 2,3,\cdots n, \qquad (17)$$

$\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigenvalues of the matrix $\mathbf{N}_0$.

Now in view of (16) we can define (see Horn and Johnson, 1986) a set of our possible solutions by the next matrix norm

$$\|\mathbf{N}_0\| = \|\mathbf{\Lambda}\| = \mathrm{Trace}(\mathbf{\Lambda}) = \sum_{i=1}^{n} \lambda_i > 0, \qquad (18)$$

because such norm is nothing else but the Euclidean norm of the matrix of the initial system (11). In fact, because $\mathbf{C}_{nn}^{-1}$ is a symmetric positive defined matrix, we can apply its spectral expansion to form the matrix $\mathbf{B} = \mathbf{C}_{nn}^{-1/2}\mathbf{A}$. Now it is evident $\mathbf{N}_0 = \mathbf{B}^{\mathrm{T}}\mathbf{B}$ and $\|\mathbf{B}\|_{E_n}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 = \mathrm{Trace}(\mathbf{N}_0)$. As a result, we come to the remarkable relationship

$$\left\|\mathbf{C}_{nn}^{-1/2}\mathbf{A}\right\|_{E_n}^2 = \|\mathbf{\Lambda}\|. \qquad (19)$$

According to the norm (18), we shall apply the factor $\|\mathbf{\Lambda}\|/n$ and rewrite (16) in the form

$$(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I})\mathbf{Y}_\gamma = \frac{n}{\|\mathbf{\Lambda}\|}\mathbf{U}, \qquad (20a)$$

$$\overline{\mathbf{\Lambda}} = \frac{n}{\|\mathbf{\Lambda}\|}\mathbf{\Lambda}, \quad \alpha = \frac{n}{\|\mathbf{\Lambda}\|}\gamma \geq 0 \qquad (20b)$$

Further next relationships are valid:

$$\|\overline{\mathbf{\Lambda}} + \alpha\mathbf{I}\| = n(1+\alpha), \qquad (21)$$

$$\left\|(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I})^{-1}\right\| = \frac{n\alpha^{n-1} + \sum_{k=1}^{n-1}(n-k)J_k\alpha^{n-k-1}}{\alpha^n + \sum_{k=1}^{n} J_k\alpha^{n-k}}, \qquad (22)$$

$$J_k = \sum\prod_{i=1}^{k} \overline{\lambda}_{l_i} \geq 0, \qquad (23a)$$

$$J_1 = \|\overline{\mathbf{\Lambda}}\| = n, \quad J_n = \prod_{i=1}^{k} \overline{\lambda}_i = \det\overline{\mathbf{\Lambda}} \geq 0, \qquad (23b)$$

where the sum in (23) covers all combinations of indexes $l_1, l_2, \cdots l_k$. It is evident that the norm $\|\overline{\mathbf{\Lambda}} + \alpha\mathbf{I}\|$ increases and the norm $\left\|(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I})^{-1}\right\|$ decreases if the parameter α increases.

The polynomial in the denominator of right-hand side (22) is nothing else but the determinant of the matrix $\overline{\mathbf{\Lambda}} + \alpha\mathbf{I}$:

$$P_n(\alpha) = \alpha^n + \sum_{k=1}^{n} J_k\alpha^{n-k} = \det(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I}), \qquad (24)$$

and the polynomial in the numerator is the first order derivative of (24) regarding α:

$$\frac{d}{d\alpha}P_n(\alpha) = P_n'(\alpha) = n\alpha^{n-1} + \sum_{k=1}^{n}(n-k)J_k\alpha^{n-k-1} \qquad (25)$$

Now considering partial sums of the polynomial (24):

$$S_{n,m}(\alpha) = \alpha^n + \sum_{k=1}^{m} J_k\alpha^{n-k}, \quad 0 \leq m \leq n, \qquad (26)$$

we can form the sequence

$$\frac{n}{\alpha} = \frac{S_{n,0}'(\alpha)}{S_{n,0}(\alpha)} \geq \frac{S_{n,1}'(\alpha)}{S_{n,1}(\alpha)} \geq \frac{S_{n,2}'(\alpha)}{S_{n,2}(\alpha)} \geq \cdots \geq \frac{S_{n,n}'(\alpha)}{S_{n,n}(\alpha)} = \frac{P_n'(\alpha)}{P_n(\alpha)}. \qquad (27)$$

Thus, we come to the expression of the norm (22) by means of the logarithmic derivative

$$\left\|(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I})^{-1}\right\| = \frac{P_n'(\alpha)}{P_n(\alpha)} = \frac{d}{d\alpha}\ln\det(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I}), \qquad (28)$$

and to the following inequalities:

$$\frac{P_n'(\alpha)}{P_n(\alpha)} \leq \frac{n}{\alpha}, \qquad \left\|(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I})^{-1}\right\| \leq \frac{n}{\alpha}. \qquad (29)$$

4. Determination of the regularization parameter

In spite of traditional regularization method (section 2) we shall try to build on this step the regularization algorithm without information about accuracy of measured data. First, we should remember that our solution belongs to such a set, which is defined by the matrix norm (18) – (19). As a matter of fact, a most stable solution of a system of linear equations in practice connects with a possibility of its expansion into a set of orthonormal vectors. This corresponds to the matrix of a system represented by a scalar or the unit matrix $\mathbf{I}$. In such case the parameter of regularization α should be equal to zero. Therefore, we may find the parameter α from an agreement of the initial matrix (14) with the corresponding unit matrix. Considering below only a problem of inversion of initial operator, we come to the following "ideal" condition

$$\left\|(\overline{\mathbf{\Lambda}} + \alpha\mathbf{I})^{-1}\right\| = \|\overline{\mathbf{\Lambda}} + \alpha\mathbf{I}\|, \qquad (30)$$

that leads to the non-linear equation, which in view of (21) and (28) may be written as

$$n(1+\alpha)=\frac{P_n'(\alpha)}{P_n(\alpha)}. \qquad (31)$$

Due to the inequalities (29), we get

$$n(1+\alpha)\le\frac{n}{\alpha}, \qquad (32)$$

that gives immediately the upper limit for α as

$$\alpha\le\alpha_0=\frac{\sqrt{5}}{2}-\frac{1}{2}\approx 0.618. \qquad (33)$$

Next, in view of (27) we come to conclusion that the solutions $\alpha_{n,m}$ of the non-linear equation

$$n(1+\alpha_{n,m})=\frac{S'_{n,m}(\alpha_{n,m})}{S_{n,m}(\alpha_{n,m})} \qquad (34)$$

for different m are connected by the inequalities

$$\frac{\sqrt{5}}{2}-\frac{1}{2}=\alpha_{n,0}\ge\alpha_{n,1}\ge\alpha_{n,2}\ge\cdots\ge\alpha_{n,n}=\alpha. \qquad (35)$$

It is evident that equation (34) may be solved in the closed form only for $m<2$. So, we can improve the estimation (33), which corresponds $m=0$. Considering equation (34) for $m=1$ and taking into account the expressions (23), we come to the equation

$$1+\alpha_{n,1}=\frac{\alpha_{n,1}+n-1}{\alpha_{n,1}(\alpha_{n,1}+n)} \qquad (36)$$

and get the upper limit of α as a function of the order n of the system (13):

$$\alpha_{n,1}=-\frac{n+1}{3}-\frac{2\sqrt{n^2-n+4}}{3}\times \times\sin\left(\frac{1}{3}\arcsin\frac{2n^3-3n^2-21n+38}{2\sqrt{(n^2-n+4)^3}}-\frac{\pi}{3}\right) \qquad (37)$$

This expression yields the regularization parameter α for the normal system *of order n and rank 1*. In particular, the case $n=1$ leads to zero α:

$$\alpha_{1,1}=0, \qquad (38)$$

that reflects the obvious fact: no regularization for one linear equation with one unknown only. In addition, the estimation (37) goes to α_0 (33) if $n\to\infty$:

$$\lim_{n\to\infty}\alpha_{n,1}=\alpha_0=\frac{\sqrt{5}}{2}-\frac{1}{2}, \qquad (39)$$

that corresponds to the case of *infinite-dimensional normal system of rank 1*. Figure 1 shows behaviour of $\alpha_{n,1}$.

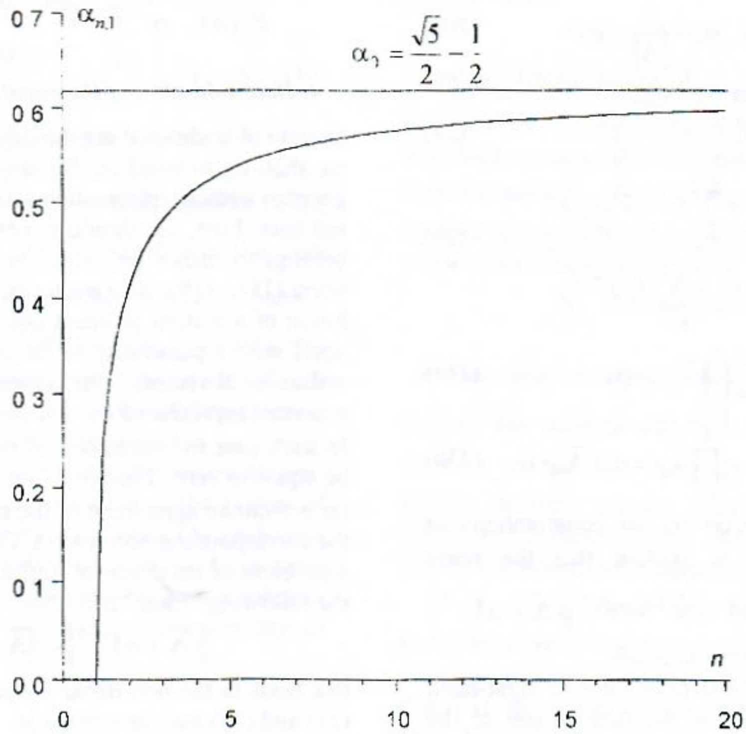

Fig.1. Upper limit of the parameter α for various n

The solution of the non-linear equation (31) may be based on the recursive formula

$$\hat{\alpha}_{m+1} = \frac{1}{n} \cdot \frac{\hat{\alpha}_m}{1+\hat{\alpha}_m} \cdot \frac{P_n'(\hat{\alpha}_m)}{P_n(\hat{\alpha}_m)}, \qquad (40)$$

with the initial value $\hat{\alpha}_0 = \alpha_{n,1}$ Convergence of iterations may be proved in elementary way on the ground of above considered properties of polynomials (24) and the inequalities (35).

5. Practical aspects

Taking into account that the norm of type (18) is invariant of normal matrix, we must note that in practice we need in spectral transformation only for stabilizer $\mathbf{\Omega}$ that leads to the system

$$(\widetilde{\mathbf{N}} + \gamma\mathbf{I})\mathbf{Z}_\gamma = \widetilde{\mathbf{U}}, \qquad (41)$$

where

$$\|\mathbf{A}\| = \|\widetilde{\mathbf{N}}\| = \mathrm{Trace}\,\widetilde{\mathbf{N}}. \qquad (42)$$

Thus, the regularized solution $\mathbf{Z}_\gamma$ of the system (41) is nothing else but well-known quasi-solution (Tikhonov and Arsenin, 1986) of corresponding system of linear equations

Using the normalization as in (20):

$$\overline{\mathbf{N}} = \frac{n}{\|\widetilde{\mathbf{N}}\|}\widetilde{\mathbf{N}}, \qquad (43)$$

we can rewrite recursive formula (40) in most appropriate form

$$\hat{\alpha}_{m+1} = \frac{\hat{\alpha}_m}{n(1+\hat{\alpha}_m)}\left\|(\overline{\mathbf{N}} + \hat{\alpha}_m\mathbf{I})^{-1}\right\| \qquad (44)$$

and then compute such estimation of $\hat{\alpha}$, for which the condition

$$\left|\frac{\left\|(\overline{\mathbf{N}} + \hat{\alpha}\mathbf{I})^{-1}\right\|}{n(1+\hat{\alpha})} - 1\right| < \varepsilon \qquad (45)$$

becomes valid with a given precision ε>0

6. Numerical results

The equation (1) was used in the case of PREM model only below the radius ℓ=5701 km. In this study we decided to use it within the whole Earth (up to $\ell=R=$ 6371 km). Thus, our data set was represented by 94 values of the function (2) in accordance with PREM. Stratification of the Earth's interior by 7 shells was taken from the paper (Marchenko, 1999).

With the mentioned stratification, the simultaneous determination of 14 parameters $\{a_i, b_i\}$ was carried out for every piecewise model (4) – (6) separately. In all computations only two additional conditions (for the mean density $\overline{\rho}$=5.514 g/cm^3 and for the dimensionless mean moment of inertia I=0.32998) were applied. In spite of small order ($n = 14$) of the normal system (13), a destruction of least squares solution ($\gamma = 0$) took place for every model. Therefore, the application of the developed regularization algorithm became necessary.

Construction of a stabilizer for the Earth's radial density distribution is the complicated separate problem, which requires special investigations. Thus, we decided to use here the simplest case $\mathbf{\Omega} = \mathbf{I}$, which is famous in the regularization theory and leads to the construction of the so-called quasi solution. After this choice of the stabilizer we have formed the system (41) in which $\widetilde{\mathbf{N}} = \mathbf{A}^T\mathbf{C}_{nn}^{-1}\mathbf{A}$ is obtained without any additional transformations.

Parameters of these models (from seismic data) based on the proposed regularization algorithm are presented in the Table 1. Note again that our algorithm is connected only with the condition (29), which provides the determination of the regularization parameter in view of inversion of initial normal operator. Therefore, we may say that stablization of our solutions depends on the total accuracy of the integral-differential operator (1) and the corresponding density parameterization (4), (5), or (6). Such a conclusion is confirmed by good agreement of the coefficients a_i for different models within the same shells, since these coefficients can be treated as the analytical continuation of partial models to the Earth's center $a_i = \rho_i(\ell = 0)$.

Table 2 consists of the values of Earth's density and density jumps at the adopted discontinuity radii according to constructed piecewise radial models. For comparison, the table shows also such values computed from PREM model. All these characteristics of radial density distribution agree for different models and we can see better agreement of the density jumps for two polynomial models (PREM and Roche's ones) and for two non-polynomial models (Gauss' and Legendre-Laplace ones). Such results may be explained by the adopted different parameterizations of the density. To our surprise, we found almost the same values of sum of jumps for all discussed models. These sums are shown in the last row of the Table 2 together with the accuracy estimations computed from the data of Table 1. Note that the density jump 0.3 g/cm^3 at the radius 6356 km was taken into account for PREM model in addition to the jumps shown in the Table 2.

Table 1. Parameters of constructed models of Earth's radial density distribution

Shells	Roche's (4)		Gauss' (5)		Legendre-Laplace (6)	
	a_i, g/cm³	b_i, g/cm³	a_i, g/cm³	b_i	a_i, g/cm³	b_i
ℓ <1221.5	13.062 ±0.004	2.980 ±0.004	13.063 ±0.002	0.828 ±0.002	13.066 ±0.003	2.023 ±0.003
1221.5< ℓ <3480.0	12.451 ±0.016	2.916 ±0.012	12.338 ±0.008	0.867 ±0.004	12.274 ±0.014	2.074 ±0.008
3480.0< ℓ <5701.0	6.386 ±0.030	1.595 ±0.008	6.600 ±0.015	0.697 ±0.003	6.571 ±0.027	1.632 ±0.006
5701.0< ℓ <5970.0	5.935 ±0.038	1.540 ±0 017	6.387 ±0.020	0.760 ±0.010	6 036 ±0.035	1.713 ±0.016
5970.0< ℓ <6151.0	5.680 ±0.040	1.585 ±0.023	6.071 ±0.020	0.820 ±0.014	5.793 ±0.036	1.825 ±0.023
6151.0< ℓ <6346.6	5.933 ±0.041	1.670 ±0 021	6.424 ±0.021	0.877 ±0.014	6.086 ±0.037	1.911 ±0.021
6346 6< ℓ <6371.0	6.592 ±0.042	2.106 ±0.034	6.662 ±0.021	1.029 ±0.020	6.625 ±0.038	2.227 ±0.032
R.m.s. of seismic data fitting, km²/s²	5.52		5.33		5.11	
Regularization parameter value	0.48		0.49		0.48	

On the ground of obtained numerical results we can conclude that the application of the developed regularization algorithm led to the reliable determination of the global trend of the Earth's radial density distribution from the seismic data with different parameterizations. It is important that this algorithm allowed to use the fundamental equation (1) within whole Earth's interior for a stable determination of parameters of discussed piecewise radial models.

Conclusions

Thus, we developed the regularization algorithm based on such fit of the normal operator, which is closed to a system of linear equations with scalar or unit matrix. The application of the famous theorem on the spectral expansion of normal matrixes led to introducing of the simplest matrix norm (18) and allowed form the condition (30), which provides the determination of the regularization parameter.

Table 2. Characteristics of the Earth's radial density distribution

Radius ℓ, km	Values of the density ρ and density jumps δρ, g/cm³							
	PREM		Roche's (4)		Gauss' (5)		Legendre-Laplace (6)	
	ρ	δρ	ρ	δρ	ρ	δρ	ρ	δρ
0.0	13.088		13.062		13.063		13.066	
1221.5	12.764 12.166	0.598	12.735 12.138	0.597	12.738 12.002	0.736	12.741 11 954	0.787
3480 0	9.903 5.566	4 337	9.914 5.627	4.287	9.860 5.709	4 151	9.812 5 734	4.078
5701.0	4.381 3.992	0 389	4.349 4.037	0.312	4.472 4.021	0.451	4.471 3 935	0.536
5971.0	3.724 3.543	0.181	3.853 3.474	0.379	3.845 3.362	0.483	3.758 3.353	0.405
6151.0	3.436 3.360	0.076	3.339 3.333	0.006	3.243 3.135	0.108	3 227 3 176	0.051
6346.6	3.381 2.900	0.481	3.165 2.192	0.973	2.993 2 328	0.665	3 022 2 381	0.641
6368.0	2.600		2.163		2.311		2.360	
$\sum \delta\rho$	6.362		6.554 ± 0.054		6.594 ± 0.032		6.498 ± 0.052	

Developed algorithm may be treated as those that yields a best approximation of normal operator by an orthogonal one in view of the norm (18) – (19). Actually, assuming in (20) $\overline{\mathbf{A}} = \mathbf{I}$, we come immediately to the condition (30) in the form

$$1/(1+\alpha) = 1+\alpha\,, \qquad (46)$$

that gives α=0, which coincides exactly with (38). Thus, in addition to (38) we can see: *no regularization for the normal system with the matrix proportional to the unit matrix.* As a result, we can say that in view of the general theory (Tikhonov and Arsenin, 1986), our approach yields the *regularization parameter, which is responsible only for an accuracy of an initial operator* and therefore, provides a stable inversion.

Numerical testing of the algorithm was performed for the construction of piecewise density models of the Earth based on Roche's, Gauss', and Legendre-Laplace laws. By the way, in processing of seismic data the regularization is using traditionally, in particular, for construction of inverse convolution operators (Hatton, et al , 1986). Finally, we should note that because proposed regularization algorithm was developed on the basis of stable inversion of a normal operator, then it may be applied for wide spectrum of problems connected with solution of ill-conditioned systems of normal equations.

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ПРО СТІЙКЕ ВИЗНАЧЕННЯ ДЕЯКИХ РАДІАЛЬНИХ МОДЕЛЕЙ ГУСТИНИ ЗЕМЛІ

Резюме

Регуляризуючий алгоритм розроблений на основі апроксимації нормального оператора, яка є близькою до системи лінійних рівнянь зі скалярною або одиничною матрицею. Застосування теореми про спектральний розклад нормальних матриць призвело до введення найпростішої матричної норми, пов'язаної з традиційною Евклідовою нормою, та спеціальної умови для визначення параметра регуляризації. Параметр регуляризації визначено лише за точністю вихідного оператора, що забезпечує його стійку інверсію. Алгоритм протестовано при побудові кусково-неперервних моделей густини Землі за законами Роша, Гауса та Лежандра-Лапласа.

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Резюме

Регуляризующий алгоритм разработан на основе аппроксимации нормального оператора, близкой к системе линейных уравнений со скалярной или единичной матрицей. Применений теоремы о спектральном разложении нормальных матриц привело к введению простейшей матричной нормы, связанной с традиционной Евклидовой нормой, и специального условия для определения параметра регуляризации. Параметр регуляризации определен только на основе точности исходного оператора, что обеспечивает его устойчивую инверсию. Алгоритм апробирован при построении кусочно-непрерывных моделей плотности Земли по законам Роша, Гаусса и Лежандра-Лапласа.