COMPUTER DESIGN SYSTEMS. THEORY AND PRACTICE

Vol. 1, No. 1, 2019

https://doi.org/10.23939/cds2019.01.050

**P. Kosobutskyy, R. Ferens**
Lviv Polytechnic National University

# SOME FEATURES OF THE DIRECT AND INVERSE TRANSFORMATION OF RANDOM VARIABLES



**Always actual tasks of obtaining and processing experimental results in complex systems. Random obstacles (errors), measurement errors, imperfections and limitations of mathematical models and data processing algorithms can change the appearance of the distribution and lead to incorrect use of algorithms, for example, as is the case with Kalman filtering in control systems. Complex methods for the identification of distribution laws require the study of quantum systems, natural phenomena, environmental, biological, etc. processes, which are characterized by the presence of singularities and multimodality of distributions. Therefore, it is often not recommended to apply separate distribution laws to simulate probabilistic experimental data distributions, but a generalized distribution as a single statistical system, which known distributions include as individual partial cases. Thus, the generalized gamma distribution includes Rayleigh, Maxwell, Weibull, Levy, Hi-Square distributions, which are widely used in applied problems associated with statistical methods of physical processes research, remote sensing, in the theory of reliability, for describing the dispersion composition of particles fragmentation and calculation of the efficiency of phase separation in gas-liquid streams.**

## Introduction

Always actual tasks of obtaining and processing experimental results in complex systems. Random obstacles (errors), measurement errors, imperfections and limitations of mathematical models and data processing algorithms can change the appearance of the distribution and lead to incorrect use of algorithms, for example, as is the case with Kalman filtering in control systems. Complex methods for the identification of distribution laws require the study of quantum systems, natural phenomena, environmental, biological, etc. processes, which are characterized by the presence of singularities and multimodality of distributions. Therefore, it is often not recommended to apply separate distribution laws to simulate probabilistic experimental data distributions, but a generalized distribution as a single statistical system, which known distributions include as individual partial cases. Thus, the generalized gamma distribution includes Rayleigh, Maxwell, Weibull, Levy, Hi-Square distributions, which are widely used in applied problems associated with statistical methods of physical processes research, remote sensing, in the theory of reliability, for describing the dispersion composition of particles fragmentation and calculation of the efficiency of phase separation in gas-liquid streams [1–3].

Using Erlang's distribution, imitation models of processes are created, the duration of which can be represented as the sum of elementary sequential components distributed in exponential law. By distributing the Relay, they simulate a change in the amplitude of the radio signals and estimate the random deviation from the point on the plane that is not co-ordinating with each other. The distribution of Rice is related to the statistics of radio frequency propagation in a multichannel conductor, which is used to process magnet resonance imaging data.

Widespread use has been made of generalized distributions. A generalized index distribution is a scalable mixture of normal distributions, and is therefore considered as the boundary for random amounts of statistics constructed from random volumes [4–5]. The generalization of the normal distribution in the form of a superposition of the normal distribution of Gauss and of the exponential distribution of Laplace allows us to map not only the diversity of statistical distributions, but also without checking the numerous hypotheses according to the criteria for approval to make the selection of the BB sample to a certain distribution law. The application of generalized divisions in aeronautics allows us to obtain a reliable estimate of errors in determining the location of a point on a plane and to make a probabilistic prediction of liquid events due to the presence of distribution tails.

In statistical simulation, they work either with random sample sizes (ABB) (abbreviation RV), or with experimental data, or with Monte Carlo simulation results. Therefore, for constructing models on the basis of the statistical analysis of the BB, the minimum of the basic provisions of the probability theory includes the estimation of the probability of the event, the function of distribution of density $f_X(x)$ probabilities, its parameters, and the connection between the BB. Function $f_X(x)$ – is integral and has a definition area $D(f)$ – the whole number axis, except perhaps a counted set (coutable set) of points where density $f_X(x)$ may not exist; change area $E(f)=[0;+\infty)$ $\Rightarrow$ $0 \le f_X(x) < +\infty$. In many practical problems, the distribution density itself is not of interest, but the integral of the product of this density on a certain function. These operations in the theory of probabilities have special notations and names and allow us to calculate the important characteristics of the BB as the position of the center of scattering as the mean value of the distribution (average, center of gravity) $E_X$ ( $m_X$ or average $\overline{X}$ ) and the parameter of the shape of the curve itself: $\sigma_X^2 = D_X$ ($\sigma_X$ >0).

If the probability theory uses the method of moments to estimate the distribution parameters, then for such distributions as normal, exponential, trapezoidal, etc., the notion of quantitative characteristics of the BB as the mean, variance, mean square deviation deviation (abbreviation SD) and others. But the method of moments is valid only for those distributions for which there are moments, that is, the corresponding integrals do not run out.

A more universal concept is the distribution center, which is defined as the center of gravity distribution or 50 % quantile. The center of gravity of the distribution of BB is a mechanical analogue of mathematical hope, assuming that the probabilities of values are the masses of points. In physics, based on the model of the center of gravity, it is substantiated that an arbitrary body in an uncertain state tries to take an equilibrium state. Similarly, an arbitrary BB, subject to a significant amount of measurements, goes to its equilibrium (in the sense of the mean) – the mathematical expectation. Such an approach only requires the existence of the zero order and the distribution width. For a symmetric distribution, as a standard (normal), the center of gravity coincides with the fashion. However, unlike fashion, the concept of the center of gravity distribution is legitimate for all distributions. So, for the Cauchy distribution there is no mathematical expectation, then the concept of the center of gravity of the distribution curve for it is lawful. There is no fashion for even distribution. Mathematical expectation $E_X$ (or $M[X]$) (mean) $m_X$ – This is a description of the situation, and the dispersion (variance of distribution) $D_X$ – is a characteristic of scattering*). An integral value of the square root $\sigma_X = +\sqrt{D_X}$ , describes (the standart deviation (abbreviation SD)).

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* In probability theory, the variance is a measure of scattering from the mean, whereas in mathematical statistics, it characterizes the degree of scattering of the quantitative values of the statistical sample relative to the average – the matrix of the expectation of the square of the deviation of the random variable from its mathematical expectation.

The dispersion characterizes scattering in relation to its mathematical expectation. It connects the first and second initial moments and characterizes the intensity of the fluctuations, so the case $D_X = 0$ has no physical content. If the second initial moment characterizes the distribution of BB in relation to the origin of the coordinates, then the variance is relative to the mean value. From the standpoint of physical interpretation, the variance for a deterministic quantity is absent, although in reality the second initial moment is not zero at the same time. Indeed, the deterministic value is located at a certain distance relative to the origin of the coordinate, which is not equal to zero, so the second starting point is different from zero.

The dispersion has the dimension of its square. This is not very convenient for simulating physical processes, therefore, another scattering parameter $\sigma_X$ is introduced. For example, in the electric circuits (circuits), the dispersion is preferably related to the average power emitted by the active electric resistance of the variable component of the applied electric voltage or electric current flowing through this resistance. Then the square root of the dispersion in this case will correspond to the displays of a voltmeter or an ammeter, if by the condenser, the component of the electrical signal has been eliminated.

Parameter $\sigma_X$ in science and technology is the unit of expected or measured scattering of explosives. Probability of scatter within the range $\pm\sigma_X$ :

$$P(-\sigma_X \le x \le +\sigma_X) = 0.63. \tag{1}$$

In military calculations, scattering x is often modeled in units $\Xi_X$ – this is such a value for which the probability of the value of BB $x$ from the middle $\Xi_X$ equals

$$P(-\Xi_X \le x \le +\Xi_X) = 0.5. \tag{2}$$

In this case, all possible range of scattering values is divided into two equal parts corresponding to the probabilities of "hit" and "failure". Values $\Xi_X$ and $\sigma_X$ are linked by a relationship: $\Xi_X = 0.675\,\sigma_X$. Accordingly, confidence intervals of deviations $x$, for which probability $P(x) \approx 0.99$, equals $3\sigma_X$, or $4\Xi_X$

$$P(-3\sigma_X \le x \le +3\sigma_X) = 0.997 \text{ or } P(-4\Xi_X \le x \le +4\Xi_X) = 0.993. \tag{3}$$

Nonlinear distributions are still accepted to characterize the coefficients of sloping (coefficient of skewness)

$$S_k = \frac{\text{3th moment about the mean}}{(\text{variance})^{3/2}} = \frac{\mu_3}{\mu_2^{3/2}} \tag{4}$$

and steepness (coefficient of kurtosis or coefficient of excess.):

$$E_k = \frac{\text{4th moment about the mean}}{(\text{variance})^{2}} = \frac{\mu_4}{\mu_2^{2}}. \tag{5}$$

Relation

$$V_X = \frac{\sigma_X}{m_X}, \tag{6}$$

characterizes the variation of the values of the BB, which is called the coefficient of variation and is considered only if $m_X > 0$. The greater its value, the greater the variance. The coefficient of variation depends on the "start time" and is convenient especially for the exponential distribution. For an exponential distribution, the coefficient (6) does not depend on its parameter and is always equal to one. The coefficient of VV variation is a relative measure of fluctuations, which characterizes the spread in relative units, while the mean square deviation is in absolute terms and is used to estimate the SKV of the mathematical expectation.

An important stage of statistical simulation is related to the direct and inverse transformation of random physical data, as a quadratic type. So, on the transformation of the type

$$g(X) = Y = \alpha X^2 . \tag{7}$$

The principle of the operation of optoelectronic transducers [11], speckle-interferometers [12], is based on the Monte Carlo method [13], is used in the theory of diffraction [14], and others. By physical nature, this type of signal converters are quadratic detectors, in which an output is obtained from the signal $aX^2$ at the input $X$ . The nonlinear transformation of a quadratic type plays an important role in quantum physics [24], where, due to the limitation of the technical capabilities of the experiment to the principle of uncertainty

$$\delta E = \frac{\hbar^2 k^2}{2m^*} = \left(\frac{\hbar^2}{2m^*}\right)\left(\frac{\pi}{a}\right)^2 , \tag{8}$$

random restrictions of the spatial motion of a quantum particle are accompanied by a fluctuation of the pulse by the amplitude $\frac{\hbar}{a}$.

Reversed transformation(7)

$$g^{-1}(Y) = Y = \beta \sqrt{X} , \tag{9}$$

allows you to find speed with the known kinetic effect – the presence of energy in the moving body $\vartheta = \sqrt{\frac{2}{m}}\sqrt{W}$ , the strength of the current in the manifestation of the thermal effect – the motion of the carriers in a medium with resistance $I = \frac{1}{\sqrt{R}}\sqrt{P}$, and others. It is possible to infer other types of direct and inverse transformations of random variables (BB), trigonometric, such as, for example, a functional connection between the parameters of the oscillatory oscillator. Depending on the setting of the task, they are direct $\sin X, \cos X$ and reversed $\arcsin X = \sin^{-1} X, \arccos X = \cos^{-1} X$ trigonometric transformations.

A direct square transform of type (7) was studied in detail see, for example, [10–17], but only in terms of constructing the probability density distribution function $f_Y(y)$, whereas the laws of the inverse to him were studied less intensively. The author [18] tried to solve such a problem, proposing for this a scheme of so-called transformation of indices in solutions of the dispersion equation. As shown in [19], the author's approach [18] turned out to be false. Therefore, for practical purposes, such studies are also relevant in the future. This work is devoted to the algorithm of the correct application of direct and inverse transformation of explosives in a probabilistic-statistical experiment.

## Algorithm for working out and discussion of results

First of all, let us draw attention to some of the problems that may be accompanied by statistical averaging of physical quantities. The function is valid, must satisfy all axioms of probability. Often, mathematical hope $E_X$ BB $x$ call it the mean value $\overline{X}$ and it is calculated as an integral:

$$\overline{X} = E_X = \int_0^{+\infty} x f_X(x) dx . \tag{10}$$

If the integral (10) is divergent, then there is no mathematical hope. In it, the boundaries of integration cover a semi-limited interval $(0;+\infty)$, since the physical values of the negative absolute values are not taken into account.

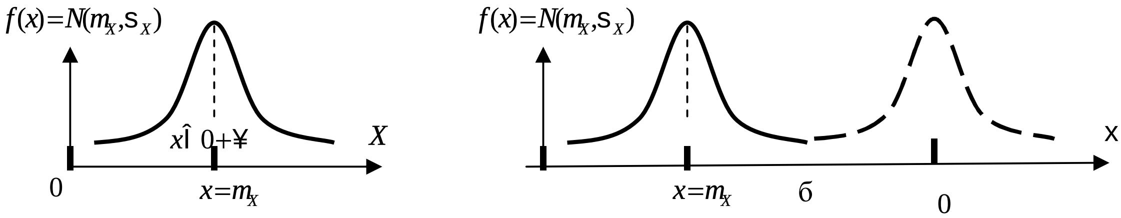


*Fig. 1*

Schematically, this is depicted in Fig. 1 (a) for the case BB $X$ with probability Gauss density $f_X(x)$.

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)=\frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-p(x-m_X)^2\right), \tag{11}$$

which is characterized by average $E_X=m_X$ and dispersion $D_X=\sigma_X^2$ ($N(m_X;\sigma_X^2)$ - Distribution with mean square deviation (MSD) $\sigma_X$).

Let`s apply to BB $X$ with distribution (11) linear transformation

$$g(Y)=Y=aX+b, \tag{12}$$

So we get BB $Y$ with probability density distribution function $f_Y(y)$.

We substantiate the analytic form of the function $f_Y(y)$.

By definition [11], the function $f_Y(y)$ The transformed BB is calculated by the formula

$$f_Y(y)=\left|\frac{d}{dy}g^{-1}(x)\right| f_X(g^{-1}(x)), \tag{13}$$

where

$$g^{-1}(x)=x=\frac{y-b}{a}\ , \tag{16}$$

the function reversed to (11) transformation, the module of the derivative from which is equal to:

$$\frac{d}{dy}g^{-1}(x)=\frac{1}{a}. \tag{15}$$

Substituting (14) into formula (10), we get that function $f_Y(y)$ looks like:

$$f_Y(y)=\frac{1}{\sqrt{2\pi\sigma_X^2}}\frac{1}{a}\exp\left(-p\left(\frac{y-b}{a}-m_X\right)^2\right)=\frac{1}{\sqrt{2\pi\sigma_X^2}}\frac{1}{a}\exp\left(-\frac{p}{a^2}\left(y-[b+am_X]\right)^2\right). \tag{16}$$

Consequently, the linear transformation (11) does not change the form of the distribution of the probability density $f_X(x)$ and distribution (16) BB $Y$ is subject to Gaussian, but with an altered coordinate of the extremum

$y=[b+am_X]$.

Having selected the displaced position of the extremum as the starting point in the centered system, the integral (10) will look like:

$$\overline{Y}=E_Y=\int_{-\infty}^{+\infty} y f_Y(y)dy \tag{17}$$

with an infinite two-sided interval of values of the integration variable $(-\infty;+\infty)$.

We draw attention to the possible problem of incorrect application of mathematical transformation in the statistical averaging of random variables. Yes, let the sample be obtained by transforming the explosives into a fractional law

$$h(x)=\sqrt{X}\ . \tag{18}$$

If you make a formal replacement $\left(\sqrt{x}\right)^2 = x$ ,so square average $\overline{\left(\sqrt{x}\right)^2}$ can be calculated as integral

$$\overline{\left(\sqrt{X}\right)^2} = \overline{x} = \int_{-\infty}^{+\infty} x f_X(x)dx \,. \tag{19}$$

Since the integral function is odd, and the limits of integration are symmetric, then the integral (15) is zero. The resulting result is incorrect, so this task will also be given attention.

Consider another example. Let the statistical analysis undergo fluctuations of the harmonic oscillator. If at the initial moment the oscillator was in a state of stable equilibrium and had a speed other than zero, then in the absence of disparative processes in the system, the amplitude of the deviation x is described by the function $x = x_0 \sin\varphi$ .

Let us assume that the fluctuations are amplitudes and the random variable X is subject to a uniform distribution with probability density

$$f_X(x) = \frac{1}{x_0} \,. \tag{20}$$

Calculating the dispersion of the phase

$$D_\Phi = \overline{\Phi^2} - \left(\overline{\Phi}\right)^2 . \tag{21}$$

To calculate statistical averages $\overline{\Phi}$ and square $\overline{\Phi^2}$

$$\overline{\Phi} = \int_{-\infty}^{\infty} \varphi\, f_\Phi(\varphi)d\varphi \,;\quad \overline{\Phi^2} = \int_{-\infty}^{\infty} \varphi^2 f_\Phi(\varphi)d\varphi \,, \tag{22}$$

it is necessary to justify the integration limits in (22) and the analytic form of the probability density distribution function $f_\Phi(\varphi)$ .

We perform transformation of BB $\Phi$

$$\Phi = g(X) = \arcsin\frac{X}{x_0} = \sin^{-1}\frac{X}{x_0} , \tag{23}$$

back to which looks

$$X = g^{-1}(X) = x_0 \sin\Phi \,, \tag{24}$$

The derivative function module $x = x_0\sin\varphi$ equals

$$\frac{d}{d\varphi}\left|\sin^{-1}\frac{x}{x_0}\right| = x_0\,|\cos\varphi| , \tag{25}$$

So the function $f_\Phi(\varphi)$ will looks like

$$f_\Phi(\varphi) = |\cos\varphi| . \tag{26}$$

Limited range of extreme values $-x_0 \le x \le x_0$ The amplitude of the deviation of the oscillator from the equilibrium position, according to (20), imposes a restriction on the phase change interval:

$$-\frac{\pi}{2} \le \varphi \le +\frac{\pi}{2} . \tag{27}$$

Therefore integrating parts (22)

$$\begin{gathered}\overline{\Phi} = 2\int_{-\pi/2}^{\pi/2}\varphi\,|\cos\varphi|\,d\varphi = 2\left|\begin{matrix} u=\varphi & d\upsilon = |\cos\varphi|\,d\varphi \\ du = d\varphi & \upsilon = |\sin\varphi| \end{matrix}\right| = 2\varphi\,|\sin\varphi|\,\Big|_{-\pi/2}^{\pi/2} - 2\int_{-\pi/2}^{\pi/2}|\sin\varphi|\,d\varphi = -2\varphi\,|\cos\varphi|\,\Big|_{-\pi/2}^{\pi/2} = 0, \\ \overline{\Phi^2} = 2\int_{-\pi/2}^{\pi/2}\varphi^2\,|\sin\varphi|\,d\varphi = 2\left|\begin{matrix} u=\varphi^2 & d\upsilon = |\sin\varphi|\,d\varphi \\ du = 2\varphi\, d\varphi & \upsilon = |\sin\varphi| \end{matrix}\right| = 2\varphi^2\,|\sin\varphi|\,\Big|_{-\pi/2}^{\pi/2} - 4\int_{-\pi/2}^{\pi/2}\varphi\,|\sin\varphi|\,d\varphi = 0.\end{gathered} \tag{28}$$

We assume that the phase dispersion equals

$$D_\Phi = \overline{\Phi^2} - \overline{\Phi} = 0 - 0 = 0 \,. \tag{29}$$

The result is incorrect. The equality of zero (29) contradicts the very physical nature of the dispersion, as the intensity of fluctuations. In addition, the Monte Carlo statistical modeling process is accompanied by an estimation of SQU. Therefore, the variance can not accept negative values, and therefore can be used as an objective characteristic of the degree of intensity of fluctuations.

The equation of quadratic transformation (7) in the range of values $y<0$ ,

it does not have the actual solutions and the density of probabilities $f_Y(y)=0$. In region $y \geq 0$, probability density $f_Y(y) \neq 0$ and equation (1) has two roots:

$$X_1 = -\sqrt{Y/\alpha}, \quad x<0, \qquad X_2 = +\sqrt{Y/\alpha}, \quad x>0. \tag{30}$$

In the domain of the monotonicity of function (1), a cumulative probability correlates with the relation:

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{Y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \tag{31}$$

From where function $f_Y(y)$ is obtained by differentiating the expression (31):

$$f_Y(y) = \frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{y}} f_X\left(+\sqrt{\frac{y}{\alpha}}\right) + \frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{y}} f_X\left(-\sqrt{\frac{y}{\alpha}}\right). \tag{32}$$

Here it is taken into account that in the range of values of the argument $(-\infty, 0)$ ,
the inverse function and the first derivative have the appearance:

$$g^{-1}(y) = -\frac{1}{\sqrt{\alpha}}\sqrt{y} \;\text{ i }\; \frac{d}{dy}g^{-1}(y) = -\frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{y}}, \tag{33}$$

But in region $(0,+\infty)$

$$g^{-1}(y) = +\frac{1}{\sqrt{\alpha}}\sqrt{y} \;\text{ i }\; \frac{d}{dy}g^{-1}(y) = +\frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{y}}. \tag{34}$$

Therefore, for strictly monotonic functions, the probability density is calculated by the formula (13).

The dispersion of the transformed BB by law (29) is defined as:

$$\begin{aligned} D_Y &= \int_{-\infty}^{\infty}\left(y-\overline{Y}\right)^2 f_Y(y)dy = \int_{-\infty}^{\infty}\left((y)^2 + \overline{Y}^2 - 2y\overline{Y}\right) f_Y(y)dy = \\ &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy + \overline{Y}^2\int_{-\infty}^{\infty} f_Y(y)dy - 2\overline{Y}\int_{-\infty}^{\infty} y f_Y(y)dy = \overline{Y^2} + \overline{Y}^2\int_{-\infty}^{\infty} f_Y(y)dy - 2\overline{Y}^2. \end{aligned} \tag{35}$$

To substantiate the final analytical form (35), we need to specify the explicit type of the distribution of the IV, which is subjected to transformation.

A case of a uniform distribution of the original random variable $X$ .

Let VV be distributed by law (16). Check the condition of normalization:

$$\int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{x_0}\int_{-\infty}^{\infty} dx = \frac{1}{x_0}\int_{0}^{x_0} dx = \frac{1}{x_0} x\Big|_0^{x_0} = 1. \tag{36}$$

Integration boundaries are adjusted to the interval $[0; x_0]$ changes in the absolute value of the amplitude, so the dispersion of the output BB will be equal:

$$D_X = \overline{X^2} + \overline{X}^2\int_0^{x_0} f_X(x)dx - 2\overline{X}^2 = \overline{X^2} + \overline{X}^2 - 2\overline{X}^2 = \overline{X^2} - \overline{X}^2. \tag{37}$$

The equation (37) is valid for statistically independent BB* [11].

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* For independent BB $W$ cavariation equals 0 [11]. The correlation coefficient of two BBs can be zero, even if the BBs are not independent. On the contrary, if the correlation coefficient is different from zero, then two BB`s can not be independent [20–22].

Calculating average $\overline{X}$ i $\overline{X^2}$ :

$$\overline{X}=\int_0^{x_0} x\, f_X(x)\,dx=\frac{1}{2x_0}x^2\Big|_0^{x_0}=\frac{x_0}{2},\quad \overline{X^2}=\int_0^{\infty} x^2\, f_X(x)\,dx=\frac{1}{3x_0}x^3\Big|_0^{x_0}=\frac{x_0^2}{3}. \qquad (38)$$

So dispertion (37) equals:

$$D_X=\overline{X^2}-\overline{X}^2=\frac{x_0^2}{3}-\frac{x_0^2}{4}=\frac{x_0^2}{12}>0, \qquad (39)$$

which is consistent with [11].

Using formulas (32)–(34), we set the analytic form of the function $f_Y(y)$. If we take into account that random changes are considered for absolute values x, then (30) has only one root $X_1=+\sqrt{Y/\alpha}$ , so

$$f_Y(y)=\frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{y}}\,f_X\left(+\sqrt{\frac{y}{\alpha}}\right)=\frac{1}{2x_0\sqrt{y}\sqrt{\alpha}}\,. \qquad (40)$$

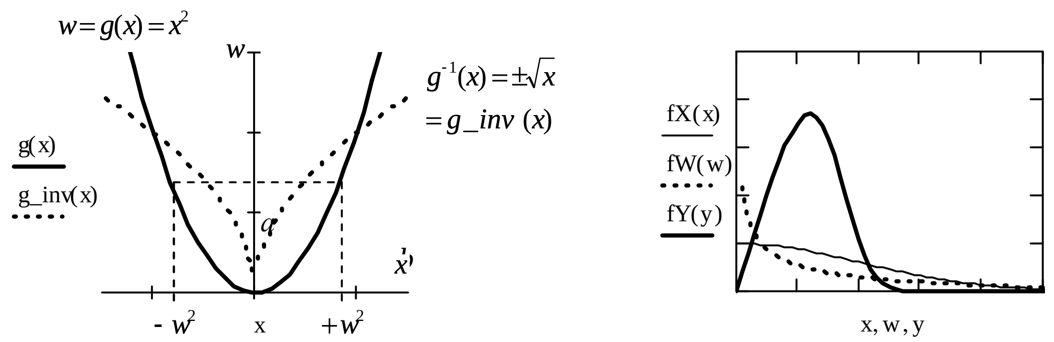
Note that the function (40) is nonlinear, therefore, in contrast to the linear transformation (12), changes the type of distribution of BB [12].

Let's check for the distribution (40) of the normalization condition:

$$\frac{1}{x_0\sqrt{\alpha}}\int_0^{\alpha x_0^2}\frac{dy}{\sqrt{y}}=\frac{2}{2x_0\sqrt{\alpha}}\sqrt{y}\Big|_0^{\alpha x_0^2}=\frac{1}{x_0\sqrt{\alpha}}\sqrt{\alpha}\,x_0=1. \qquad (41)$$

Define the boundaries of integration:

$$y=\alpha x^2 \Rightarrow x=\frac{1}{\alpha}\sqrt{y} \Rightarrow \begin{cases} x=0, & y=0,\\ x=x_0, & y=\alpha x_0^2. \end{cases} \qquad (42)$$




*a* *b*

*Fig. 2*

So average $\overline{Y}$ and $\overline{Y^2}$ calculating as integrals:

$$\overline{Y}=\frac{1}{2x_0\sqrt{\alpha}}\int_0^{\alpha x_0^2}\frac{y}{\sqrt{y}}dy=\frac{1}{2x_0\sqrt{\alpha}}\int_0^{\alpha x_0^2}\sqrt{y}\,dy=\frac{2}{6x_0\sqrt{\alpha}}\,y\Big|_0^{\alpha x_0^2}=\frac{1}{3x_0\sqrt{\alpha}}\alpha x_0^2=\frac{1}{3}\sqrt{\alpha}\,x_0,$$

$$\overline{Y^2}=\frac{1}{2x_0\sqrt{\alpha}}\int_0^{\alpha x_0^2}\frac{y^2}{\sqrt{y}}dy=\frac{1}{2x_0\sqrt{\alpha}}\int_0^{\alpha x_0^2}y^{3/2}dy=\frac{2}{2\times 5x_0\sqrt{\alpha}}\,y^{5/2}\Big|_0^{\alpha x_0^2}=\frac{1}{5}\alpha x_0^4, \qquad (43)$$

And dispersion (35) will be:

$$D_Y=\overline{Y^2}-\overline{Y}^2=\frac{1}{5}\alpha x_0^4-\left(\frac{1}{3}\sqrt{\alpha}\,x_0\right)^2=\frac{1}{30}\alpha x_0^4>0. \qquad (44)$$

In the literature, for example [10–12], the quadratic transformations of type (7) were analyzed, taking into account both roots of the function (30).

Case of uneven distribution of output BB.

Let BB X is in the range $[0;x_0]$ standardized $N(0,\sigma_X^2)$ (в (4) $m_X=0$). Adjust for interval $[0;x_0]$ norm valuation:

$$C\int_{-\infty}^{\infty} f_X(x)dx = \frac{C}{\sqrt{2\pi\sigma_X^2}}\int_0^{x_0}\exp\left(-px^2\right)dx = \frac{C}{\sqrt{2\pi\sigma_X^2}}\int_0^{x_0}\exp\left(-\left(\sqrt{p}x\right)^2\right)dx =$$

$$= \frac{C\sqrt{2}\sigma_X}{\sqrt{2\pi\sigma_X^2}}\int_0^{\sqrt{p}x_0}\exp\left(-t^2\right)dt = \frac{C}{\sqrt{\pi}}\left|\begin{matrix} u=1 & d\nu=\exp\left(-t^2\right)dt \\ du=0 & \nu=\sqrt{\frac{\pi}{4}}erf(t)\end{matrix}\right| = \frac{C}{2}erf(t)\Big|_0^{\sqrt{p}x_0} = \quad (45)$$

$$= \frac{C}{2}erf\left(\sqrt{p}x_0\right)=1 \quad \Rightarrow \quad C = 2\left[erf\left(\sqrt{p}x_0\right)\right]^{-1},$$

де $t=\frac{x}{\sqrt{2}\sigma_X} \Rightarrow dx=\sqrt{2}\sigma_X dt$, and the boundaries of integration are defined as

$$t=\sqrt{p}x_0 \Rightarrow \begin{cases} x=0, & t=0, \\ x=x_0, & t=\sqrt{p}x_0. \end{cases}$$

In (45) when converted

$d\nu=\exp\left(-t^2\right)dt \Rightarrow \nu=\int\exp\left(-t^2\right)dt$ taken into account the table integral [23]:

$$\int\exp\left(-cx^2\right)dx=\sqrt{\frac{\pi}{4c}}erf\left(\sqrt{c}x\right), \quad erf \textit{ is the Error function}. \quad (46)$$

Then, in order to provide for a given case the condition of valuation in the form

$$\int_0^{x_0} ff_X(x)dx=1, \quad (47)$$

the standard distribution function (10) is renormalized:

$$ff_X(x)=\frac{C}{\sqrt{2\pi\sigma_X^2}}\exp\left(-px^2\right). \quad (48)$$

Thus, for the considered problem, the variance of the output random variable with the standard distribution (48) in the change interval $[0;x_0]$ is equal to:

$$D_X=\int_0^{x_0}\left(x-\overline{X}\right)^2 ff_X(x)dx=\int_0^{x_0}\left((x)^2+\overline{X}^2-2x\overline{X}\right)ff_X(x)dx=\int_0^{x_0}x^2 ff_X(x)dx+\overline{X}^2\int_0^{x_0}ff_X(x)dx-$$

$$-2\overline{X}\int_0^{x_0}x ff_X(x)dx=\overline{X^2}+\overline{X}^2\int_0^{x_0}ff_X(x)dx-2\overline{X}^2=\overline{X^2}+\overline{X}^2-2\overline{X}^2=\overline{X^2}-\overline{X}^2. \quad (49)$$

Calculating average $\overline{X}$ :

$$\overline{X}=\int_0^{x_0}x\, ff_X(x)dx=\frac{C}{\sqrt{2\pi\sigma_X^2}}\int_0^{x_0}x\exp\left(-px^2\right)dx=\frac{C}{\sqrt{2\pi\sigma_X^2}}\left|\begin{matrix} u=x & d\nu=\exp\left(-px^2\right)dx \\ du=dx & \nu=\sqrt{\sigma_X^2\pi/2}\,erf\left(\sqrt{p}x\right)\end{matrix}\right|=$$

$$=\frac{C}{\sqrt{2\pi\sigma_X^2}}\sqrt{\sigma_X^2\pi/2}\;x\,erf\left(\sqrt{p}x\right)\Big|_0^{x_0}-\frac{C\sigma_X}{\sqrt{2}}\int_0^{x_0}erf\left(\sqrt{p}x\right)d\left(\sqrt{p}x\right)=x_0-x_0- \quad (50)$$

$$-\frac{1}{\sqrt{2\pi\sigma_X^2}}\frac{1}{\sqrt{\pi}}\left(\exp\left(-px_0^2\right)-1\right)=\frac{1}{\sqrt{2\pi}\sigma_X}\left(1-\exp\left(-px_0^2\right)\right).$$

Here the table integral has been taken into account [23]:

$$\int_0^x erf(t)\,dt = x\,erf(x) + \frac{1}{\sqrt{\pi}}\left(\exp\left(-x^2\right) - 1\right). \qquad (51)$$

Let`s calculate the mean square $\overline{X^2}$ :

$$\overline{X^2} = \int_0^{x_0} x^2\, ff_X(x)\,dx = \frac{C}{\sqrt{2\pi\sigma_X^2}}\int_0^{x_0} x^2 \exp\left(-px^2\right)dx = \frac{C}{\sqrt{2\pi\sigma_X^2}}\left|\begin{matrix} u = x^2 & d\nu = \exp\left(-px^2\right)dt \\ du = 2x\,dx & \nu = \sqrt{\frac{\sigma_X^2\pi}{2}}\,erf\left(\sqrt{p}x\right)\end{matrix}\right| =$$

$$= \frac{C}{2}x^2 erf\left(\sqrt{p}x\right)\Big|_0^2 - C\int_0^{x_0} x\,erf\left(\sqrt{p}x\right)dx = x_0^2 - \left(1 - \exp\left(-px_0^2\right)\right).$$

So, dispersion

$$D_X = x_0^2 - \left(1 - \exp\left(-px_0^2\right)\right) - \left[\frac{1}{\sqrt{2\pi}\sigma_X}\left(1 - \exp\left(-px_0^2\right)\right)\right]^2 =$$

$$= x_0^2 - \left(1 - \exp\left(-px_0^2\right)\right)\left[1 + \frac{1 - \exp\left(-px_0^2\right)}{2\pi^2\sigma_X^2}\right]. \qquad (52)$$

## Conclusions

In conclusion, let's note this. The study of the influence of the restriction of the interval of values of the BB on the laws of its probabilities was started long ago [25], and the corresponding results in the future were used to model the reliability of physical and technical accuracy and accuracy of production [13, 17, 26].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**П. С. Кособуцький, Р. А. Ференс**
Національний університет "Львівська політехніка"

## ПРО ДЕЯКІ ОСОБЛИВОСТІ ПРЯМОГО І ОБЕРНЕНОГО ПЕРЕТВОРЕННЯ ВИПАДКОВИХ ВЕЛИЧИН



**Завжди актуальні задачі отримання і опрацювання експериментальних результатів в складних системах. Випадкові завади, похибки вимірювань, недосконалість та обмеженість математичних моделей та алгоритмів обробки даних здатні змінювати вигляд розподілу і призводити до некоректності використання алгоритмів, наприклад, як це має місце з фільтрації по Калману в системах керування. Складні методи ідентифікації законів розподілу потребують дослідження квантових систем, природніх явищ, екологічних, біологічних, тощо процесів, для яких характерна наявність сингулярностей і багатомодовості розподілів. Тому часто для моделювання ймовірнісних розподілів експериментальних даних рекомендують застосовувати не окремі закони розподілів, а узагальнений розподіл як єдину статистичну систему, яка відомі розподіли включає в себе як окремі часткові випадки. Так узагальнений гамма-розподіл включає в себе розподіли Релея, Максвелла, Вейбулла, Леві, хі-квадрат, які широко використовують в прикладних задачах, зв'язаних із статистичними методами досліджень фізичних процесів, дистанційним зондуванням, в теорії надійності, для опису дисперсійного складу частинок дроблення та розрахунку ефективності розділення фаз у газорідинних потоках.**