ON THE MODEL AND OPERATIONAL APPROACHES TO THE CONSTRUCTION OF THE EARTH'S RADIAL DENSITY

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Abstract. *Legendre-Laplace, Roche and Gauss continuous radial density distributions representing the global trend of piecewise density profile were used for the creation of simplest stabilizers and the construction of radial density distribution. In addition, the operational approach was studied that leads to the traditional least squares collocation method.*

Introduction

Recently, instead of the standard polynomial representation of a piecewise radial density (PREM-model (Dziewonski. and Anderson 1981), etc.) some famous hypotheses for density distribution (see, Bullen, 1975) were analyzed especially in view of Clairaut and Williamson-Adams equations. The latter leads to the study of the *hydrostatic/adiabatic Earth* and goes back to the *Gaussian distribution* of the *Earth's radial density* (Marchenko, 1999; Marchenko, 2000). In contrast to Laplace-Legendre and Roche's models, the created continuous Gauss' radial profile is in a good agreement with the piecewise PREM (see Figure 1) with mean deviation ≈ 0.1 g/cm^3. This fact leads to the idea to apply such simple trend of the radial density as additional information for the inversion of seismic data. Thus, the goal of this paper is the construction of appropriate stabilizers of solution of the mentioned ill-posed inverse problem in the frame of regularization approach.

According to (Bullen, 1975) the oldest solutions of *Clairaut's* equation for the radial density ρ is *Legendre - Laplace law*

$$\rho(x) = \rho_0 \frac{\sin(\gamma x)}{\gamma x}, \qquad \gamma\text{=const}, \qquad (1)$$

where $x = \ell/R$ is the dimensionless "radius-vector" regarding to the mean Earth's radius R=6371 km; ρ_o = const and may be considered here as the density at the origin. The second one is *Roche's law*

$$\rho(x) = \rho_0\left(1 - Kx^2\right) = a + bx^2, \quad (a = \rho_0 > 0,\ b = \rho_0 K < 0) \qquad (2)$$

Omitting here Darwin's law (1884) of density, we

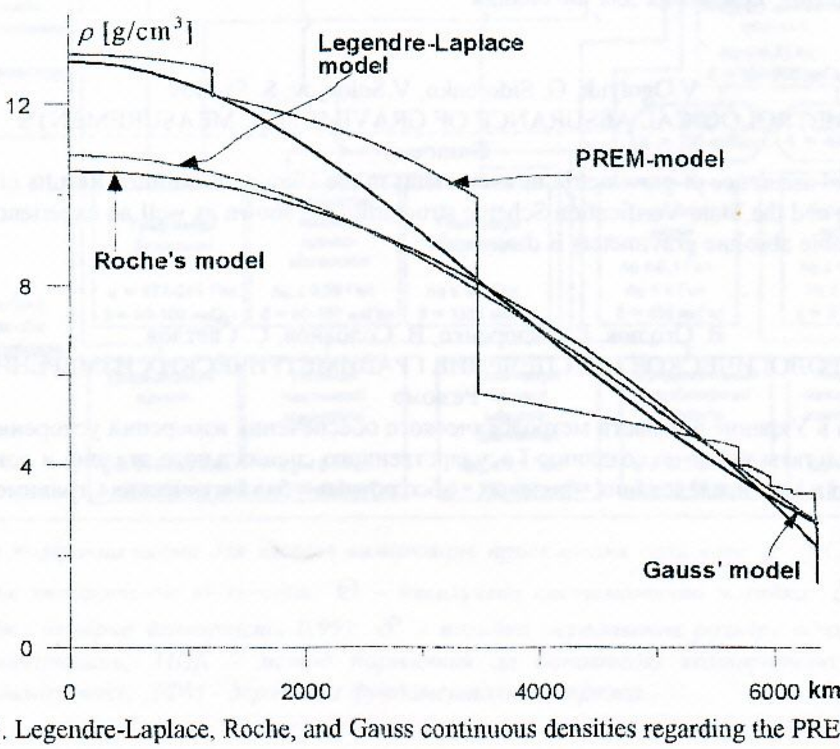

Figure 1. Legendre-Laplace, Roche, and Gauss continuous densities regarding the PREM-model

note again that special consideration of *Williamson-Adams* equation for the parameter Φ (which connects in standard way with seismic velocities V_p and V_s as initial data $\Phi = \Phi(\ell) = V(\ell)_P^2 - \frac{4}{3}V(\ell)_S^2$) leads to its partial solution as *Gauss'* function

$$\rho(x) = \rho_0 \exp\left(-\beta^2 x^2\right), \quad \beta = \text{const}, \qquad (3)$$

where the power 2 is the lowest power for which we may get a non-zero value Φ at the origin. Note also that Taylor series expansion of (1) or (3) yields in practice Roche's model (2) if we disregard other higher powers of x. As a result, we try to construct below the corresponding stabilizers of solution for every mentioned model. On the other hand, instead of such model approach we shall consider the operational approach in view of the stable solution of studying inverse problem.

1. Linearization

In accordance with the traditional operational approach of physical geodesy we shall start from the "measurements as nonlinear functionals" (Moritz, 1980). In such a formulation seismic observations will depend on the Earth's density distribution, so that for each seismic measurement $l = \Phi$ we may write

$$l = F[\rho(x)]. \qquad (4)$$

Here $\rho(x)$ denotes the Earth's radial density distribution (continuous or piecewise). Further we shall assume that $\rho(x)$ may be represented as the sum of it certain "normal" or "trend" part $\rho_t(x)$ and the radial density anomaly:

$$\rho(x) = \rho_t(x) + \Delta\rho(x). \qquad (5)$$

The function $\rho(x)$ belongs on the whole to some set of the space $L_2(x\in[0,1])$ of square integrable functions. In such a formulation the functional F will present a mapping of $L_2(x\in[0,1])$ into the set of real numbers. Since we may have some different kinds of measurements, in the total we shall obtain the different functionals F, which in the general case are nonlinear ones. Because $\rho(x)$ belongs to the infinitely dimensional space L_2, and a number of measurements or functionals of $\rho(x)$ always finite, the inverse problem has not a unique solution. As a rule, an instability of solution is accompanied to the absence of unique solution that is caused, for example, by properties of initial operator and errors of initial information. In other words, this means in practice that we have an improperly posed or ill-posed problem of the determination of the Earth's density.

Clearly that the direct using (5) may be complicated in practice. It is possible to show that in the frame of our assumptions one of the simplest way is the consideration this inverse problem with respect to the mean density $D(x)$ because the function $D(x)$ is continuous. In any case such possibility is the subject of separate paper and we recall here that traditional way of a solution of nonlinear problems consists of it linearization by Taylor's theorem and further direct solution of linear problem. Such approach is the base for methods of data processing (that connects with possible great number of measurements). Obviously, a nonlinear problem can be split into *linearization and solution of system of linear equations.* Thus, we come to the standard iterative nature of that: a solution of any nonlinear problem may be treated as the differential improvement of a set of suitable parameters in the frame of linear problem.

Therefore we start from the linearization of nonlinear functionals by introducing the approximation $\rho_t(x)$ to the radial density $\rho(x)$ and rewrite (5) as

$$\Delta\rho(x) = \rho(x) - \rho_t(x). \qquad (6)$$

where the difference (6) is considered to be small in (5) as some anomalous density. We assume also that averaging of (6) for the studying domain is equal to zero .

All above-mentioned models (1), (2), and (3) have only $p=2$ parameters that we reflected below by the 2-dimensional column vector $\mathbf{X}$ of these parameters of "trend models" and the corresponding column vector $\Delta\mathbf{X}$ of their corrections. So, the following expression is valid for (4):

$$l(x) = F[\mathbf{X} + \Delta\mathbf{X}, \rho_t(x) + \Delta\rho(x)]. \qquad (7)$$

Next, Taylor's expansion of (7) gives

$$l = F[\mathbf{X}_0, \rho_t] + \mathbf{A}^{\mathrm{T}}\Delta\mathbf{X} + L\Delta\rho, \qquad (8)$$

by neglecting the second and higher order terms. Here $\mathbf{A}$ is the column vector that consists of the ordinary partial derivatives A_k (k=1, 2) of $F[\mathbf{X}_0, \rho_t]$ with respect to the component X_k of the vector $\mathbf{X}$; the approximate values $\mathbf{X}_0$ and $\rho_t(x)$ are taken instead of the values $\mathbf{X}$ and $\rho(x)$:

$$A_k = \frac{\partial F[\mathbf{X}_0, \rho_t]}{\partial X_k}. \qquad (9)$$

The term ($\mathbf{A}^{\mathrm{T}}\Delta\mathbf{X}$) is the scalar product; $L\Delta\rho$ expresses the operator L acting on the anomalous density $\Delta\rho(x)$. Then the result of linearization represents the next simple difference

$$\Delta l(x) = F[\mathbf{X}, \rho(x)] - F[\mathbf{X}_0, \rho_t(x)], \qquad (10)$$

where $F[\mathbf{X},\rho(x)]$ is the observation; $F[\mathbf{X}_0,\rho_t(x)]$ is it approximate value; Δl is the *„observation minus computation"*. The linearized system (without errors of observations) can be written now as

$$\left.\begin{aligned} \Delta l_1 &= \mathbf{A}_1^{\mathrm{T}}\Delta\mathbf{X} + L_1\Delta\rho, \\ \Delta l_2 &= \mathbf{A}_2^{\mathrm{T}}\Delta\mathbf{X} + L_2\Delta\rho, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta l_q &= \mathbf{A}_q^{\mathrm{T}}\Delta\mathbf{X} + L_q\Delta\rho. \end{aligned}\right\} \quad (11)$$

where q denotes the number of observations. On the whole our function $\rho_t(x)$ may be differed from one of the continuous distributions (1)-(3). For this reason we shall admit to consider instead of $\rho_t(x)$ a more complicated function then (1)-(3) for that the traditional old denotation $\rho_t(x)$ is keeping.

2. Least squares collocation

Now we rewrite the system (11) of linear equations, which was found by the linearization

$$\Delta\mathbf{l} = \mathbf{A}\Delta\mathbf{X} + \mathbf{B}\Delta\rho + \mathbf{n}\ . \quad (12)$$

Here $\Delta\mathbf{X}$ is the p-vector (p=2 for one of continuous models) that consists of the corrections ΔX_i to parameters of initial model; $\Delta\mathbf{l}$ is the q-vector that consists of the components (10). The matrix $\mathbf{A}$ of partial derivatives has the dimension (q x p) and B is the operator, which can be formed from q functionals L_i. Thus, we put

$$\Delta l = \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \ldots \\ \Delta l_q \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}} \\ \mathbf{A}_2^{\mathrm{T}} \\ \ldots \\ \mathbf{A}_q^{\mathrm{T}} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} L_1 \\ L_2 \\ \ldots \\ L_q \end{bmatrix}, \quad (13)$$

assuming that $\mathbf{A}$ has a full rank ($p<q$). The q-vector $\mathbf{n}$ reflects here the influence of measuring errors („noise"); the vector $\Delta\mathbf{X}$ and the anomalous density $\Delta\rho$ should be determined from various kinds of measurements in (12). Thus, to our own surprise we have got a basic model (12) for the determination of the parameters $\Delta\mathbf{X}$ and $\Delta\rho$ in the frame of least squares collocation method.

As a result, the operational approach to the considered inverse problem leads to well-known solution of the system (12), which may be represented formally in the traditional form

$$\Delta\mathbf{X} = \left[\mathbf{A}^{\mathrm{T}}(\mathbf{C}_{tt}+\mathbf{C}_{nn})^{-1}\mathbf{A}\right]^{-1}\mathbf{A}^{\mathrm{T}}(\mathbf{C}_{tt}+\mathbf{C}_{nn})^{-1}\Delta\mathbf{l}\,, \quad (14)$$

$$\Delta\rho = \mathbf{C}_{st}(\mathbf{C}_{tt}+\mathbf{C}_{nn})^{-1}(\Delta\mathbf{l}-\mathbf{A}\Delta\mathbf{X}), \quad (15)$$

and that are obtained in (Moritz, 1980) for solution of basic problems of physical geodesy under the least-squares principle with the condition (12):

$$\mathbf{s}^{\mathrm{T}}\mathbf{C}_{ss}^{-1}\mathbf{s} + \mathbf{n}^{\mathrm{T}}\mathbf{C}_{nn}^{-1}\mathbf{n} = \min\ . \quad (16)$$

Here the following notations are adopted: we use in (14), (15) two "signal" vectors

$$\mathbf{s} = \Delta\rho \qquad \mathbf{t} = \mathbf{B}\Delta\rho\ , \quad (17)$$

and the next covariance and cross-covariance matrixes

$$\mathbf{C}_{ss} = \mathrm{cov}(s,s)\,, \quad \mathbf{C}_{st} = \mathrm{cov}(s,t)\,,$$

$$\mathbf{C}_{tt} = \mathrm{cov}(t,t)\,, \quad \mathbf{C}_{nn}\ , \quad (18)$$

where the last covariance matrix represents the mathematical expectation of "noise" $\mathbf{n}$ and reflects influence of measuring errors. Thus we come to the *least-squares collocation solution* (14), (15) of the system (12) as the corresponding *variational problem* (16). As well-known this solution admits a certain statistical treatment.

Note also that collocation method can be considered as a special version of Tikhonov's regularization of a solution of improperly posed problems (Tikhonov and Arsenin, 1974). For this reason, we shall treat collocation as a stable linear estimation that was developed in the frame of the operational approach. As a matter of fact, a real achievement of stable solutions is possible only by involving some *additional information* about the Earth's density distribution. The latter is reflected by the equations (14), (15), where such additional information represented in the generalized form by the *covariance function and covariance matrix* $\mathbf{C}_{ss}$ *of the anomalous density*. Other covariance matrixes $\mathbf{C}_{st}$ and $\mathbf{C}_{tt}$ can be constructed by means of the covariance propagation rule (Moritz, 1980). Now we note that the determination only the Earth's density trend part in the considered way (14) requires also the necessity of the above covariance matrixes, just for the stable linear estimation of this part. Naturally that such solution (14) (and (15)) leads to additional difficulties, if these covariance function of the anomalous density and covariance matrixes must be constructed. For this reason, we shall consider some another approach to the determination of $\Delta\mathbf{X}$ or corrections-vector in one of the initial models in the frame of the standard regularization method.

3. Application of the regularization method

Now we start from the traditional correction of the parameters in (1), (2), and (3) in the frame of the well-known linearized model

$$\Delta\mathbf{l} = \mathbf{A}\Delta\mathbf{X} + \mathbf{v}\ , \quad (19)$$

or in practice the system of linear equations in the traditional least-squares adjustment by parameters with standard vector v, which is reflected an influence of measuring errors and accuracy of the approximation. The model (12) may be treated as the generalization of the standard version (19). It is obvious that the first terms of (12) or (19) can be considered in the q-dimensional Euclidean space $\mathbf{R}^q$. The second term of (12) includes the anomalous density $\Delta\rho$ and can be treated as an element of the Hilbert space L_2.

For this reason we return to the solution of the traditional system (19) and recollect again that an inverse problem is called by the properly posed according to *Hadamar* if it solution satisfies to the following requirements:

- *existence*;
- *uniqueness*;
- *stability*.

It is well-known that the system (19) has a unique solution in the following form

$$\Delta\hat{\mathbf{X}} = \left(\mathbf{A}^T\mathbf{C}_{nn}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{C}_{nn}^{-1}\Delta\mathbf{l}\ , \qquad (20)$$

if the matrix **A** has a full rank ($p<q$). The expression (20) is the solution of the *normal* system of linear equations that coincides with the *main or normal pseudosolution*. Thus we come to the conclusion that first two conditions of the *properly posed problem* are satisfied for (20), if parameterization of density distribution is chosen. The third requirement connects in practice with continuity of the solution $\Delta\hat{\mathbf{X}}$ on the „initial data" $\Delta\mathbf{l}$. Such dependence is not always continuous. Hence the third requirement may be violated (Tikhonov and Arsenin, 1974) and the corresponding problem transforms to the *improperly posed problem*. There are many examples (Marchenko, 1999; Marchenko, 2000, etc.) where the construction of normal solutions for the Earth's piecewise radial profiles (1), (2), and (3) leads just to the unstable results if all (two) parameters of models for every shell are determined. Note also that this problem was solved by means of the determination only one parameter for every shell with the application of some additional conditions. If the basic covariance function of the anomalous density is absent, we can expect that such a situation should be better if instead of the least squares condition (16) some another similar principle will be applied (Tikhonov and Arsenin, 1974), for instance, in the following traditional form

$$\mathbf{v}^T\mathbf{C}_{nn}^{-1}\mathbf{v} + \lambda\Omega = \min\ , \qquad (21)$$

where Ω is the so-called stabilizer of solution, which represents on the whole non-negative functional; λ is the parameter of regularization. There exist some basic approaches for λ determination. For the problem of the radial density determination one of the simplest methods was build recently by (Abrikosov, 2000) and further we shall consider such a situation when this parameter λ is chosen.

Next, the stabilizers for each shell separately will be applied as the following quadratic functionals

$$\mathbf{\Omega}_1 = \int_{x_i}^{x_{i+1}} \Delta\rho(x)^2\,dx\,, \qquad (22)$$

$$\mathbf{\Omega}_2 = \int_{x_i}^{x_{i+1}} \left[\Delta\rho(x)^2 + \left(\frac{d\Delta\rho(x)}{dx}\right)^2\right]dx\ , \qquad (23)$$

which correspond to the squared norms of Sobolev spaces $W_2^0 = L_2$ and W_2^1 respectively. Here x_i and x_{i+1} represent two boundary of one shell. It is evident that for the whole Earth as one shell x_1=0 and x_2=1. It is evident that these stabilizers Ω_1 and Ω_2 are different for conditions of smoothness and the approximation of derivatives and functions within every shell.

As well-known the solution of the variational problem (21) has the next unique form

$$\Delta\tilde{\mathbf{X}}_\lambda = \left[\mathbf{A}^T\mathbf{C}_{nn}^{-1}\mathbf{A} + \lambda\mathbf{F}\right]^{-1}\mathbf{A}^T\mathbf{C}_{nn}^{-1}\Delta\mathbf{l}\,, \qquad (24)$$

where the matrix **F** is nothing else but some stabilizer quasidiagonal matrix (shell by shell) with dominated main diagonal, which reflects the including of the stabilization regarding to (20); $\Delta\tilde{\mathbf{X}}_\lambda$ is the estimation of corrections into parameters by means of iterations in the frame of such regularization technique. Since the formula (24) reflects only one step of iterative estimation, the construction of the matrix **F** should be considered as limited case at the neighborhood of zero corrections to the parameters corresponding (1) or (2) or (3) separately. In other words we shall consider (24) as the same standard algorithm for every step.

Thus we come to three possible stabilizers for every studying model and start here from simplest *Roche's law*. In this case the difference between initial and improving density models can be written as

$$\Delta\rho(x) = \Delta a + \Delta b x^2\ , \qquad (25)$$

where the coefficients Δa and Δb should be determined in the frame of (24) for one iteration. Substitution (25) into (22) and (23) after integration leads immediately to the following stabilizer matrixes respectively

$$\mathbf{F}_1^R = \begin{pmatrix} x_{i+1} - x_i & \frac{1}{3}\left(x_{i+1}^3 - x_i^3\right) \\ \frac{1}{3}\left(x_{i+1}^3 - x_i^3\right) & \frac{1}{5}\left(x_{i+1}^5 - x_i^5\right) \end{pmatrix}, \tag{26}$$

$$\mathbf{F}_2^R = \begin{pmatrix} x_{i+1} - x_i & \frac{1}{3}\left(x_{i+1}^3 - x_i^3\right) \\ \frac{1}{3}\left(x_{i+1}^3 - x_i^3\right) & \frac{4}{3}\left(x_{i+1}^3 - x_i^3\right) + \frac{1}{5}\left(x_{i+1}^5 - x_i^5\right) \end{pmatrix}. \tag{27}$$

It is obvious that the application of (27) in (24) gives better results in comparison with (26): we get smaller elements of the main diagonal in (26) than in (27).

Omitting here all auxiliary assumptions and computations, note that in the case of *Legandre - Laplace law* (1) and *Gauss'* normal law (3) we can apply the next approximate diagonal stabilizers matrixes

$$\mathbf{F}_1^L = \mathbf{F}_2^L \approx \begin{pmatrix} x_{i+1} - x_i & 0 \\ 0 & 0 \end{pmatrix} \tag{28}$$

$$\mathbf{F}_1^G = \mathbf{F}_2^G \approx \begin{pmatrix} x_{i+1} - x_i & 0 \\ 0 & 0 \end{pmatrix}$$

that reflect a weak dependence on the corrections in the parameter γ in Legendre-Laplace law and the parameter β in Gauss' model respectively. Finally, such a conclusion we have got, in fact, in the practical estimations of Legendre-Laplace and Gauss piecewise density profiles (see, for instance, Marchenko, 1999; Marchenko, 2000; Marchenko, 2000a).

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ПРО МОДЕЛЬНИЙ ТА ОПЕРАЦІЙНИЙ ПІДХОДИ ДО ПОБУДОВИ РАДІАЛЬНОЇ ГУСТИНИ ЗЕМЛІ

Резюме

Неперервні радіальні розподіли Лежандра-Лапласа, Роша і Гауса, які описують глобальний тренд кусково-неперервного профіля густини, були використані для побудови найпростіших стабілізаторів та визначення радіальної густини. Крім цього був розглянутий операційний підхід, який приводить до методу середньої квадратичної колокації.

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О МОДЕЛЬНОМ И ОПЕРАЦИОННОМ ПОДХОДЕ К ПОСТРОЕНИЮ РАДИАЛЬНОЙ ПЛОТНОСТИ ЗЕМЛИ

Резюме

Непрерывные радиальные распределения Лежандра-Ларласа, Роша и Гаусса, описывающие глобальный тренд кусочно-непрерывного распределения плотности, были использованы для построения простейших стабилизаторов и определения радиальной плотности Земли. Отдельное рассмотрение задачи в рамках операционного подхода приводит к необходимости использования метода средней квадратической коллокации.