ЗАСОБИ ВИМІРЮВАНЬ ЕЛЕКТРИЧНИХ ТА МАГНІТНИХ ВЕЛИЧИН

НЕПЕВНОСТІ БАГАТОПАРАМЕТРОВИХ НЕПРЯМИХ ВИМІРЮВАНЬ ЗА ДОПОМОГОЮ ЕЛЕКТРИЧНИХ СХЕМ ПОСТІЙНОГО СТРУМУ

UNCERTAINTIES OF MULTIVARIABLE INDIRECT MEASUREMENTS OF DC ELECTRICAL CIRCUITS

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Abstract. In this paper three examples of processing uncertainties of a indirect multi-variable measurement system are considered. It was proposed to extend the vector method of estimating measurement uncertainties, given in Supplement 2 to GUM, on the statistical description of the accuracy of whole ranges of indirect multivariable measurement system. Formula for the covariance matrix of relative uncertainties of the vector measurand is given. The covariance matrixes of uncertainties of few DC electrical measurement circuits are presented, i.e.: for indirect measurement of three resistances with using them in three variants of balanced Wheatstone bridge or without disconnection this circuit but with apply unconventional current supplies; the measurement of three internal resistances of the star circuit from its terminals and estimation of uncertainty of powers of two currents if two other currents are measured and their uncertainties are known. Formulas for absolute and relative uncertainties and their correlation coefficients are given. The general conclusion is that in the description accuracy of multivariable measurement systems the relative uncertainties are sometimes preferable than the absolute ones, and uncertainties of their main measurement functions have been also considered.

Key words: непевністьUncertainty, Electrical circuit, Multiplicative measurement equations.

Анотація. Висвітлено особливості опрацювання результатів вимірювання за допомогою непрямої багатопараметрової вимірювальної системи. Розглянуто три випадки вивчення непевностей вимірювання. Запропоновано поширити векторний метод оцінювання непевностей вимірювання, поданий у додатку 2 до GUM, на статистичне описання параметрів точності непрямої багатопараметрової вимірювальної системи. Подано формули для коваріантної матриці відносних непевностей векторного межеранда. Матриця непевностей для випадку декількох електричних систем вимірювання на постійному струмі підлягає аналізу. Інакше кажучи, аналізують непрямі вимірювання електричного опору за допомогою моста Вінстона, причому вимірювання внутрішніх опорів зіркової конфігурації з оцінюванням непевності струму напруги живлення підлягають розгляду. Наведені формули для абсолютної та відносної непевностей, а також їх коефіцієнти кореляції. Основний висновок такий: описання параметрів точності багатопараметрової вимірювальної системи відносними непевностями іноді краще порівняно із подібним описанням на основі абсолютних непевностей, що й повинно враховуватись під час розгляду.

Ключові слова: , електричне коло, багатопараметрові вимірювальні рівняння.

Introduction

In indirect methods of measurements, results the tested quantities (observables) and their accuracy is determined from direct measurements of the set of jointed other quantities, named the multivariate or vector measurand. In general case of multivariable measurements, the relation between values $\boldsymbol{X}$ of quantities measured in input and data of $\boldsymbol{Y}$ quantities obtained on output after processing is a functional $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{Y}) = \mathbf{0}$. Usually it can be formulated as the following multivariable function

$$\boldsymbol{Y} = \boldsymbol{F}(\boldsymbol{X})\ . \tag{1}$$

The multivariable vector dependence (1) can be linear or non-linear. The general flow chart of multi-variable measurement system is shown in Fig. 1.

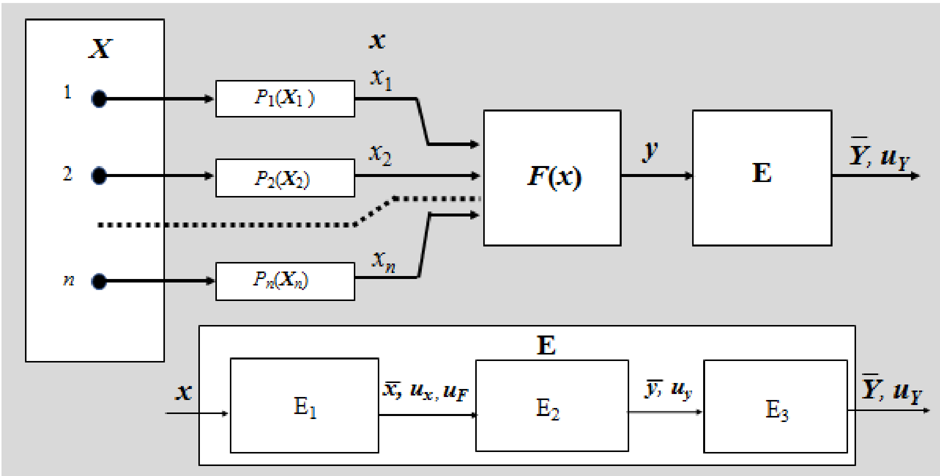


Fig. 1. Signal processing in multi-variable indirect measurement system

In Fig. 1 and in the text below the next symbols are used: X, Y – vectors of values of input and output variables; $x=[x_1, x_2,\ldots x_n]$, $\boldsymbol{y}=[y_1, y_2,\ldots y_m]$ – vectors of input x_i and output y_i signals; $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{y}}$, u_x, $u_{\delta x}$, u_y, $u_{\delta y}$ – vectors of estimators of values of input x and output y signals and their absolute and relative standard uncertainties; U_X, U_Y, $U_{\delta x}$, $U_{\delta y}$ – covariance matrixes; $\boldsymbol{F}(x)$, u_F, $u_{\delta F}$, U_F, $U_{\delta F}$ – function of processing signals $\boldsymbol{x}$ to $\boldsymbol{y}$, their absolute and relative uncertainties and its covariance matrixes, $\mathbf{E}$ – processing unit of $\boldsymbol{y}$ to obtain vectors of $\bar{\boldsymbol{Y}}$, $\boldsymbol{u_y}$, $\boldsymbol{u_{\delta y}}$ and covariance matrixes $\boldsymbol{U_Y}$, $\boldsymbol{U_{\delta Y}}$.

The method of estimation uncertainties in indirect multivariable measurements are given in GUM Supplement 2 [1]. A collection of n individual quantities X_i of the input vector measurand X are measured directly, and from their signals x_i the output vector $\boldsymbol{Y}$ of m estimators $\overline{Y_i}$ (named observables [2–3]) and covariance matrixes U_X, U_Y of absolute standard uncertainties $u_x \equiv \sigma_x$, $u_y \equiv \sigma_y$ are calculated. The vector $\boldsymbol{Y}$ depends on whether the functional $\boldsymbol{F}(X)$ is linear ($m \le n$) or nonlinear ($m \le$ sum of linear and nonlinear equations). The relation between output and input covariance matrixes is

$$\boldsymbol{U_Y} = \boldsymbol{S}\,\boldsymbol{U_X}\boldsymbol{S}^{\mathrm{T}} \tag{2}$$

where:

$$\boldsymbol{U_X} = \begin{bmatrix} \sigma_{x1}^2 & \cdots & \rho_{x1n}\sigma_{x1}\sigma_{xn} \\ \cdots & \cdots & \cdots \\ \rho_{xn1}\sigma_{xn}\sigma_{x1} & \cdots & \sigma_{xn}^2 \end{bmatrix},$$

$$\boldsymbol{S} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix},$$

$$\boldsymbol{U_Y} = \begin{bmatrix} \sigma_{y1}^2 & \cdots & \rho_{y1m}\sigma_{y1}\sigma_{ym} \\ \cdots & \cdots & \cdots \\ \rho_{y1m}\sigma_{ym}\sigma_{y1} & \cdots & \sigma_{ym}^2 \end{bmatrix} \quad \text{(2a, b, c)}$$

Covariance matrixes $\boldsymbol{U_X}$, $\boldsymbol{U_Y}$ are symmetric, i.e. correlation coefficients $\rho_{xij} = \rho_{xji}$, and $\rho_{yij} = \rho_{yji}$.

$\boldsymbol{S}$ is the sensitivity matrix obtained after linearization of function $\boldsymbol{F}$ for small changes of $\boldsymbol{X}$ elements.

All, or some results of components of the measurand $\boldsymbol{Y}$ can be used further separately or jointly. In the latter case it is necessary to take also in considerations the correlation coefficients between the uncertainties of its components y_i, which are in non-diagonal elements of the covariance matrix $\boldsymbol{U_Y}$.

For calculations made *of line*, i.e. after collection all data, the uncertainties of function $\boldsymbol{F}$ can be obtained negligible. Recommendation for estimation uncertainties of this case is in the GUM Supplement 2. The realization of indirect measurements of m – components of the multivariate measurand $\boldsymbol{Y}$ can be made also *on line* by automatic instrumental measurement systems. In this case the uncertainties of $\boldsymbol{F}$ should be also considered.

The metrological description of the multivariable instrumental measurement system needs the accuracy for the whole values of measurement ranges of input and output signals. The accuracy of each range is up to now described by the maximal value (worth case) of limited absolute error $\left|\Delta_{y_i}\right|_{max}$.The absolute error of any output signal y_i of $\boldsymbol{Y}$ may be presented in the similar two component form as for digital voltmeter, i.e. $\Delta_{y_i}=\Delta_{y_{i0}} + \Delta_{(y_i - y_{i0})}$ and the limited absolute error is

$$|\Delta_{y_i}| \leq |\Delta_{y_{i0}}| + y_i |\varepsilon_{s_i}(y_i)| \quad \text{for } i = 1, \dots m \qquad (3)$$

where: $\Delta_{y_{i0}}$ - absolute error of initial value y_{i0} of the range, $\varepsilon_{s_i} \equiv \Delta(y_i - y_{i0})/y_i$ - relative error of the difference $(y_i - y_{i0})$ of output signal or reading.

If $|\Delta_{y_{i0}}| \ll |\Delta_{(y_i - y_{i0})}|$, then the relative limited error (worth case) of the component y_i is

$$|\Delta_{y_i}/y_i|_{max} \cong |\varepsilon_{s_i}|_{max} \qquad (3a)$$

The probability of existence the maximal limited error in each range is very low. Then the randomized description as in GUM [1] by uncertainties type B can be more valuable. It may be made in the similar two component form as for limited errors in the equation (3). Then it should contain the expanded absolute uncertainty U_0 of the initial value y_{i0} of each range and expanded relative uncertainty U_r of its increase $(y_i - y_{i0})$ for all values of the range. Both these uncertainties should be given for defined P probability of the confidence level, e.g. for P=0.95 is $U_0 \approx 0.95u$ and $U_r \approx 0.95u_r$ (u, u_r - standard uncertainties marked as in GUM). In the most cases these components of uncertainty are non-correlated, and very often relative uncertainty U_r is constant for the whole range or its function or maximal its value can be used. Moreover, the type B standard uncertainty is significantly smaller then maximal limited error because it is estimated as the square root of possible values of components, and not as their sum. The standard uncertainty of the single output value y_i is

$$u_{y_i} = \sqrt{u_{y_{i0}}^2 + (y_i - y_{i0})^2 u_{r\,y_i}^2} \qquad (4)$$

If $u_{y_{i0}}^2 \ll (y_i - y_{i0})^2 u_{r\,y_i}^2$, then the accuracy of y_i is described only by the single value of relative uncertainty $u_{y_i} \cong u_{ry_i} \equiv \delta_{y_i}$, unchanged in almost whole measuring range, or as $\delta_{y_i} \leq u_{ry_i max}$.

Up to date there are no internationally accepted regulations how to describe statistically by the uncertainties the accuracy of different kind of instrumental systems for indirect multivariable measurements. For the multiplicative type of measurement equations we found that is possible to use given below the new vector formula (5) between covariance matrixes $\boldsymbol{U}_{\delta X}$ and $\boldsymbol{U}_{\delta Y}$ of relative standard uncertainties $u_{rxi} \equiv \delta_{xi}$, $u_{ryi} \equiv \delta_{yi}$. Their correlation coefficients are the same as in (2).

$$\boldsymbol{U}_{\delta Y} = \boldsymbol{S}_\delta \boldsymbol{U}_{\delta X} \boldsymbol{S}_\delta^T \qquad (5)$$

where:

$$\boldsymbol{U}_{\delta X} = \begin{bmatrix} \delta_{x1}^2 & \dots & \rho_{x1n}\delta_{x1}\delta_{xn} \\ \dots & \dots & \dots \\ \rho_{xn1}\delta_{xn}\delta_{x1} & \dots & \delta_{xn}^2 \end{bmatrix},$$

$$\boldsymbol{S}_\delta = \begin{bmatrix} \frac{x_1}{y_1}\frac{\partial y_1}{\partial x_1} & \dots & \frac{x_n}{y_1}\frac{\partial y_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{x_1}{y_m}\frac{\partial y_m}{\partial x_1} & \dots & \frac{x_n}{y_m}\frac{\partial y_m}{\partial x_n} \end{bmatrix},$$

$$\boldsymbol{U}_{\delta Y} = \begin{bmatrix} \delta_{y1}^2 & \dots & \rho_{y1m}\delta_{y1}\delta_{ym} \\ \dots & \dots & \dots \\ \rho_{y1m}\delta_{ym}\delta_{y1} & \dots & \sigma_{ym}^2 \end{bmatrix} \qquad (5a\text{-}c)$$

All that should be clearer on analysis of few examples of indirect multivariate measurements with the multiplicative and additive type of functional $\boldsymbol{F}$ given below, i.e. indirect measurements of three arm resistances of the Wheatstone bridge and measurement of star circuit internal resistances from its terminals. The description of the uncertainty of active power measurements will be also discussed. Some general conclusions are given in the end.

1. Case of multiplicative measurement equations

The unknown values R_2, R_3, R_4 of three resistors can be determined without use the high accuracy digital ohmmeter. Two cases of indirect measurements are possible. The first, when these resistors and the regulated multi-decade resistor R_1 are connected as the Wheatstone bridge circuit but three times in three different orders in its loop, i.e. R_2, R_3, R_4 (Fig 2a), or R_2, R_4, R_3 and R_3, R_2, R_4. Three bridge' balances U_{CD} = 0 give three settings R_{x1}, R_{x2}, R_{x3} of R_1. The settings R_{x2}, R_{x3} can be obtained also without disconnection this bridge, by unconventional supply by current sources $\boldsymbol{J_1}$=$\boldsymbol{J_3}$ connected parallelly to opposite arms 1, 3 and balancing outputs AB or DC on diagonals (Fig 2b).
In both circuits the same settings R_{x1}, R_{x2}, R_{x3} on multi decade resistor R_1satisfied circuit balances, i.e.:

$$R_{x1} = R_2\frac{R_4}{R_3}, R_{x2} = R_2\frac{R_3}{R_4}, R_{x3} = R_3\frac{R_4}{R_2} \qquad (6a, b, c)$$

From above relations the unknown resistances as elements of the output vector $\boldsymbol{Y}$ can be calculated

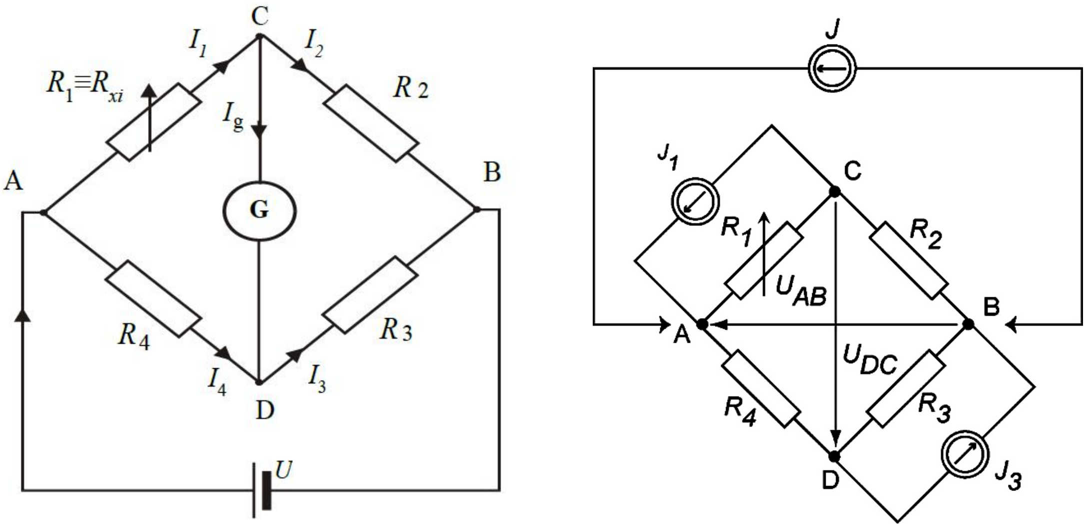
$$R_2 = \sqrt{R_{x1}\,R_{x2}}, \qquad R_3 = \sqrt{R_{x2}\,R_{x3}},$$

$$R_4 = \sqrt{R_{x1}\,R_{x3}} \qquad (7a, b, c)$$

As solutions (26) are of the multiplicative type the equation (3) for direct calculating the relative uncertainties can be used. The measurement sensitivity function $\boldsymbol{S}_\delta$ for relative uncertainties is

$$\boldsymbol{S}_\delta = \begin{bmatrix} \frac{R_{x1}}{R_2}\frac{\partial R_2}{\partial R_{x1}} & \frac{R_{x2}}{R_2}\frac{\partial R_2}{\partial R_{x2}} & \frac{R_{x3}}{R_2}\frac{\partial R_2}{\partial R_{x3}} \\ \frac{R_{x1}}{R_3}\frac{\partial R_3}{\partial R_{x1}} & \frac{R_{x2}}{R_3}\frac{\partial R_3}{\partial R_{x2}} & \frac{R_{x3}}{R_3}\frac{\partial R_3}{\partial R_{x3}} \\ \frac{R_{x1}}{R_4}\frac{\partial R_4}{\partial R_{x1}} & \frac{R_{x2}}{R_4}\frac{\partial R_4}{\partial R_{x2}} & \frac{R_{x3}}{R_4}\frac{\partial R_4}{\partial R_{x3}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad (8)$$



a *b*

Fig. 2. The structures of DC bridges for measurement three resistances R2, R3, R4: a) the first of arm resistances of three variants of the balanced Wheatstone bridge; b) two different the bridge loop circuit supplies: classic one as in a) from single source J (or U) and in balance UDC = 0 is Rx1R3 = R2R4; unconventional double current supply J1=J3 in parallel to opposite arms 1 and 3 (or J3 = 0 and J1 switched between these arms) and then eq. of outputs DC and AB balances are used for measurement, i.e. if UAB =0, R1R4 =R2R3 or if UD C = 0 R1R2 = R3R4

For estimation uncertainty of measured resistances, we assumed firstly that input variables R_{x1}, R_{x2}, R_{x3} are not correlated. and using formula (3) relative uncertainties, i.e. $\boldsymbol{U}_{\delta y} = \boldsymbol{S}_{\delta}\, \boldsymbol{U}_{\delta x}\, \boldsymbol{S}_{\delta}^{\mathrm{T}}$ is:

$$u_{\delta Y} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta^2_{R_{x1}} & 0 & 0 \\ 0 & \delta^2_{R_{x1}} & 0 \\ 0 & 0 & \delta^2_{R_{x1}} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \frac{1}{4}\begin{bmatrix} \delta^2_{R_{x1}}+\delta^2_{R_{x2}} & \delta^2_{R_{x2}} & \delta^2_{R_{x1}} \\ \delta^2_{R_{x2}} & \delta^2_{R_{x2}}+\delta^2_{R_{x3}} & \delta^2_{R_{x3}} \\ \delta^2_{R_{x1}} & \delta^2_{R_{x3}} & \delta^2_{R_{x1}}+\delta^2_{R_{x3}} \end{bmatrix} \qquad (9)$$

So, the standard relative uncertainties of output quantities are defined**:**

$$\delta_{R_2} = \frac{1}{2}\sqrt{\delta^2_{R_{x1}}+\delta^2_{R_{x2}}}, \quad \delta_{R_3} = \frac{1}{2}\sqrt{\delta^2_{R_{x2}}+\delta^2_{R_{x3}}},$$

$$\delta_{R_3} = \frac{1}{2}\sqrt{\delta^2_{R_{x1}}+\delta^2_{R_{x3}}} \qquad (10a,b,c)$$

and correlations coefficients:

$$\rho_{R_2R_3} = \frac{\frac{1}{4}\delta^2_{R_{x2}}}{\frac{1}{2}\sqrt{\delta^2_{R_{x1}}+\delta^2_{R_{x2}}}\,\frac{1}{2}\sqrt{\delta^2_{R_{x2}}+\delta^2_{R_{x3}}}} =$$

$$= \frac{1}{\sqrt{1+\left(\frac{\delta_{R_{x1}}}{\delta_{R_{x2}}}\right)^2}\sqrt{1+\left(\frac{\delta_{R_{x3}}}{\delta_{R_{x2}}}\right)^2}} > 0$$

$$\rho_{R_2R_4} = \frac{\frac{1}{4}\delta^2_{R_{x1}}}{\frac{1}{2}\sqrt{\delta^2_{R_{x1}}+\delta^2_{R_{x2}}}\,\frac{1}{2}\sqrt{\delta^2_{R_{x1}}+\delta^2_{R_{x3}}}} =$$

$$= \frac{1}{\sqrt{1+\left(\frac{\delta_{R_{x2}}}{\delta_{R_{x1}}}\right)^2}\sqrt{1+\left(\frac{\delta_{R_{x3}}}{\delta_{R_{x1}}}\right)^2}} > 0 \qquad (11a, b, c)$$

$$\rho_{R_3R_4} = \frac{\frac{1}{4}\delta^2_{R_{x3}}}{\frac{1}{2}\sqrt{\delta^2_{R_{x2}}+\delta^2_{R_{x3}}}\,\frac{1}{2}\sqrt{\delta^2_{R_{x1}}+\delta^2_{R_{x3}}}} =$$

$$= \frac{1}{\sqrt{1+\left(\frac{\delta_{R_{x2}}}{\delta_{R_{x3}}}\right)^2}\sqrt{1+\left(\frac{\delta_{R_{x1}}}{\delta_{R_{x3}}}\right)^2}} > 0$$

All above correlations coefficients are positive.

If $\delta_{R_{x1}} = \delta_{R_{x2}} = \delta_{R_{x3}} = \delta$,

$$\delta_{R_2} = \delta_{R_3} = \delta_{R_4} = \frac{\sqrt{2}}{2}\delta \,;\; \rho_{R_2R_3} = \rho_{R_2R_4} = \rho_{R_3R_4} = \frac{1}{2},$$

then the coverage region is ellipsoid of parameter $w = 1 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} > 0$

In this case the ellipsoidal coverage region for relative uncertainties of $\boldsymbol{Y}$ with probability 0.95 determines the ellipsoid with half axis a =2,8δ, b =1,4δ, c =1,4δ. In formulas it is used that the coverage factor/extension/coefficient for 95% of coverage region in 3D (three-dimensional) Gauss distribution is equal k_p=2,8. This ellipsoid is contiguous in six points to the cube with edges d=2·2,8$\frac{\sqrt{2}}{2}\delta = 3{,}96\,\delta$. The relations between capacity of ellipsoid and cube is 4π abc/ (3d^3) =37 %.

2.1. Example of the additive type of multivariate measurement equations

In many practical situations the star circuit of resistances connection is applied and there is no possibility of disconnection them from the common point 0 and even this point is not available, or star structure is the equivalent circuit only. So, three values of star resistances must be determined indirectly from measurements of three input resistances between terminals A, B, C (Fig 3). If changes of star resistances must be remote monitored from a distance, then the special measurement circuit **E** is used for these indirect measurements. Let us as the first step assume that the values of resistances of star are determined precisely without any disturbances and modifications by A/D converters and arithmetical modules located in **E.**

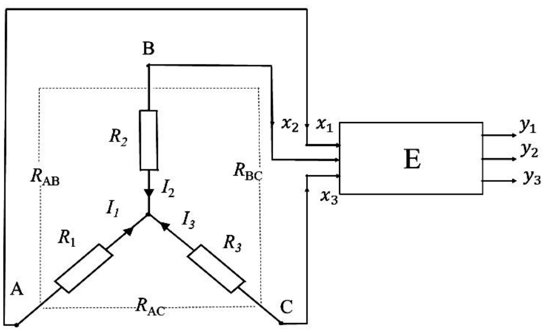


Fig. 3. The diagram of the star circuits with module of performance measurements

The main measurement equations are:

$$R_{AB} = R_1 + R_2, \qquad R_{BC} = R_2 + R_3, \qquad R_{AC} = R_1 + R_3 \tag{12}$$

or in the matrix form:

$$\begin{bmatrix} R_{AB} \\ R_{BC} \\ R_{AC} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \tag{12a}$$

To obtain solutions, the both sides of eq.(6a) are multiplicated by the inverse to above matrix and the main formula (1) has here the matrix form

$$\boldsymbol{Y} = \boldsymbol{F} \cdot \boldsymbol{X} \tag{13}$$

Where: $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} R_{AB} \\ R_{BC} \\ R_{AC} \end{bmatrix}$, $\boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$,

$$\boldsymbol{F} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \tag{13a, b, c}$$

The star circuit resistances are

$$R_1 = \frac{R_{AB}}{2} - \frac{R_{BC}}{2} + \frac{R_{AC}}{2};$$

$$R_2 = \frac{R_{AB}}{2} + \frac{R_{BC}}{2} - \frac{R_{AC}}{2};$$

$$R_3 = -\frac{R_{AB}}{2} + \frac{R_{BC}}{2} + \frac{R_{AC}}{2} \tag{14a, b, c}$$

Then corrections are implemented for known systematic errors. Unknown systematic errors are randomized and estimated as components of the type B uncertainty u_B. Next the results of absolute standard uncertainties σ_{AB}, σ_{BC}, σ_{AC} are find as a square of quadratic values of uncertainties u_A and u_B (type A and B), and relative uncertainties δ_{AB}, δ_{BC}, δ_{AC} should be calculated.

To find the absolute uncertainties and correlation coefficients of star resistances as output quantities, the vector method given in Supplement 2 to GUM is used [1]. Covariance matrices are related by formula (2), i.e. $\boldsymbol{U_Y} = \boldsymbol{S} \cdot \boldsymbol{U_X} \cdot \boldsymbol{S}^T$. In which: $\boldsymbol{U_Y}$, $\boldsymbol{U_X}$ – covariance matrixes of output vector $\boldsymbol{Y}$ and input vector $\boldsymbol{X}$, $\boldsymbol{S}$ - the Jacobian matrix sensitivity coefficients of absolute uncertainties.

For the resistances of stair circuit

$$\boldsymbol{S} = \boldsymbol{F} = \begin{bmatrix} \partial R_1/\partial R_{AB} & \partial R_1/\partial R_{BC} & \partial R_1/\partial R_{AC} \\ \partial R_2/\partial R_{AB} & \partial R_2/\partial R_{BC} & \partial R_2/\partial R_{AC} \\ \partial R_3/\partial R_{AB} & \partial R_3/\partial R_{BC} & \partial R_3/\partial R_{AC} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \tag{15}$$

Ø correlated variables in the input

Let us consider the general case when absolute uncertainties of input quantities σ_{AB}, σ_{BC}, σ_{AC} are correlated. Then in the covariance matrix $\boldsymbol{U_X}$ of input quantities in such case the non-zero elements in non-diagonal positions are appearing. They are defined with correlation coefficients ρ_{AB}, ρ_{BC}, ρ_{AC}

$$\boldsymbol{U_X} = \begin{bmatrix} \sigma_{AB}^2 & \rho_{AB}\sigma_{AB}\sigma_{BC} & \rho_{BC}\sigma_{AB}\sigma_{AC} \\ \rho_{AB}\sigma_{AB}\sigma_{BC} & \sigma_{BC}^2 & \rho_{AC}\sigma_{BC}\sigma_{AC} \\ \rho_{BC}\sigma_{AB}\sigma_{AC} & \rho_{AC}\sigma_{BC}\sigma_{AC} & \sigma_{AC}^2 \end{bmatrix} \tag{16}$$

If the relative uncertainties of measured resistances on stair circuit terminals are the same, i.e.:

$\delta_{AB} = \delta_{BC} = \delta_{AC} = \delta$, then absolute uncertainties of input quantities are: $\sigma_{AB} = \delta \cdot R_{AB}$, $\sigma_{AC} = \delta \cdot R_{AC}$ and $\sigma_{BC} = \delta \cdot R_{BC}$. Then output absolute uncertainties are:

$$\sigma_{y1} = \frac{\delta}{2}\sqrt{R_{AB}^2 + R_{BC}^2 + R_{AC}^2 + 2(\rho_{BC} R_{AB} R_{AC} - \rho_{AB} R_{AB} R_{BC} - \rho_{AC} R_{BC} R_{AC})}$$

$$\sigma_{y2} = \frac{\delta}{2}\sqrt{R_{AB}^2 + R_{BC}^2 + R_{AC}^2 + 2(\rho_{AB} R_{AB} R_{BC} - \rho_{BC} R_{AB} R_{AC} - \rho_{AC} R_{BC} R_{AC})} \tag{17a, b, c}$$

$$\sigma_{y3} = \frac{\delta}{2}\sqrt{R_{AB}^2 + R_{BC}^2 + R_{AC}^2 + 2(\rho_{AC} R_{BC} R_{AC} - \rho_{BC} R_{AB} R_{AC} - \rho_{AB} R_{AB} R_{BC})}$$

The output relative uncertainties:

$$\delta_{y1} = \frac{\delta\sqrt{1 + \beta^2 + \gamma^2 + 2(\rho_{BC}\,\gamma - \rho_{AB}\,\beta - \rho_{AC}\,\beta\gamma)}}{1 - \beta + \gamma}$$

$$\delta_{y2} = \frac{\delta\sqrt{1 + \beta^2 + \gamma^2 + 2(\rho_{AB}\,\beta - \rho_{BC}\,\gamma - \rho_{AC}\,\beta\gamma)}}{1 + \beta - \gamma} \tag{18a, b, c}$$

$$\delta_{y3} = \frac{\delta\sqrt{1 + \beta^2 + \gamma^2 + 2(\rho_{AC}\,\beta\gamma - \rho_{BC}\,\gamma - \rho_{AB}\,\beta)}}{\beta + \gamma - 1}$$

where: $\beta = \frac{R_{BC}}{R_{AB}}$ and $\gamma = \frac{R_{AC}}{R_{AB}}$.

The correlations coefficients of output quantities are defined as follows

$$\rho_{y1y2} = \frac{\delta^2}{4}\frac{R_{AB}^2 - R_{BC}^2 - R_{AC}^2 + 2\rho_{AC} R_{BC} R_{AC}}{\sigma_1\sigma_2},$$

$$\rho_{y1y3} = \frac{\delta^2}{4}\frac{R_{AC}^2 - R_{BC}^2 - R_{AB}^2 + 2\rho_{AB} R_{BC} R_{AC}}{\sigma_1\sigma_3}$$

$$\rho_{y2y3} = \frac{\delta^2}{4}\frac{R_{BC}^2 - R_{AC}^2 - R_{AB}^2 + 2\rho_{BC} R_{AB} R_{AC}}{\sigma_2\sigma_3} \quad (19a, b, c)$$

If $R_{AB} = R_{BC} = R_{AC} = R$, the uncertainties are

$$\sigma_{y1} = \frac{\delta R}{2}\sqrt{3 + 2(\rho_{BC} - \rho_{AB} - \rho_{AC})},$$

$$\sigma_{y2} = \frac{\delta R}{2}\sqrt{3 + 2(\rho_{AB} - \rho_{BC} - \rho_{AC})}$$

$$\sigma_{y3} = \frac{\delta R}{2}\sqrt{3 + 2(\rho_{AC} - \rho_{AB} - \rho_{BC})} \quad (20a, b, c)$$

- **non-correlated variables in the input**

For non-correlated variables $\rho_{AB} = \rho_{BC} = \rho_{AC} = 0$, and from (20) the absolute uncertainties are

$$\sigma_{y_1} = \sigma_{y_2} = \sigma_{y_3} = \frac{1}{2}\sqrt{\sigma_{R_{AB}}^2 + \sigma_{R_{AC}}^2 + \sigma_{R_{BC}}^2} \quad (21)$$

And from (13) correlation coefficients

$$\rho_{y1y2} = \frac{\sigma_{AB}^2 - \sigma_{BC}^2 - \sigma_{AC}^2}{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2},$$

$$\rho_{y1y3} = \frac{-\sigma_{AB}^2 - \sigma_{BC}^2 + \sigma_{AC}^2}{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2},$$

$$\rho_{y2y3} = \frac{-\sigma_{AB}^2 + \sigma_{BC}^2 - \sigma_{AC}^2}{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2} \quad (22a, b, c)$$

If $\sigma_{AB} = \sigma_{BC} = \sigma_{AC} = \sigma$ then

$$\sigma_{y1} = \sigma_{y2} = \sigma_{y3} = \frac{\sqrt{3}}{2}\sigma$$

$$, \rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = -\frac{1}{3}.$$

It can be show that determinant of matrix $\boldsymbol{U}_y$ by the sign of parameter w

$$w = 1 - \rho_{y1y2}^2 - \rho_{y1y3}^2 - \rho_{y2y3}^2 +$$

$$+2\cdot \rho_{y1y2}\,\rho_{y1y3}\,\rho_{y2y3} > 0 \quad (23)$$

is always positive.

Defining $\varkappa = \frac{\sigma_{BC}^2 + \sigma_{AC}^2}{\sigma_{AB}^2} > 0$, $v = \frac{\sigma_{BC}^2 - \sigma_{AC}^2}{\sigma_{AB}^2}$ we express correlation coefficients by

$$\rho_{y1y2} = \frac{1-\varkappa}{1+\varkappa};\quad \rho_{y1y3} = -\frac{1+v}{1+\varkappa};\quad \rho_{y2y3} = \frac{v-1}{1+\varkappa} \quad (24a, b, c)$$

so, the parameter $w = 4\frac{\varkappa^2 - v^2}{(1+\varkappa)^3}$ must be $\varkappa^2 > v^2$, what is always fulfilled.

That is why the characteristic equation of inverse matrix has three positive roots.

The border of cover region for values of results with given probability $P \le 0{,}95$ is ellipsoid, closed in solid cubic, and contiguous in six points the wall of cubic with edge distance

$$k_p\sqrt{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2} \quad (25)$$

k_p=2,8 – cover factor/ extension coefficients.

Summary of solutions of some cases / $P \le 0{,}95$

- if $\sigma_{AB} = \sigma_{BC} = \sigma_{AC} = \sigma_{in}$, $\rho_{AB} = \rho_{BC} = \rho_{AC} = \rho_{in}$;
- $\sigma_{out} = \sigma_{y1} = \sigma_{y2} = \sigma_{y3} = \frac{1}{2}\sigma_{in}\sqrt{3 - 2\rho_{in}}$;
- $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = \frac{2\rho_{in} - 1}{3 - 2\rho_{in}}$; $\rho_{y1y2}, \rho_{y2y3}, \rho_{y1y3} \le 0$ for $\rho_{in} \le$ ½.
- if $\rho_{in} = 0$: $\sigma_{out} = \frac{\sqrt{3}}{2}\sigma_{in}$, $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = -\frac{1}{3}$; half axes: $1{,}4\sigma_{in}$, $2{,}8\,\sigma_{in}$, $2{,}8\,\sigma_{in}$
- if $\rho_{in} = \frac{1}{2}$: min $\sigma_{out} = \frac{1}{2}\sigma_{in}$, $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = 0$; radius $1{,}4\,\sigma_{in}$
- if $\rho_{in} = -1$: max $\sigma_{out} = \frac{\sqrt{5}}{2}\sigma_{in}$, $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = -\frac{3}{5}$; $w < 0$.

2.2. Influence of uncertainties u_F of matrix *F* in stair circuit measurements

In the instrumental system for measurements the stair circuit resistances processing of output values and their uncertainties is made in digital unit **E**. The main matrix equation ***Y*** =***F*** ·***X*** was given in (13) and (13a, b, c). Solution of vector ***Y*** elements is in (14). Let us now consider uncertainties of amplification/attenuation of signals in measurement channels. The realization of signals processing has linear disturbances in channels changing levels of signals e.g.:

$$X_S = \begin{bmatrix} k_1 x_1 \\ k_2 x_2 \\ k_3 x_3 \end{bmatrix} \quad (26)$$

where k_1, k_2, k_3 is amplifying coefficients.

The analog/digital processing input signals has uncertainty $\boldsymbol{u_F}$. Therefore, the functional matrix ***F*** is must be modified, and a new matrix is defined as follows:

$$F_S = \frac{1}{2}\begin{bmatrix} k_1(1+\delta_1) & -k_2(1+\delta_1) & k_3(1+\delta_1) \\ k_1(1+\delta_2) & k_2(1+\delta_2) & k_3(1+\delta_2) \\ -k_1(1+\delta_3) & k_2(1+\delta_3) & k_3(1+\delta_3) \end{bmatrix} \quad (27)$$

where: $\delta_1, \delta_2, \delta_3$ - coefficients dedicated to the components of output quantities.

The vector function of output quantities is additionally perturbed by uncertainties associated with zero set errors ($\frac{\Delta_{10}}{1+\delta_1}$; $\frac{\Delta_{20}}{1+\delta_2}$; $\frac{\Delta_{30}}{1+\delta_3}$) and results the output components now are:

$$y_1 = (1+\delta_1)\left(\frac{k_1x_1 - k_2x_2 + k_3x_3}{2} + \frac{\Delta_{10}}{1+\delta_1}\right)$$

$$y_2 = (1+\delta_2)\left(\frac{k_1x_1 + k_2x_2 - k_3x_3}{2} + \frac{\Delta_{20}}{1+\delta_2}\right)$$

$$y_3 = (1+\delta_3)\left(\frac{-k_1x_1 + k_2x_2 + k_3x_3}{2} + \frac{\Delta_{30}}{1+\delta_3}\right) \quad \text{(28a, b, c)}$$

Using formulas which are derived for absolute uncertainties of star circuit and modifying it, the absolute uncertainties after modification with zero set errors are more complicated as:

$$\sigma_{y_1} = (1+\delta_1)\sqrt{k_1^2\sigma_{x_1}^2 + k_2^2\sigma_{x_2}^2 + k_3^2\sigma_{x_3}^2 + 2\left(k_1k_3\,\rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3} - k_1k_2\,\rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} - k_2k_3\,\rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3}\right) + \sigma^2\left(\frac{\Delta_{10}}{1+\delta_1}\right)}$$

$$\sigma_{y_2} = (1+\delta_2)\sqrt{k_1^2\sigma_{x_1}^2 + k_2^2\sigma_{x_2}^2 + k_3^2\sigma_{x_3}^2 + 2\left(k_1k_2\,\rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} - k_1k_3\,\rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3} - k_2k_3\,\rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3}\right) + \sigma^2\left(\frac{\Delta_{20}}{1+\delta_2}\right)}$$

$$\sigma_{y_3} = (1+\delta_3)\sqrt{k_1^2\sigma_{x_1}^2 + k_2^2\sigma_{x_2}^2 + k_3^2\sigma_{x_3}^2 + 2\left(k_2k_3\,\rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3} - k_1k_2\,\rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} - k_1k_3\,\rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3}\right) + \sigma^2\left(\frac{\Delta_{30}}{1+\delta_3}\right)} \quad \text{(29a, b, c)}$$

2.3. Power indirect measurements

Now the considerations of the case of directly measurements, where two of values of currents $I_1$and I_2 $(I_2 > I_1)$and non-correlated values of its uncertainties σ_{I1}, σ_{I2} conducted from measurement are done. Assuming the linear dependence of estimated current $I(k)$ as a function of both measured currents in the form:

$$I(k) = kI_1 + (1-k)\,I_2 \quad (30)$$

where: $0 \le k \le 1$.

The outside range of above defined values of *k* leads on the linear dependence too, however the limits of the interval of *k* for extrapolation of values of current in the range $I_L \le I(k) \le I_H$ must be explicit evaluated e.g. $\frac{I_H - I_2}{I_1 - I_2} \le k \le \frac{I_L - I_2}{I_1 - I_2}$.

Active power $P(k)$ in the case of flow of current $I(k)$ through resistance *R* is as follow as:

$$P(k) = (kI_1 + (1-k)I_2)^2R =$$
$$= [k^2{I_1}^2 + (1-k)^2{I_2}^2 + 2k(1-k)I_2I_1]R \quad (31)$$

The goal of this part is estimation of power uncertainties of σ_{P1}, σ_{P2} both of current $I(k_1)$, $I(k_2)$ and its correlation coefficient.

The covariance matrix of non-correlated input quantities e.g. measured currents $I_1$and I_2 has following form:

$$U_X = \begin{bmatrix} \sigma_{I1}^2 & 0 \\ 0 & \sigma_{I2}^2 \end{bmatrix} \quad (32)$$

The matrix of linearized functional of both active powers $P(k_1)$ i $P(k_2)$ emitted on the resistor *R* as the output quantities is the following:

$$S = \begin{bmatrix} 2k_1[I_1k_1 + (1-k_1)I_2] & 2(1-k_1)[I_2(1-k_1) + k_1I_1] \\ 2k_2[I_1k_2 + (1-k_2)I_2] & 2(1-k_2)[I_2(1-k_2) + k_2I_1] \end{bmatrix}$$

$$S^T = \begin{bmatrix} 2k_1[I_1k_1 + (1-k_1)I_2] & 2k_2[I_1k_2 + (1-k_2)I_2] \\ 2(1-k_1)[I_2(1-k_1) + k_1I_1] & 2(1-k_2)[I_2(1-k_2) + k_2I_1] \end{bmatrix} \quad \text{(33a,b)}$$

The covariance matrix of output quantities is described by formulae:

$$U_Y \equiv \begin{bmatrix} \sigma_{P1}^2 & \rho_P\sigma_{I1}\sigma_{I2} \\ \rho_P\sigma_{I1}\sigma_{I2} & \sigma_{P2}^2 \end{bmatrix} = S\,U_X\,S^T \quad (34)$$

After simply transformations the variance of both uncertainties of active powers is given:

$$\sigma_{P1}^2 = 4[I_1k_1 + (1-k_1)I_2]^2({k_1}^2\sigma_{I1}^2 + (1-k_1)^2\sigma_{I2}^2)$$
$$\sigma_{P2}^2 = 4[I_1k_2 + (1-k_2)I_2]^2({k_2}^2\sigma_{I1}^2 + (1-k_2)^2\sigma_{I2}^2) \quad (35)$$

and its correlation coefficient:

$$\rho_P = \frac{k_1k_2\sigma_{I1}^2 + (1-k_1)(1-k_2)\sigma_{I2}^2}{\sqrt{{k_1}^2\sigma_{I1}^2 + (1-k_1)^2\sigma_{I2}^2}\,\sqrt{{k_2}^2\sigma_{I1}^2 + (1-k_2)^2\sigma_{I2}^2}} \quad (36)$$

Above formulae is identical as correlation coefficient for intensity currents of $I(k_1)$ and $I(k_2)$.

Conclusion

Few examples of determining the uncertainties in case of multi-parameter linear and nonlinear formulas have been presented, for example indirect measurements of resistance by Wheatstone bridge, a star circuit and module of difference of magnetic field induction vector. Covariance matrix of relative uncertainties is also applied. We proposed of using the covariance matrix for relative uncertainties and corresponding measured function dedicated for relative uncertainties as well as relative errors were applied in the classical approach.

It is shown that in the case when two or more parameters /for example element of electronic circuits/ are measured together, the uncertainties of above parameters are correlated. So, if the above correlated elements will be used without disconnection in the next circuits of device then in the determination of uncertainties of such new device we should consider the corresponding correlations coefficients obtained from first measurements.

Information about uncertainty calculations of multivariable AC measurements is in [3]. Supplement 2 of GUM [1] does not cover situations existing in instrumental systems, when realization of functional $\boldsymbol{F}(\boldsymbol{x})$ is not accurate. Such inaccuracy can be due to approximation of transfer functions and limited their frequency ranges, using in signal processing an A/D converters, analogue multipliers, and other functional elements, necessary in indirect measurements. Therefore $\boldsymbol{F}(\boldsymbol{x})$ is also saddled with own uncertainties $\boldsymbol{u}_{\boldsymbol{F}}$.[1] Even in the most precise measurements the rounding of results also becomes essential, including one resulting from the precision of digital circuits [2-3].

In the last days authors have developed the vector method for the description of the accuracy of multivariate measurements systems with considering uncertainties $\boldsymbol{u}_{\boldsymbol{F}}$, $\boldsymbol{u}_{\boldsymbol{\delta F}}$ of the functional $\boldsymbol{F}$ parameters. This method is wider then recommendations given in GUM Supplement 2 [1], which do not consider inaccuracy of $\boldsymbol{F}$. Details and new formulas will be provided in the next authors' work.

References

[1] JCGM 102:2011, Evaluation of measurement data – Supplement 2 to the Guide to the expression of uncertainty in measurement"– Extension to any number of output quantities.

[2] M. Dorozhovets, Processing the measurement results. Lviv, Ukraine: Publ. House of Lviv Pol. Nat. Univ., 2007.

[3] Warsza, Z. L.; Puchalski, J.: Estimation of vector uncertainties of multivariable indirect instrumental measurement systems on the star circuit example. Congress IMEKO 2018 CD Proceedings PO-062

[4] L. Finkelstein, "Fundamental concepts of measurement", ACTA IMEKO, vol. 3, no. 1, p. 10–15 May 2014.

[1]Such problems of measurement technology are included in the ***measurement science,*** discipline wider then metrology. The concept and term of this discipline were proposed by prof. L. Finkelstein from the City University of London in 1970-s years during his IMEKO activity [4].