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A.N. MARCHENKO
Lviv Polytechnic National University

A NOTE ON THE EIGENVALUE – EIGENVECTOR PROBLEM: APPLICATION TO THE GRAVITATIONAL GRADIENT TENSOR[*)]

В роботі розглянутий загальний аналітичний розв'язок задачі знаходження власних чисел та власних векторів симетричного тензора другого рангу. Виконана їх оцінка точності на основі правила строгого перетворення коваріацій. Розв'язання проілюстровано на прикладі знаходження власних чисел та власних векторів тензора гравітаційного градієнту.

В роботе рассмотрено общее аналитическое решение задачи нахождения собственных чисел и собственных векторов симметричного тензора второй валентности. Выполнена их оценка точности на основании правила строгого преобразования ковариаций. Решение проилюстрировано на примере нахождения собственных чисел и собственных векторов тензора гравитационного градиента.

Introduction

The study of the Earth as a planet leads in practice of geosciences to well chosen mathematical methods solving various tasks of Geodesy, Astronomy, Geodynamics, etc. Very often different applications in natural sciences may have the same mathematical tool. Tensor analysis represents one of such general approaches: "A particularly important role in applications (physics, engineering) is played by tensors represented by 3×3 *symmetric matrices*: strain tensor, stress tensor, inertia tensor, etc." (Moritz and Hofmann-Wellenhof, 1993). The Earth's inertia tensor, in view of the planet's rotation theory and density distribution, was considered by (Moritz and Mueller, 1987) and Moritz (1990) using mainly the coordinate system of the principal axes of inertia. Determination of the principal deformations and the orientation of the principal directions is one of central tasks of the strain tensor theory. Hence, the transformation of the mentioned 3×3 symmetric matrices to the diagonal form requires the solution of the eigenvalue – eigenvector problem that is the point of departure to obtain the principal values and principal directions of the corresponding tensors in various applications.

A suitable analytical solution of such eigenvalue – eigenvector problem for one partial case of the matrix–deviator was given by Marchenko and Abrikosov (2001) and was developed essentially by Marchenko and Schwintzer (2003) for the rigorous error propagation from the 2nd degree harmonic coefficients of the geopotential and dynamical ellipticity to some fundamental parameters of the Earth. The last paper represents rather a geodetic treatment of the problem.

Permanent using of the remarkable textbooks of Professor Helmut Moritz, my translation into Russian of his book "The Figure of the Earth", and numerous scientific discussions in TU Graz and different meetings (1983, 1988, 1990, 1991, 1992, 1993, 1994, 1996, 2000, 2001) together with the mentioned investigations are provided, in particular, the author's general viewpoint on the above problem. That is why the main goal of this contribution is to extend the previous studies to the analytical solution of the eigenvalue – eigenvector problem, including the rigorous error propagation, for the general case of a tensor of order 2 represented by 3×3 *symmetric matrix*. As an example, the derived solution is applied to the gravitational gradient tensor (Moritz, 1989) using the dynamical model of the Earth's gravitational potential.

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Eigenvalue-eigenvector problem

As well-known, the principal values and principal directions of the tensor $\mathbf{T}=[t_{ij}]$ of the second order represented by the 3×3 symmetric matrix (Kochin, 1965):

$$\mathbf{T}=\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{bmatrix}=\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{xy} & t_{yy} & t_{yz} \\ t_{xz} & t_{yz} & t_{zz} \end{bmatrix}, \quad (1)$$

and defined here in an adopted Cartesian coordinate system (*x*, *y*, *z*) can be found from the solution of eigenvalue-eigenvector problem. In order to prepare the error propagation we prefer to use the analytical approach to this problem and the suitable separation of the tensor **T** to the following two parts

$$\mathbf{T}=\widetilde{\mathbf{T}}+\mathbf{D}. \quad (2)$$

The first one is described by the factor χ:

$$\widetilde{\mathbf{T}}=\chi\cdot\mathbf{I}=\frac{\mathrm{trace}\mathbf{T}}{3}\cdot\mathbf{I}, \quad \mathrm{trace}\mathbf{T}=t_{xx}+t_{yy}+t_{zz}, \quad (3)$$

and the diagonal unit (3×3) matrix **I** or the Kronecker tensor. The second one represents the so-called deviatoric part or the deviatoric tensor (Kochin, 1965):

$$\mathbf{D}=\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix}=\begin{bmatrix} \frac{2t_{xx}-t_{yy}-t_{zz}}{3} & t_{xy} & t_{xz} \\ t_{xy} & \frac{-t_{xx}+2t_{yy}-t_{zz}}{3} & t_{yz} \\ t_{xz} & t_{yz} & \frac{-t_{xx}-t_{yy}+2t_{zz}}{3} \end{bmatrix}. \quad (4)$$

The matrix (4) has remarkable properties. For instance, the first invariant of (4) is $I_1=\mathrm{trace}\mathbf{D}=0$. Note also that both tensors **T** and $\mathbf{D}=[d_{ij}]$ have the same principal directions. The following relationships for the second invariant of (4):

$$I_2=-\frac{1}{2}\|\mathbf{D}\|^2_{R^3} \Rightarrow I_2\le 0, \quad (5)$$

is valid for an arbitrary matrix-deviator and can be found straightforward by comparison of I_2 and the squared Euclidean norm $\|\mathbf{D}\|^2_{R^3}=\sum_{i=1}^{3}\sum_{j=1}^{3}d_{ij}^2$ of the matrix **D**. The latter leads to the identical algebraic equations solving the *eigenvalue* problem for the *deviator* (4):

$$\lambda^3+I_2\lambda-I_3=0, \quad \lambda^3-\frac{1}{2}\|\mathbf{D}\|^2_{R^3}\lambda-I_3=0 \quad (6)$$

taking into account the relationship (5). We prefer to use below the solution of the second equation of (6) and get the eigenvalues Λ_1, Λ_2, and Λ_3 of the symmetric matrix (1) in the following form

$$\left.\begin{matrix}\Lambda_1\\ \Lambda_2\\ \Lambda_3\end{matrix}\right\}=\chi+\left\{\begin{matrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{matrix}\right., \quad \left.\begin{matrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{matrix}\right\}=\sqrt{\frac{2}{3}}\cdot\|\mathbf{D}\|_{R^3}\cdot\left\{\begin{matrix}\tilde{\lambda}_1\\ \tilde{\lambda}_2\\ \tilde{\lambda}_3\end{matrix}\right., \quad \left.\begin{matrix}\tilde{\lambda}_1\\ \tilde{\lambda}_2\\ \tilde{\lambda}_3\end{matrix}\right\}=\left\{\begin{matrix}\sin\left(\frac{\varphi}{3}-\frac{\pi}{3}\right)\\ -\sin\frac{\varphi}{3}\\ \sin\left(\frac{\varphi}{3}+\frac{\pi}{3}\right)\end{matrix}\right., \quad (7)$$

where the eigenvalues $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$ are the so-called normalized solution (see, Marchenko and Abrikosov, 2001); the characteristic angle φ is expressed

$$\varphi = \sin^{-1}\left(\frac{3\sqrt{6}I_3}{\|\mathbf{D}\|_{R^3}^3}\right), \tag{8}$$

via the usual Euclidean norm of the matrix **D** and the third invariant I_3:

$$I_3 = \det \mathbf{D}\,. \tag{9}$$

Then any *eigenvector* $\mathbf{X}_j$ corresponding to a certain eigenvalue λ_j or Λ_j:

$$\mathbf{X}_j = \frac{1}{\sqrt{\mathbf{Z}_j^{\mathrm{T}}\mathbf{Z}_j}}\mathbf{Z}_j, \quad (j=1,2,3), \tag{10}$$

can be found according to (Marchenko and Schwintzer (2003) through the non-unit vectors $\mathbf{Z}_j$ (j=1,2,3):

$$\mathbf{Z}_j = \mathbf{P} + \lambda_j\mathbf{S} + \lambda_j^2\mathbf{E}, \tag{11}$$

$$\mathbf{P} = \mathbf{t}_1 \times \mathbf{t}_2 + \mathbf{t}_2 \times \mathbf{t}_3 + \mathbf{t}_3 \times \mathbf{t}_1 = \mathbf{T}_2^{\mathrm{T}} \cdot \mathbf{t}_1 + \mathbf{T}_3^{\mathrm{T}} \cdot \mathbf{t}_2 + \mathbf{T}_1^{\mathrm{T}} \cdot \mathbf{t}_3, \tag{12}$$

$$\mathbf{S} = \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3, \tag{13}$$

$$\mathbf{E} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\mathrm{T}}, \tag{14}$$

(hereafter the symbol T denotes transposition) where the matrices $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ are defined below by (32)–(34) and

$$\mathbf{t}_1 = \begin{bmatrix} \dfrac{2t_{xx} - t_{yy} - t_{zz}}{3} \\ t_{xy} \\ t_{xz} \end{bmatrix}, \quad \mathbf{t}_2 = \begin{bmatrix} t_{xy} \\ \dfrac{2t_{yy} - t_{xx} - t_{zz}}{3} \\ t_{yz} \end{bmatrix}, \quad \mathbf{t}_3 = \begin{bmatrix} t_{xz} \\ t_{yz} \\ \dfrac{2t_{zz} - t_{xx} - t_{yy}}{3} \end{bmatrix}, \tag{15}$$

are the column-vectors of (4); $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit vectors of the matrix **I**.

Error propagation for the eigenvalue problem

The result of solution of the discussed problem is a set of closed expressions reflecting a functional dependence between eigenvalues and eigenvectors of the matrices (4) and (1) and their components. This fact allows considering the error propagation from given elements to the principal values and principal directions.

We start from the deviatoric part (4) using the following auxiliary vector

$$\mathbf{t} = \begin{bmatrix} t_{xx}, t_{yy}, t_{zz}, t_{xy}, t_{xz}, t_{yz} \end{bmatrix}^{\mathrm{T}}, \tag{16}$$

which is formed by the six components of the tensor (1). Thus, the vector **t** and its (6×6) variance-covariance matrix $\mathbf{C}_{tt}$ are adopted as given initial information. Then, following to (Marchenko and Schwintzer, 2003) and setting the vectors

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1(\mathbf{J}), \lambda_2(\mathbf{J}), \lambda_3(\mathbf{J}) \end{bmatrix}^{\mathrm{T}}, \qquad \mathbf{J} = \begin{bmatrix} \|\mathbf{D}(\mathbf{t})\|_{R^3}^2, I_3(\mathbf{t}) \end{bmatrix}^{\mathrm{T}} \tag{17}$$

we introduce the (3×6) matrix

$$\frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{t}} = \frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{J}} \cdot \frac{\partial \mathbf{J}}{\partial \mathbf{t}}, \tag{18}$$

of partial derivatives of the eigenvalues λ_1, λ_2, and λ_3 of the deviator (4) with respect to the elements of the vector (16). After simple manipulations we get the first component in the right hand side of (18) as the (3×2) matrix

$$\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{J}}=\frac{1}{2\|\mathbf{D}(\mathbf{t})\|^2_{R^3}}\bar{\boldsymbol{\lambda}}+\frac{1}{3\sqrt{3}}\mathbf{Q}\cdot\frac{\partial\varphi(\mathbf{J})}{\partial\mathbf{J}},\quad \bar{\boldsymbol{\lambda}}=[\boldsymbol{\lambda},\mathbf{0}],\quad \mathbf{Q}=\begin{bmatrix}\sqrt{2\|\mathbf{D}\|^2_{R^3}-3\lambda_1^2}\\ -\sqrt{2\|\mathbf{D}\|^2_{R^3}-3\lambda_2^2}\\ \sqrt{2\|\mathbf{D}\|^2_{R^3}-3\lambda_3^2}\end{bmatrix}, \tag{19}$$

$$\mathbf{0}=[0\quad 0\quad 0]^{\mathrm{T}},\qquad \frac{\partial\varphi(\mathbf{J})}{\partial\mathbf{J}}=\tan\varphi\cdot\left[-\frac{3}{2\|\mathbf{D}\|^2_{R^3}},\frac{1}{I_3}\right]. \tag{20}$$

The second component or the (2×6) matrix may be formed as

$$\frac{\partial\mathbf{J}}{\partial\mathbf{t}}=\begin{bmatrix}\dfrac{\partial\|\mathbf{D}(\mathbf{t})\|^2_{R^3}}{\partial\mathbf{t}}\\ \dfrac{\partial I_3(\mathbf{t})}{\partial\mathbf{t}}\end{bmatrix}=\begin{bmatrix}\mathbf{t}^{\mathrm{T}}\cdot\mathbf{a}\\ (\mathbf{t}_1\times\mathbf{t}_2)^{\mathrm{T}}\mathbf{A}_3+(\mathbf{t}_2\times\mathbf{t}_3)^{\mathrm{T}}\mathbf{A}_1+(\mathbf{t}_3\times\mathbf{t}_1)^{\mathrm{T}}\mathbf{A}_2\end{bmatrix}, \tag{21}$$

where $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ are the column-vectors defined above by the relationships (15). Using the standard expression for the squared Euclidean norm $\|\mathbf{D}\|^2_{R^3}$, the (6×6) matrix **a** is found by direct differentiating $(\partial\mathbf{J}/\partial\mathbf{t})$:

$$\mathbf{a}=\frac{2}{3}\begin{bmatrix}2&-1&-1&0&0&0\\-1&2&-1&0&0&0\\-1&-1&2&0&0&0\\0&0&0&6&0&0\\0&0&0&0&6&0\\0&0&0&0&0&6\end{bmatrix} \tag{22}$$

The (3×6) matrices $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are represented by the following derivatives:

$$\mathbf{A}_1=\frac{\partial\mathbf{t}_1(\mathbf{t})}{\partial\mathbf{t}}=\frac{1}{3}\begin{bmatrix}2&-1&-1&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\end{bmatrix}, \tag{23}$$

$$\mathbf{A}_2=\frac{\partial\mathbf{t}_2(\mathbf{t})}{\partial\mathbf{t}}=\frac{1}{3}\begin{bmatrix}0&0&0&3&0&0\\-1&2&-1&0&0&0\\0&0&0&0&0&3\end{bmatrix}, \tag{24}$$

$$\mathbf{A}_3=\frac{\partial\mathbf{t}_3(\mathbf{t})}{\partial\mathbf{t}}=\frac{1}{3}\begin{bmatrix}0&0&0&0&3&0\\0&0&0&0&0&3\\-1&-1&2&0&0&0\end{bmatrix} \tag{25}$$

Thus, the relationships (19)–(25) allow the error propagation for the computation of the (2×2) covariance matrix $\mathbf{C}_{\mathrm{JJ}}$ of the invariants $\|\mathbf{D}\|^2_{R^3}$ and I_3:

$$\mathbf{C}_{\mathrm{JJ}}=\frac{\partial\mathbf{J}}{\partial\mathbf{t}}\mathbf{C}_{\mathrm{tt}}\left(\frac{\partial\mathbf{J}}{\partial\mathbf{t}}\right)^{\mathrm{T}}, \tag{26}$$

the (3×3) covariance matrix $\mathbf{C}_{\lambda\lambda}$ of the eigenvalues λ_1, λ_2, λ_3 of the deviator (4):

$$\mathbf{C}_{\lambda\lambda}=\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{t}}\mathbf{C}_{\mathrm{tt}}\left(\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{t}}\right)^{\mathrm{T}}=\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{J}}\mathbf{C}_{\mathrm{JJ}}\left(\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{J}}\right)^{\mathrm{T}}, \tag{27}$$

the variance var(χ) of the parameter χ=TraceT/3:

$$\mathrm{var}(\chi) = \frac{\partial \chi}{\partial \mathbf{t}} \mathbf{C}_{tt} \left(\frac{\partial \chi}{\partial \mathbf{t}} \right)^{\mathrm{T}}, \quad \frac{\partial \chi}{\partial \mathbf{t}} = \frac{1}{3}[1;\ 1;\ 1;\ 0;\ 0;\ 0], \tag{28}$$

and the (3×3) covariance matrix $C_{\Lambda\Lambda}$ of the eigenvalues $\Lambda_1, \Lambda_2, \Lambda_3$ of the initial symmetric matrix (1):

$$\mathbf{C}_{\Lambda\Lambda} = \left(\mathbf{E}\frac{\partial \chi}{\partial \mathbf{t}} + \frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{t}} \right) \mathbf{C}_{tt} \left(\mathbf{E}\frac{\partial \chi}{\partial \mathbf{t}} + \frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{t}} \right)^{\mathrm{T}}, \tag{29}$$

where the vector **E** is defined by the expression (14).

Error propagation for the eigenvector problem

The eigenvectors $\mathbf{X}_j$ (j=1,2,3) are nothing else but the unit vectors with the same directions as $\mathbf{Z}_j$. Nevertheless it is sufficient to use $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$, which give access to simpler expressions for required derivatives (see, Marchenko and Schwintzer, 2003). Remembering the definition (15) of $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ we come to the non-linear functional dependence (11) on the six components of the vector **t**:

$$\mathbf{Z}_j(\mathbf{t}) = \mathbf{P}(\mathbf{t}) + \lambda_j(\mathbf{t}) \cdot \mathbf{S}(\mathbf{t}) + \lambda_j^2(\mathbf{t}) \cdot \mathbf{E}. \tag{30}$$

The differentiation of (30) with respect to the elements of **t** and some elementary transformations gives the (3×6) matrices of partial derivatives

$$\frac{\partial \mathbf{Z}_j}{\partial \mathbf{t}} = \frac{\partial \mathbf{P}}{\partial \mathbf{t}} + \lambda_j \frac{\partial \mathbf{S}}{\partial \mathbf{t}} + (\mathbf{S} + 2\lambda_j \mathbf{E}) \frac{\partial \lambda_j}{\partial \mathbf{t}}, \quad (j=1,2,3). \tag{31}$$

where the (3×6)-matrix $\partial\boldsymbol{\lambda}/\partial\mathbf{t}$ is determined already by expressions (17)–(25).

Then we introduce the auxiliary skew-symmetric matrices

$$\mathbf{T}_1 = \begin{bmatrix} 0 & -t_{xz} & t_{xy} \\ t_{xz} & 0 & -(2t_{xx} - t_{yy} - t_{zz})/3 \\ -t_{xy} & (2t_{xx} - t_{yy} - t_{zz})/3 & 0 \end{bmatrix}, \tag{32}$$

$$\mathbf{T}_2 = \begin{bmatrix} 0 & -t_{yz} & (2t_{yy} - t_{xx} - t_{zz})/3 \\ t_{yz} & 0 & -t_{xy} \\ -(2t_{yy} - t_{xx} - t_{zz})/3 & t_{xy} & 0 \end{bmatrix}, \tag{33}$$

$$\mathbf{T}_3 = \begin{bmatrix} 0 & -(2t_{zz} - t_{xx} - t_{yy})/3 & t_{yz} \\ (2t_{zz} - t_{xx} - t_{yy})/3 & 0 & -t_{xz} \\ -t_{yz} & t_{xz} & 0 \end{bmatrix}, \tag{34}$$

constructed for every vectors $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ respectively. Omitting here algebraic manipulations, we get the final relationships for the required (3×6) matrices as

$$\frac{\partial \mathbf{P}}{\partial \mathbf{t}} = (\mathbf{T}_3 - \mathbf{T}_1)\mathbf{A}_1 + (\mathbf{T}_1 - \mathbf{T}_3)\mathbf{A}_2 + (\mathbf{T}_2 - \mathbf{T}_1)\mathbf{A}_3, \tag{35}$$

$$\frac{\partial \mathbf{S}}{\partial \mathbf{t}} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3, \tag{36}$$

where A_1, A_2, and A_3 are defined by (23)–(25). Therefore the relationships (18)–(25) and (35)–(36) allow computations of the partial derivatives (31) and the variance-covariance matrices $\mathbf{C}_{\mathbf{Z}_j\mathbf{Z}_j}$:

$$\mathbf{C}_{\mathbf{Z}_j\mathbf{Z}_j} = \frac{\partial \mathbf{Z}_j}{\partial \mathbf{t}} \mathbf{C}_{tt} \left(\frac{\partial \mathbf{Z}_j}{\partial \mathbf{t}} \right)^{\mathrm{T}}. \tag{37}$$

Let $\overline{\varphi}_j$ and $\overline{\lambda}_j$ denote now the geocentric latitude and longitude of some pole where the straight line $\mathbf{Z}_j$ cuts the unit sphere. These coordinates can be expressed through the components of the vectors $\mathbf{Z}_j = \left[z_1^j, z_2^j, z_3^j\right]^{\mathrm{T}}$:

$$\overline{\varphi}_j = \sin^{-1}\left(\frac{z_3^j}{\rho_j}\right), \quad \overline{\lambda}_j = \tan^{-1}\left(\frac{z_2^j}{z_1^j}\right), \quad \rho_j = \sqrt{\mathbf{Z}_j^{\mathrm{T}}\mathbf{Z}_j}, \tag{38}$$

that gives the partial derivatives of the elements $\overline{\varphi}_j$, $\overline{\lambda}_j$ of $\psi_j = \left[\overline{\varphi}_j, \overline{\lambda}_j\right]^{\mathrm{T}}$ with respect to the components of the vector $\mathbf{Z}_j$. The result is the (2×3)-matrix

$$\frac{\partial \psi_j}{\partial \mathbf{Z}_j} = \begin{bmatrix} -\dfrac{z_1^j z_3^j}{\widetilde{r}_j \rho_j^2} & -\dfrac{z_2^j z_3^j}{\widetilde{r}_j \rho_j^2} & \dfrac{\widetilde{r}_j}{\rho_j^2} \\ -\dfrac{z_2^j}{\widetilde{r}_j^2} & \dfrac{z_1^j}{\widetilde{r}_j^2} & 0 \end{bmatrix}, \quad \widetilde{r}_j = \sqrt{(z_1^j)^2 + (z_2^j)^2}, \tag{39}$$

and the required variance-covariance matrix $\mathbf{C}_{\psi_j\psi_j}$:

$$\mathbf{C}_{\psi_j\psi_j} = \frac{\partial \psi_j}{\partial \mathbf{Z}_j}\mathbf{C}_{Z_jZ_j}\left(\frac{\partial \psi_j}{\partial \mathbf{Z}_j}\right)^{\mathrm{T}} = \frac{\partial \psi_j}{\partial \mathbf{Z}_j}\frac{\partial \mathbf{Z}_j}{\partial \mathbf{t}}\mathbf{C}_{\mathrm{tt}}\left(\frac{\partial \psi_j}{\partial \mathbf{Z}_j}\frac{\partial \mathbf{Z}_j}{\partial \mathbf{t}}\right)^{\mathrm{T}}. \tag{40}$$

Gradient tensor and concluding remarks

Note in conclusion that the analytical solution of the eigenvalue-eigenvector problem and error propagation are given above for the symmetric matrix (1) of general kind. In the case of the error propagation for the eigenvalue problem the expressions (17)–(29) represent the generalization of the previous results from (Marchenko and Schwintzer, 2003). Excluding a general form for (32)–(34), the error propagation for the eigenvector problem is performed in the same manner. Since the inequality $\pi/2 \le \varphi \le \pi/2$ is valid for the parameter (8), the consideration of the limited cases is necessary. We will illustrate briefly a possible example using the dynamical model of the gravitational potential of a planet.

Let the external gravitational potential V of the Earth is given in the global body-fixed Cartesian coordinate system $Oxyz$ together with the derivatives $\partial V/\partial x$, $\partial V/\partial y$, and $\partial V/\partial z$ at the current point P. To reduce the algebraic manipulations we apply according to (Marchenko, 1991) the following model representation of the function V by the potential of one non-central point mass

$$V(P) = \frac{GM}{\ell}, \quad \ell = \ell(P) = \sqrt{(x - x_C)^2 + (y - y_C)^2 + (z - z_C)^2}, \tag{41}$$

where M is the Earth's mass; G is the gravitational constant; x, y, z are the coordinates of the point P; x_C, y_C, z_C are the coordinates of the instantaneous attracting center located at the point C of the *planet's attraction*; ℓ is the distance between the points C and P. These coordinates can be found in the following way

$$x_C = x + \ell\frac{F_x}{F}, \quad y_C = y + \ell\frac{F_y}{F}, \quad z_C = z + \ell\frac{F_z}{F}, \tag{42}$$

$$\ell = \frac{V(P)}{F(P)}, \quad F(P) = \sqrt{\left(\frac{\partial V(P)}{\partial x}\right)^2 + \left(\frac{\partial V(P)}{\partial y}\right)^2 + \left(\frac{\partial V(P)}{\partial z}\right)^2} = \frac{GM}{\ell^2}. \tag{43}$$

The expressions (41)–(43) represent exactly the external Newtonian potential of the solid Earth and the corresponding attraction vector $\mathbf{F}$ (with the magnitude F and the components F_x, F_y, F_z):

$$\mathbf{F} = \left(F_x, F_y, F_z\right) = \left(\partial V/\partial x, \partial V/\partial y, \partial V/\partial z\right) = \mathrm{grad}V, \tag{44}$$

by the potential of attracting center in the frame of Newton's law of gravitation.

The *non-deviatoric* gravity gradient tensor or Eötvös tensor **W**, which is considered traditionally in the local coordinate system (Eötvös, 1896; Moritz and Hofmann-Wellenhof, 1993), consists of the gravitational **V** (*deviatoric* as a consequence of the Laplace equation) and rotational **Φ** parts. We will form the tensor **V** in the global coordinate system (see, Moritz, 1989) using the 2nd order partial derivatives $[V_{ij}]$ of the model (41) with respect to the coordinates (x, y, z):

$$\mathbf{V}=\begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12} & V_{22} & V_{23} \\ V_{13} & V_{23} & V_{33} \end{bmatrix}=\begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{xy} & V_{yy} & V_{yz} \\ V_{xz} & V_{yz} & V_{zz} \end{bmatrix}=$$

$$=\frac{GM}{\ell^5}\begin{bmatrix} 3(x-x_C)^2-\ell^2 & 3(x-x_C)(y-y_C) & 3(x-x_C)(z-z_C) \\ 3(x-x_C)(y-y_C) & 3(y-y_C)^2-\ell^2 & 3(y-y_C)(z-z_C) \\ 3(x-x_C)(z-z_C) & 3(y-y_C)(z-z_C) & 3(z-z_C)^2-\ell^2 \end{bmatrix}. \quad (45)$$

Then the substitution of (42)–(43) into (45) gives

$$\mathbf{V}=\frac{1}{V}\begin{bmatrix} 3F_x^2-F^2 & 3F_xF_y & 3F_xF_z \\ 3F_xF_y & 3F_y^2-F^2 & 3F_yF_z \\ 3F_xF_z & 3F_yF_z & 3F_z^2-F^2 \end{bmatrix}, \quad (46)$$

the gravitational gradient tensor corresponding to the model (41). Because all invariants of the matrix (46) are independent of the linear transformations of the coordinate system by applying the relationships (3)–(15) we get at any P:

$$I_1=\mathrm{trace}\mathbf{V}=0,\ \ \|\mathbf{V}\|^2_{R^3}=\frac{6F^4}{V^2}=\frac{6F^2}{\ell^2}=\frac{6V^2}{\ell^4},\ \ I_3=\frac{2F^6}{V^3}=\frac{2F^3}{\ell^3}=\frac{2V^3}{\ell^6}, \quad (47)$$

the constant value of the characteristic angle φ:

$$\varphi=\sin^{-1}(1)=\pi/2, \quad (48)$$

the diagonal form of the deviator (46):

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}=\frac{1}{\ell}\begin{bmatrix} -F & 0 & 0 \\ 0 & -F & 0 \\ 0 & 0 & 2F \end{bmatrix}=\frac{1}{\ell^2}\begin{bmatrix} -V & 0 & 0 \\ 0 & -V & 0 \\ 0 & 0 & 2V \end{bmatrix}, \quad (49)$$

and only one eigenvectors $\mathbf{X}_3$:

$$\mathbf{X}_3=F^{-1}[F_x,\ F_y,\ F_z]^{\mathrm{T}}=F^{-1}\cdot\mathrm{grad}V. \quad (50)$$

The eigenvalues $\lambda_1=\lambda_2$ and λ_3, given by (49), can be proven using the Hamilton-Cayley identity written as $(\mathbf{V}-\lambda_1\mathbf{I})^2\cdot(\mathbf{V}-\lambda_3\mathbf{I})=\mathbf{0}$. But the equations (10) – (15) lead in this case to impossible by the definition zero eigenvectors $\mathbf{X}_1=\mathbf{X}_2=\mathbf{0}$ caused by the model (41). Other difficulty is that the derivatives (18) have uncertainties for $\varphi=\pm\pi/2$ caused by $\tan(\varphi)$ in the expression (20). Hence, the error propagation when $\varphi=\pm\pi/2$ is unworkable and requires a special attention.

Remembering the important role of the gravity gradient tensor **W** in the differential geometry of the gravity field, we consider a possible geometric treatment of the gravitational gradient tensor **V** given by (49). For this reason, the Gaussian fundamental quantities of the first $(\widetilde{E},\widetilde{F},\widetilde{G})$ and second $(\widetilde{L},\widetilde{M},\widetilde{N})$ kinds are derived for the equipotential surface $V(x, y, z)=V_0$=const, passing through the point P, by applying (41) explicitly

$$\widetilde{E}=1+\left(\frac{F_x}{F_z}\right)^2,\quad \widetilde{F}=\frac{F_xF_y}{F_z^2},\quad \widetilde{G}=1+\left(\frac{F_y}{F_z}\right)^2, \quad (51)$$

$$\tilde{L} = \frac{F^2 - F_y^2}{F_z^2 \ell}, \quad \tilde{M} = \frac{F_x F_y}{F_z^2 \ell}, \quad \tilde{N} = \frac{F^2 - F_x^2}{F_z^2 \ell} \tag{52}$$

The Gaussian fundamental quantities (51) and the unite vector (50) allow to determine the orthonormal eigenvectors $\mathbf{X}_1$ and $\mathbf{X}_2$:

$$\mathbf{X}_1 = p^{-1}\left[F_z, 0, -F_x\right]^{\mathrm{T}} \quad \mathbf{X}_2 = F^{-1}\left[-F_x F_y p^{-1}, p, -F_y F_z p^{-1}\right]^{\mathrm{T}}, \quad p = \sqrt{F_x^2 + F_z^2}\ , \tag{53}$$

taking into account the obvious condition for the vector product $\mathbf{X}_1 \times \mathbf{X}_2 = \mathbf{X}_3$. These vectors ($\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$) form a moving together with the attraction center C orthogonal triad where the unite force vector $\mathbf{X}_3$ is normal to the surface $V(P)$=const and two vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ span the tangent plane at the point P.

Then, considering P as belonging to this equipotential surface, the Gaussian curvature at P can be found in the following way

$$K = \frac{\tilde{L}\tilde{N} - \tilde{M}^2}{\tilde{E}\tilde{G} - \tilde{F}^2} = \frac{1}{\ell^2} = \frac{\lambda_1 \lambda_2}{F^2}. \tag{54}$$

The mean curvature at P is expressed now via the distance to attraction center

$$J = \frac{\tilde{E}\tilde{N} - 2\tilde{F}\tilde{M} + \tilde{G}\tilde{L}}{2(\tilde{E}\tilde{G} - \tilde{F}^2)} = \frac{1}{\ell} = \frac{F}{V} = -\frac{(\lambda_1 + \lambda_2)}{2F} \tag{55}$$

Since the principal curvatures $k_1 \neq k_2$ of the equipotential surfaces of the *actual* potential V are assumed to be always positive (no singularities in the gravitational field), they characterize surfaces of the elliptic kind. In the frame of the chosen *model* (41) we get $k_1 = k_2$ at the point P due to $\lambda_1 = \lambda_2$ and only *point approximation* of the surface $V(P)$=const of the elliptic type by surfaces of the spherical kind defined by the model (41) (despite of the exact representation of the given F_x, F_y, F_z, and V). The mean curvature (55) in the case (41) can be treated as a simple ratio "Attraction/Potential" taken at the point P. The radius of this curvature is nothing else but the distance $\ell(P)$ between P and the attracting center C. The Gaussian (54) and mean (55) curvatures of the equipotential surface are proportional at P to the values F and V, respectively. In the case of the gravity gradient tensor $\mathbf{W} = [W_{ij}]$ or Marussi tensor $[w_{ij} = -W_{ij}/g]$, the same results can be found by applying the limited case $\omega \rightarrow 0$ for the angular velocity ω of the Earth's rotation when the gravity $g \rightarrow F$ is transformed to the force F.

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