MATHEMATICAL MODELING AND COMPUTING, Vol. 8, No. 1, pp. 11–21 (2021)

Properties of the beta coefficient of the global minimum variance portfolio

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(Received 16 January 2020; Revised 28 August 2020; Accepted 2 October 2020)

The paper is devoted to the investigation of statistical properties of the sample estimator of the beta coefficient in the case when the weights of benchmark portfolio are constant and for the target portfolio, the global minimum variance portfolio is taken. We provide the asymptotic distribution of the sample estimator of the beta coefficient assuming that the asset returns are multivariate normally distributed. Based on the asymptotic distribution we construct the confidence interval for the beta coefficient. We use the daily returns on the assets included in the DAX index for the period from 01.01.2018 to 30.09.2019 to compare empirical and asymptotic means, variances and densities of the standardized estimator for the beta coefficient. We obtain that the bias of the sample estimator converges to zero very slowly for a large number of assets in the portfolio. We present the adjusted estimator of the beta coefficient for which convergence of the empirical variances to the asymptotic ones is not significantly slower than for a sample estimator but the bias of the adjusted estimator is significantly smaller.

Keywords: *global minimum variance portfolio, beta coefficient, test theory, asymptotic distribution, parameter uncertainty, sample estimator.*

2010 MSC: 91G10, 62F03, 62F12 **DOI:** 10.23939/mmc2021.01.011

1. Introduction

Diversification of income sources and risks is one of the key concepts not only of financial institutions activity but also of the activity of the majority of people. From a theoretical point of view, diversification can be realized by the construction of a portfolio. The first paper, where the problem of portfolio construction had been considered, was the paper by G. Markowitz [1]. In this work, the method of portfolio construction based on the optimization of its characteristics was described from a scientific point of view. The main concept introduced by Markowitz is an efficient portfolio, that is, a portfolio for which it is impossible to reduce the risk without decreasing its return, or equivalent, it is impossible to increase the return without increasing the risk. The set of such portfolios forms an efficient frontier [2]. Portfolios from the efficient frontier play a crucial role in the portfolio theory. Many papers are devoted to the analysis of these portfolios: the maximum expected utility portfolio is considered in [3–5]; the portfolio with the maximum Sharpe ratio is analyzed in [6–9]; the minimum VaR and the minimum CVaR portfolios are considered in [10–12]. But the most important role among the portfolios of the efficient frontier plays the global minimum variance portfolio. The properties of weights ($\mathbf{w}_{GMV}$) and the characteristics of this portfolio have been studied by many scientists [13–16]. It is shown that the expected return and the variance of the GMV portfolio are two out of three parameters that uniquely determine the efficient frontier [17]. The GMV portfolio risk is the smallest among the portfolios on the efficient frontier, that is, it is impossible to construct a portfolio with a lower level of risk than the risk of GMV portfolio in the set of the selected assets. It is clear that the return of this portfolio is also the lowest among the portfolios of the efficient frontier. However, the investors who are interested in portfolio return will not choose GMV portfolio as an investing strategy. Naturally, the question arises, how can an investor check whether the GMV portfolio risk differs significantly from the risk of another portfolio? If it turns out that, the risk of considered portfolios does not differ significantly then there is an opportunity to construct a portfolio with the risk level that is statistically equal to the minimum possible level. The goal of the paper is to construct such a test. We construct the test based on the β-coefficient, a quantitative characteristic that describes the relationship between portfolios risk levels (or equivalently between the portfolios expected returns).

The β-coefficient plays a crucial role in CAPM theory and is an important practical tool for portfolios comparison. In [18] probabilistic properties of the β-coefficient estimator are investigated in the case of constant weights of the compared portfolios. These results are not applicable in the case of the global minimum variance portfolio since the weights of this portfolio depend on unknown parameters of the asset returns distribution. Therefore, in practice investor uses estimates of these weights. From the probability theory and mathematical statistics, it is known that the estimators of unknown parameters of the asset returns distribution and, as consequence, values, which depend on them, are random values. From this point of view in order to investigate the properties of the β-coefficient, it is sufficient to know the distribution of its estimator. In the paper, we find the asymptotic distribution of the sample estimator of the beta coefficient in the case when the weights of the benchmark portfolio are constant and the investor's portfolio is the GMV portfolio provided that the asset returns are multivariate normally distributed. The assumption of normality is often used in financial literature because of its attractive theoretical properties [19]. The asymptotic distribution is recommended for considering in the case when the exact distribution can not be obtained [20].

The rest of the paper is organized as follows. In the next chapter, we present the theoretical results of the paper. We deduce the asymptotic distribution of the sample estimator of beta coefficient and based on this result we present the test for values of beta. The empirical results based on the returns of 30 stocks included in the DAX index are presented in the section 3. Here we check the convergence of the empirical distributions of the beta-coefficient estimator to the asymptotic one. The concluding remarks are given in the section 4.

2. Theoretical results

Let us assume that the investor's and benchmark portfolios consist of the same k assets. We denote by $\mathbf{X}_t = (X_{1t}, X_{2t}, \ldots, X_{kt})'$ the k-dimensional vector of asset returns at time point t, i.e., $X_{it} = 100\ln(P_{it}/P_{i(t-1)})$ stands for the return of i-th asset at time point t, where P_{it} denotes the price of the i-th asset at time point t. From [18] assuming that $\mathbf{X}_t$ follows a weakly stationary process with the mean $E(\mathbf{X}_t) = \boldsymbol{\mu}$ and the covariance matrix $\mathrm{Var}(\mathbf{X}_t) = \boldsymbol{\Sigma}$ we get that the beta coefficient can be calculated in the following way

$$\beta = \frac{\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}}{\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b}, \tag{1}$$

where $\mathbf{w}$ stands for the weights of an investor portfolio and $\mathbf{w}_b$ stands for the weights of the benchmark portfolio. The results in [18] were obtained for the case when the weights $\mathbf{w}$ and $\mathbf{w}_b$ are constant and do not depend on parameters of the asset return process. In the current paper, we consider the case where the investor portfolio is the global minimum variance portfolio. In other words an investor constructs one's portfolio by unconditional minimization of the variance of the portfolio. The weights of such portfolio are given by [13]

$$\mathbf{w} = \mathbf{w}_{GMV} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}, \tag{2}$$

where $\mathbf{1}$ denotes the k-dimensional vector of ones. Inserting (2) instead of the vector of weights of investor's portfolio $\mathbf{w}$ in (1) we get

$$\beta_{GMV} = \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)}, \tag{3}$$

The method for beta coefficient calculation presented in (3) can not be used directly, because the distribution parameters of the vector of asset returns $\mathbf{X}_t$, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown in practice. They should be first somehow estimated. We make use of the sample estimators of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Let the sample of the historical values of the asset return vector $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be available, then the sample estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given by

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i, \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{X}_i - \hat{\boldsymbol{\mu}})(\mathbf{X}_i - \hat{\boldsymbol{\mu}})'. \tag{4}$$

Using the estimators (4) the estimator of β_{GMV} can be written in the following way

$$\hat{\beta}_{GMV} = \frac{1}{\left(\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\right)\left(\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b\right)}, \tag{5}$$

It is a well known fact of the mathematical statistics that the estimator (5) is a random variable. The best way to investigate its properties is to determine its distribution. In the next theorem the asymptotic distribution of the sample estimator of beta coefficient (5) is presented. In [20] it is recommended to consider an asymptotic distribution of an estimator in case where a finite sample distribution can not be obtained.

Theorem 1. *Let us form a portfolio within k assets. Denote by $\mathbf{X}_t$ k-dimensional vector of asset returns included into the portfolio at time point t. Assume that $\mathbf{X}_t$ follows k-dimensional normal distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Then for $n \to \infty$*

$$\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big) \xrightarrow{d} N(0, \sigma^2),$$

where

$$\sigma^2 = \frac{4}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2} - \frac{4}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^3(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^3}, \tag{6}$$

n is the size of the sample of historical values of the vector of asset returns $\mathbf{X}_t$ used for constructing the sample estimators of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (4), $\hat{\beta}_{GMV}$ is given in (5), β_{GMV} is the precise value of the beta coefficient, the symbol $\xrightarrow{d}$ denotes the convergence in distribution, $\mathbf{w}_b$ are the weights of the benchmark portfolio.

Proof. Denote by $\boldsymbol{\theta} = (\boldsymbol{\mu}', \mathrm{vech}(\boldsymbol{\Sigma})')'$ a vector of unknown parameters of the asset returns distribution and by $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\mu}}', \mathrm{vech}(\hat{\boldsymbol{\Sigma}})')'$ its sample estimator. The operator vech transforms an arbitrary square symmetric matrix $= (a_{ij})$ of dimension $k \times k$ into a $k(k+1)/2$-dimension vector $\mathrm{vech}(\mathbf{A}) = (a_{11}, \ldots, a_{k1}, \ldots, a_{ii}, \ldots, a_{ki}, \ldots, a_{kk})'$ [21]. From the delta method [22] we obtain that

$$\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big) \xrightarrow{d} N(0, \mathbf{G}'\boldsymbol{\Omega}\mathbf{G}),$$

where $\mathbf{G} = (\partial\beta_{GMV}/\partial\boldsymbol{\mu}, \partial\beta_{GMV}/\partial\mathrm{vech}(\boldsymbol{\Sigma}))'$ is $k(k+3)/2$-dimensional vector and $\boldsymbol{\Omega}$ is an asymptotic covariance matrix of the random vector $\sqrt{n}(\hat{\beta} - \beta)$ and is given by [22]

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0}_{k\times k(k+1)/2} \\ \mathbf{0}_{k(k+1)/2\times k} & \mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}_k^{+\prime} \end{pmatrix}, \tag{7}$$

where $\mathbf{0}_{m\times l}$ is $m \times l$-dimensional null matrix, $\mathbf{I}_{k^2}$ is $k^2 \times k^2$-dimensional identity matrix, $\mathbf{D}_k^{+} = (\mathbf{D}_k'\mathbf{D}_k)^{-1}\mathbf{D}_k'$, $\mathbf{D}_k$ is $k^2 \times k(k+1)/2$-dimensional matrix such that $\mathbf{D}_k\mathrm{vech}(\mathbf{A}) = \mathrm{vec}\,\mathbf{A}$ for arbitrary square symmetric matrix of dimension $k \times k$, the oherator vec transforms an arbitrary $m \times l$-dimensional matrix $\mathbf{B}$ into a k^2-dimensional vector by stacking matrix columns one under another $\mathrm{vec}\,\mathbf{B} = (b_{11}, \ldots, b_{m1}, \ldots, b_{1i}, \ldots, b_{mi}, \ldots, b_{ml})'$, $\mathbf{K}_k$ — $k^2 \times k^2$-dimensional matrix such that for an arbitrary matrix $\mathbf{C}$ of dimension $k \times k$ it holds that $\mathbf{K}_k \operatorname{vec} \mathbf{C} = \operatorname{vec} \mathbf{C}'$. The additional information concerning matrix operators can be found in [21].

Using the rules of matrix differential calculus [21] we obtain

$$\frac{\partial \beta_{GMV}}{\partial \boldsymbol{\mu}} = \mathbf{0}_k, \tag{8}$$

$$\frac{\partial \beta_{GMV}}{\partial \operatorname{vech}(\boldsymbol{\Sigma})} = -\frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2}\mathbf{D}_k'(\mathbf{w}_b \otimes \mathbf{w}_b) + \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)}\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{1} \otimes \mathbf{1}). \tag{9}$$

Here we use the facts described in [21]

$$\frac{\partial(\mathbf{a}'\boldsymbol{\Sigma}\mathbf{b})}{\partial \operatorname{vech}(\boldsymbol{\Sigma})} = \mathbf{D}_k'(\mathbf{b} \otimes \mathbf{a}), \quad \frac{\partial(\mathbf{a}'\boldsymbol{\Sigma}^{-1}\mathbf{b})}{\partial \operatorname{vech}(\boldsymbol{\Sigma})} = -\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{b} \otimes \mathbf{a}).$$

Taking into account the form of matrix $\boldsymbol{\Omega}$ (7) we obtain

$$\mathbf{G}'\boldsymbol{\Omega}\mathbf{G} = (\partial\beta_{GMV}/\partial\boldsymbol{\mu})'\boldsymbol{\Sigma}(\partial\beta_{GMV}/\partial\boldsymbol{\mu}) + (\partial\beta_{GMV}/\partial\operatorname{vech}(\boldsymbol{\Sigma}))'\mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}(\partial\beta_{GMV}/\partial\operatorname{vech}(\boldsymbol{\Sigma})).$$

The equality (8) implies that

$$(\partial\beta_{GMV}/\partial\boldsymbol{\mu})'\boldsymbol{\Sigma}(\partial\beta_{GMV}/\partial\boldsymbol{\mu}) = 0.$$

From (9) we obtain

$$\begin{aligned}
&(\partial\beta_{GMV}/\partial\operatorname{vech}(\boldsymbol{\Sigma}))'\mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}(\partial\beta_{GMV}/\partial\operatorname{vech}(\boldsymbol{\Sigma}))\\
&= \left(\frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)}\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{1} \otimes \mathbf{1}) - \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2}\mathbf{D}_k'(\mathbf{w}_b \otimes \mathbf{w}_b)\right)'\\
&\qquad \times \mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\\
&\times \left(\frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)}\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{1} \otimes \mathbf{1}) - \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2}\mathbf{D}_k'(\mathbf{w}_b \otimes \mathbf{w}_b)\right)\\
&= \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^4(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2}(\mathbf{1}' \otimes \mathbf{1}')\mathbf{D}_k\mathbf{D}_k^{+}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k\mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\\
&\qquad \times \mathbf{D}_k'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{1} \otimes \mathbf{1})\\
&+ \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^4}(\mathbf{w}_b' \otimes \mathbf{w}_b')\mathbf{D}_k\mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{w}_b \otimes \mathbf{w}_b)\\
&- \frac{2}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^3(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^3}(\mathbf{w}_b' \otimes \mathbf{w}_b')\mathbf{D}_k\mathbf{D}_k^{+}(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\\
&\qquad \times \mathbf{D}_k'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\mathbf{1} \otimes \mathbf{1}) = \frac{1}{4(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^4(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2}\\
&\qquad \times (\mathbf{1}' \otimes \mathbf{1}')(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(\mathbf{I}_{k^2} + \mathbf{K}_k)^3(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(\mathbf{1} \otimes \mathbf{1})\\
&\qquad + \frac{1}{4(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^4}(\mathbf{w}_b' \otimes \mathbf{w}_b')(\mathbf{I}_{k^2} + \mathbf{K}_k)^3(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})(\mathbf{w}_b \otimes \mathbf{w}_b)
\end{aligned}$$

$$- \frac{2}{4\big(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\big)^3\big(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b\big)^3}(\mathbf{w}_b' \otimes \mathbf{w}_b')\big(\mathbf{I}_{k^2} + \mathbf{K}_k\big)^3(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\big(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}\big)(\mathbf{1} \otimes \mathbf{1}).$$

Here we use the following properties of matrix operators $\mathbf{D}_k\mathbf{D}_k^+ = \frac{1}{2}\big(\mathbf{I}_{k^2}+\mathbf{K}_k\big) = \mathbf{N}_k$, $\mathbf{N}_k(\mathbf{A}\otimes\mathbf{A}) = (\mathbf{A} \otimes \mathbf{A})\mathbf{N}_k$, $\mathbf{N}_k = \mathbf{N}_k^2 = \mathbf{N}_k'$, $\mathbf{D}_k\mathbf{D}_k^+(\mathbf{A} \otimes \mathbf{A})\mathbf{D}_k = (\mathbf{A} \otimes \mathbf{A})\mathbf{D}_k$ for an arbitrary $k \times k$-dimensional matrix $\mathbf{A}$. Taking into account that $\mathbf{K}_k(\mathbf{A} \otimes \mathbf{A}) = (\mathbf{A} \otimes \mathbf{A})$ we obtain

$$(\partial\beta_{GMV}/\partial\mathrm{vech}(\boldsymbol{\Sigma}))'\mathbf{D}_k^+\big(\mathbf{I}_{k^2} + \mathbf{K}_k\big)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}(\partial\beta_{GMV}/\partial\mathrm{vech}(\boldsymbol{\Sigma})) = \frac{4}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^2} - \frac{4}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^3(\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b)^3},$$

which completes the proof of the theorem. ■

Table 1. Empirical $n \in \{120, 250, 500, 1000, 2000, 3000, 5000\}$ and asymptotic means and variances of $\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big)$ for $k \in \{5, 10, 15, 20, 25, 30\}$ equally weighted portfolios taken as benchmark and GMV portfolios taken as target.

		$k=5$	$k=10$	$k=15$	$k=20$	$k=25$	$k=30$
$n=120$	Mean	-0.2424	-0.4017	-0.6515	-0.8492	-1.1155	-1.2746
	Variance	0.5557	0.4983	0.4766	0.4334	0.4268	0.3749
$n=250$	Mean	-0.1664	-0.2733	-0.4445	-0.5859	-0.7675	-0.8757
	Variance	0.5586	0.5176	0.5056	0.4715	0.4775	0.4351
$n=500$	Mean	-0.1153	-0.1929	-0.3097	-0.4119	-0.5374	-0.6137
	Variance	0.56764	0.5247	0.5175	0.4880	0.4991	0.4648
$n=1000$	Mean	-0.0839	-0.1377	-0.2251	-0.2889	-0.3825	-0.4318
	Variance	0.5622	0.5285	0.5255	0.4977	0.5116	0.4804
$n=2000$	Mean	-0.0601	-0.0954	-0.1568	-0.2012	-0.2725	-0.3059
	Variance	0.5652	0.5298	0.5264	0.5047	0.5155	0.4792
$n=3000$	Mean	-0.0504	-0.0819	-0.1272	-0.1657	-0.2187	-0.2496
	Variance	0.5652	0.5295	0.5291	0.5083	0.5146	0.4849
$n=5000$	Mean	-0.0366	-0.0635	-0.0982	-0.1243	-0.1694	-0.1959
	Variance	0.5632	0.5270	0.5247	0.5090	0.5211	0.4860
Asymptotic	Mean	0	0	0	0	0	0
	Variance	0.5639	0.5311	0.5309	0.5083	0.5212	0.4900

In practice an investor should use the estimator of asymptotic variance (6), i.e.

$$\hat{\sigma}^2 = \frac{4}{(\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1})^2(\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b)^2} - \frac{4}{(\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1})^3(\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b)^3}, \tag{10}$$

but the theorem 1.14 [23] implies that the sample estimator of σ (10) is consistent, i.e. for $n \to \infty$

$$\hat{\sigma} \to \sigma.$$

The result of Theorem 1 provides the statistical test for significance of difference between characteristics of portfolios with the weights $\mathbf{w}_b$ and $\mathbf{w}_{GMV}$. If $(1-\gamma)$-confidence interval

$$\left[\hat{\beta}_{GMV} - \frac{\hat{\sigma}}{\sqrt{n}}z_{1-\gamma/2},\, \hat{\beta}_{GMV} + \frac{\hat{\sigma}}{\sqrt{n}}z_{1-\gamma/2}\right],$$

where z_γ stands for γ-quantile of the standard normal distribution contains one, than the risks of both portfolios do not significantly differ. In this case, an investor has the possibility to use a portfolio with the nonrandom weights and the portfolio risk does not significantly differ from the minimum portfolio risk available for the chosen set of assets.

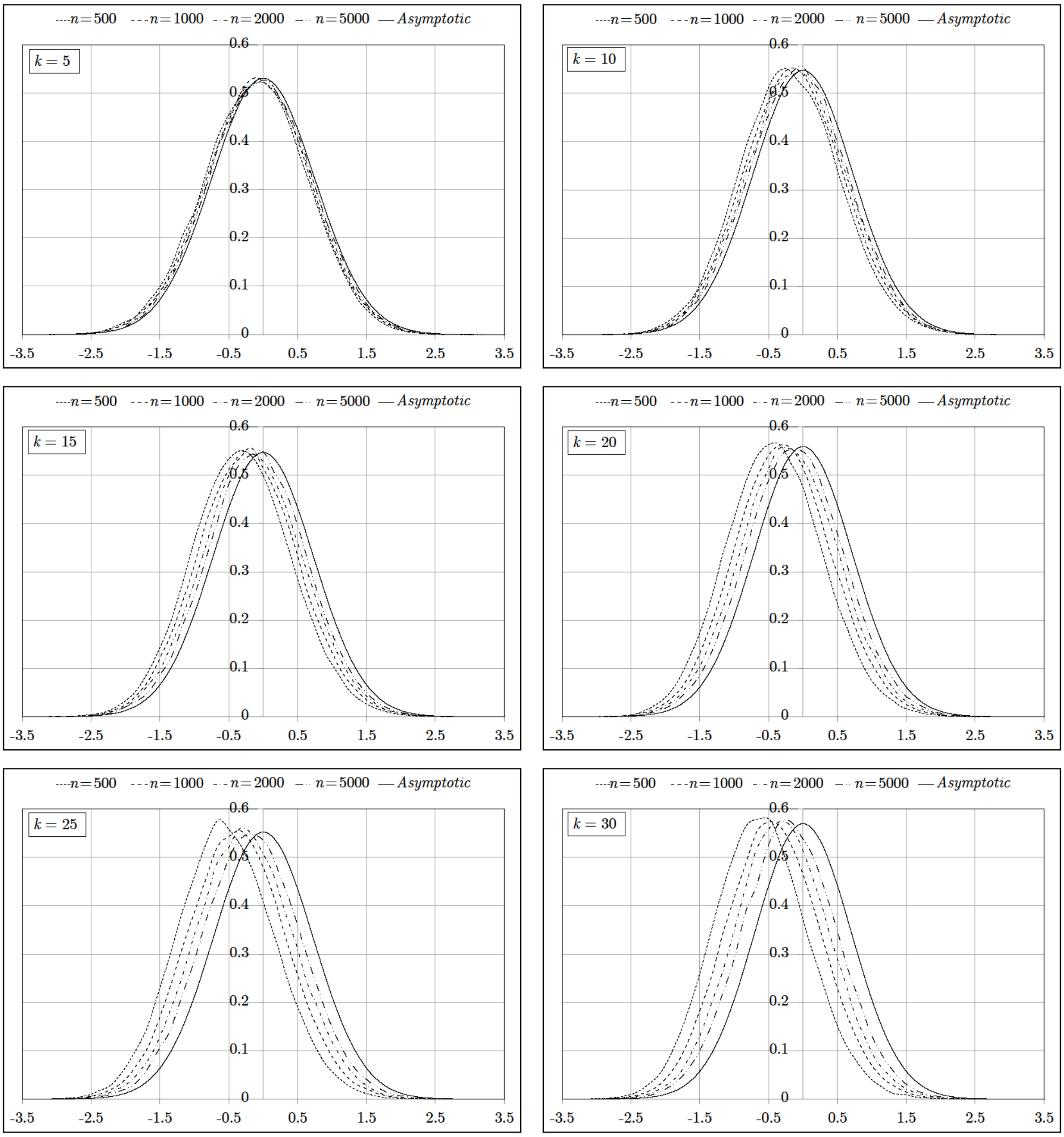

Fig. 1. Kernel density estimators for $n \in \{500, 1000, 2000, 5000\}$ and asymptotic density of $\sqrt{n}(\hat{\beta}_{GMV} - \beta_{GMV})$ for different values of k.

3. Empirical study

In this section, we provide an empirical presentation of the results from the previous section. First, we investigate the convergence of the distribution of the random variable $\sqrt{n}(\hat{\beta}_{GMV} - \beta_{GMV})$ to the asymptotic normal distribution provided by the Theorem 1. For this purpose, we use the daily returns on the assets included into the DAX index for the period from 01.01.2018 to 30.09.2019 (440 observations). We consider six GMV portfolios with dimensions $k \in \{5, 10, 15, 20, 25, 30\}$ included corresponding k assets included into the DAX index in alphabetical order. We take these portfolios as target portfolios. As a benchmark portfolio in every case, we use the equally weighted portfolio consisting of k corresponding assets. To provide empirical distributions of $\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big)$, we use the Monte Carlo study. We draw a random sample of size $n \in \{120, 250, 500, 1000, 2000, 3000, 5000\}$ from k-dimensional normal distribution with the parameters equal to the sample mean vector and to the sample covariance matrix computed from the daily returns on the assets included into the DAX index for the period from 01.01.2018 to 30.09.2019 for the corresponding values of k. Based on simulated data the value of $\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big)$ is computed and this procedure is repeated 100000 times for different values of n and k. We use the samples of values of $\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big)$ to estimate the corresponding distributions. The results are summarized in Table 1 and Fig. 1. In Table 1 we present the empirical means and variances of the random variable $\sqrt{n}\big(\hat{\beta}_{GMV} - \beta_{GMV}\big)$. In Fig. 1 we plot the empirical densities together with the corresponding asymptotic density. From the results presented in Table 1 and Fig. 1, we conclude that the empirical variances converge fast enough to the asymptotic ones but the bias goes to zero slowly, especially for larger values of k.

Table 2. Empirical $n \in \{120, 250, 500, 1000, 2000, 3000, 5000\}$ and asymptotic means and variances of $\sqrt{n}\big(\hat{\beta}_{GMV;adj} - \beta_{GMV}\big)$ for $k \in \{5, 10, 15, 20, 25, 30\}$ equally weighted portfolios taken as benchmark and GMV portfolios taken as target.

		$k=5$	$k=10$	$k=15$	$k=20$	$k=25$	$k=30$
$n=120$	Mean	-0.1040	-0.0539	-0.0554	-0.0480	-0.0508	-0.0504
	Variance	0.5757	0.5648	0.5918	0.5981	0.6435	0.6264
$n=250$	Mean	-0.0711	-0.0353	-0.0296	-0.0324	-0.0363	-0.0339
	Variance	0.5702	0.5406	0.5568	0.5493	0.5719	0.5432
$n=500$	Mean	-0.0501	-0.0293	-0.0259	-0.0230	-0.0256	-0.0210
	Variance	0.5648	0.5360	0.5407	0.5224	0.5488	0.5152
$n=1000$	Mean	-0.0348	-0.0196	-0.0191	-0.0180	-0.0186	-0.0122
	Variance	0.5637	0.5330	0.5343	0.5189	0.5339	0.5026
$n=2000$	Mean	-0.0219	-0.0142	-0.0135	-0.0126	-0.0131	-0.0100
	Variance	0.5672	0.5356	0.5306	0.5127	0.5247	0.4966
$n=3000$	Mean	-0.0182	-0.0080	-0.0109	-0.0098	-0.0107	-0.0080
	Variance	0.5653	0.5366	0.5336	0.5076	0.5267	0.4927
$n=5000$	Mean	-0.0163	-0.0075	-0.0075	-0.0061	-0.0051	-0.0037
	Variance	0.5649	0.5340	0.5313	0.5102	0.5217	0.4968
Asymptotic	Mean	0	0	0	0	0	0
	Variance	0.5639	0.5311	0.5309	0.5083	0.5212	0.4900

We present an adjusted estimator of the β coefficient. Note, that the random variable $(n-1)\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}}$ is χ^2_{n-k} distributed [17] and the random variable $(n-1)\frac{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}{\mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b}$ is χ^2_{n-1} distributed [24]. If these two random variables are independent then the finite sample distribution of $\hat{\beta}_{GMV}/\beta_{GMV}$ is $F_{n-1;n-k}$ with $E\hat{\beta}_{GMV} = \frac{n-k}{n-3}\beta_{GMV}$. We consider the adjusted estimator of the beta coefficient of the form $\hat{\beta}_{GMV;adj} = \frac{n-3}{n-k}\hat{\beta}_{GMV}$. In Table 2 and Fig. 2 it is presented the empirical and the corresponding asymptotic means, variances and densities of the random variable $\sqrt{n}\big(\hat{\beta}_{GMV;adj} - \beta_{GMV}\big)$. From the results of Table 2 and Fig. 2 we conclude that the performance of the adjusted estimator is better than the sample one. The convergence of empirical variances to the asymptotic ones is not significantly slower than for sample estimator but the bias of the adjusted estimator is significantly smaller.

Secondly, we construct the asymptotic confidence intervals at significance level $(1-\gamma) \in \{0.9, 0.95, 0.99\}$ for six beta coefficients (six GMV portfolios with dimensions $k \in \{5, 10, 15, 20, 25, 30\}$ taken as target portfolio and six equally weighted portfolio consisting of k corresponding assets taken

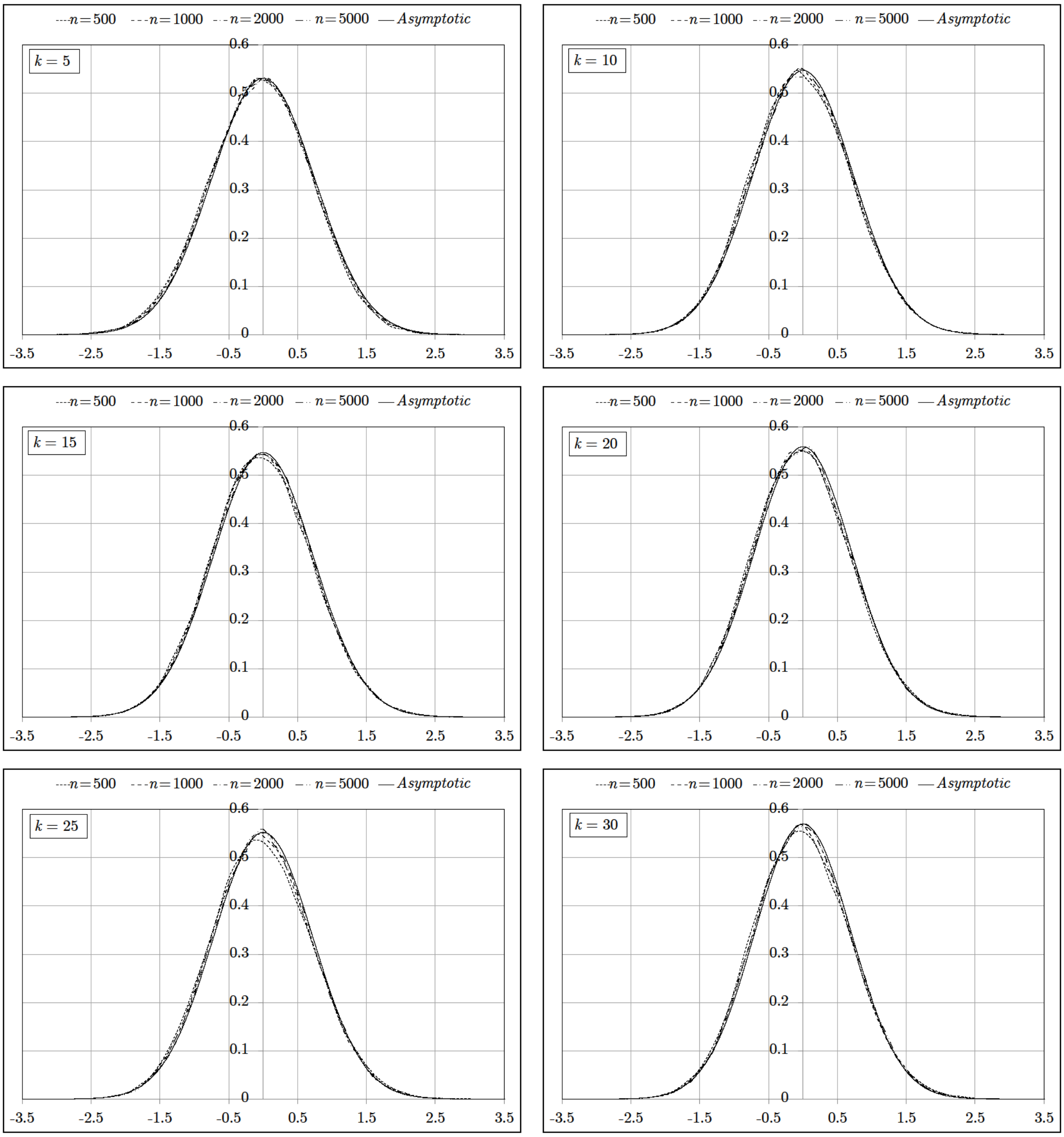

Fig. 2. Kernel density estimators for $n \in \{500, 1000, 2000, 5000\}$ and asymptotic density of $\sqrt{n}(\hat{\beta}_{GMV;adj} - \beta_{GMV})$ for different values of k.

as benchmark portfolio). We use the running window with the length $n = 250$. The results are presented in Fig. 3 for the sample estimator. We observe that in almost all cases the value of beta coefficient deviates significantly from one. Based on this observation we conclude that GMV portfolio risk is significantly smaller than the risk of the equally weighted portfolios. The results for the adjusted estimator are quite similar and the plots are not included in the paper. They are available from the authors for request.

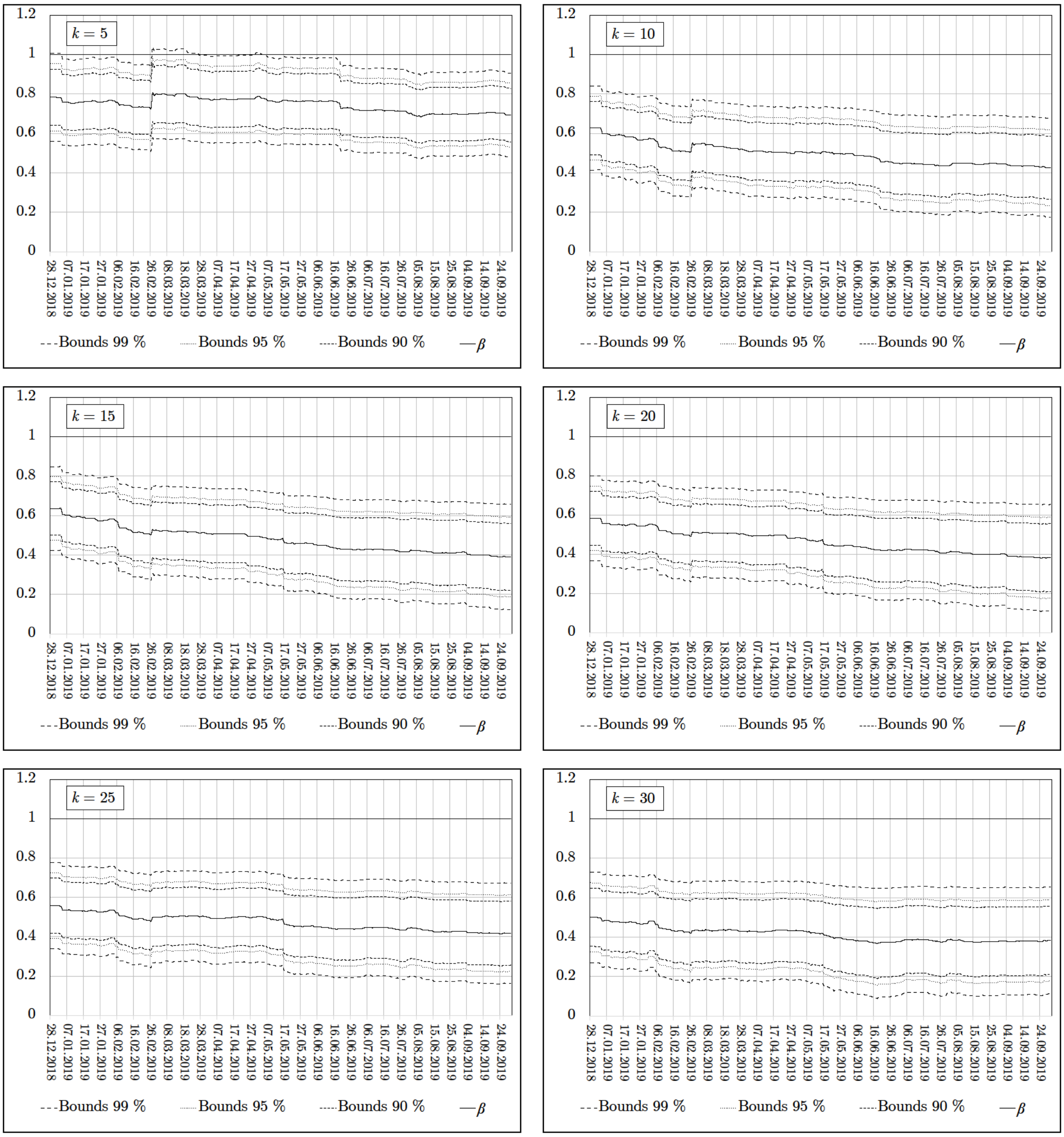

Fig. 3. Sample estimator and confidence intervals for β_{GMV} for different values of k.

4. Conclusions

In the paper, we consider the properties of the sample estimator of the beta coefficient in the case where the weights of the benchmark portfolio are constant and the weights of the target (investor's) portfolio are random. We deal with the situation when investor choose a global minimum variance portfolio. We provide an asymptotic analysis of the sample estimator by deriving its asymptotic density. Based on the asymptotic distribution of the sample estimator we construct an asymptotic interval estimator for the beta coefficient.

We provide empirical presentation of the theoretical results. Based on the daily returns on the assets included into the DAX index for the period from 01.01.2018 to 30.09.2019. We conclude that departures from asymptotic density, mean and variance are small compared to the empiric values even if the sample size is relatively small. We obtain that the bias of the sample estimator remains significantly different from 0 even for large sample size and it increases with the number of assets in portfolio. For example, for sample size $n = 5000$ and number of assets $k = 30$, the bias is equal to -0.1959. We present the adjusted estimator of the beta coefficient for which the bias is significantly smaller than in the case of the sample estimator and its values do not depend on the number of assets in portfolio. In the case of sample size $n = 5000$ and number of assets $k = 30$ the bias of the adjusted estimator is equal to -0.0037. Note that in practice this adjusted estimator should be used very carefully because its performance is not strictly substantiated.

[1] Markowitz H. Portfolio selection. Journal of finance. **7**, 77–91 (1952).

[2] Merton R. C. An analytical derivation of the efficient frontier. Journal of financial and quantitative analysis. **7** (4), 1851–1872 (1972).

[3] Okhrin Y., Schmid W. Distributional properties of optimal portfolio weights. Journal of econometrics. **134** (1), 235–256 (2006).

[4] Okhrin Y., Schmid W. Estimation of optimal portfolio weights. International journal of theoretical and applied finance. **11** (3), 249–276 (2008).

[5] Ingersoll J. E. Theory of financial decision making. New York, Rowman & Littlefield Publishers (1987).

[6] Sharpe W. F. The Sharpe ratio. The journal of portfolio management. **21** (1), 49–58 (1994).

[7] Schmid W., Zabolotskyy T. On the existence of unbiased estimators for the portfolio weights obtained by maximizing the Sharpe ratio. ASTA - Advances in Statistical Analysis. **92**, 29–34 (2008).

[8] Bodnar T., Zabolotskyy T. Maximization of the Sharpe ratio of an asset portfolio in the context of risk minimization. Economic annals - XXI. **11-12** (1), 110–113 (2013).

[9] Bodnar T., Zabolotskyy T. How risky is the optimal portfolio which maximizes the Sharpe ratio? ASTA - Advances in statistical analysis. **101** (1), 1–28 (2017).

[10] Alexander G. J., Baptista M. A. Economic implication of using a mean-VaR model for portfolio selection: a comparison with mean-variance analysis. Journal of economic dynamics & control. **26** (7-8), 1159–1193 (2002).

[11] Alexander G. J., Baptista M. A. A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. Management Science. **50** (9), 1261–1273 (2004).

[12] Bodnar T., Schmid W., Zabolotskyy T. Minimum VaR and Minimum CVaR optimal portfolios: estimators, confidence regions, and tests. Statistics & Risk Modeling. **29** (4), 281–314 (2012).

[13] Bodnar T., Schmid W. A test for the weights of the global minimum variance portfolio in an elliptical model. Metrika. **67**, 12–143 (2008).

[14] Bodnar T., Mazur S., Okhrin Y. Bayesian estimation of the global minimum variance portfolio. European journal of operational research. **256** (1), 292–307 (2017).

[15] Kan R., Smith D. R. The distribution of the sample minimum-variance frontier. Management. **54** (7), 1364–1380 (2008).

[16] Chan L. K. C., Karceski J., Lakonishok J. On portfolio optimization: forecasting and choosing the risk model. The review of financial study. **12** (5), 937–974 (1999).

[17] Bodnar T., Schmid W. Econometrical analysis of the sample efficient frontier. The European journal of finance. **15** (3), 317–335 (2009).

[18] Bodnar T., Gupta A. K., Vitlinskiy V., Zabolotskyy T. Statistical inference for the β coefficient. Risks. **7** (2), 56 (2019).

[19] Markowitz H. Foundations of portfolio theory. Journal of finance. **7**, 469–477 (1991).

[20] Ling S., McAleer M. Asymptotic theory for a vector ARMA-GARCH model. Econometric theory. **19** (2), 280–310 (2003).

[21] Harville D. A. Matrix algebra from a statistician's perspective. New York, Springer Science+Business Media (2008).

[22] Brockwell P. J., Davis R. A. Time series: theory and methods. New York, Springer Science+Business Media (2006).

[23] DasGupta A. Asymptotic theory of statistics and probability. New York, Springer (2008).

[24] Muirhead R. J. Aspects of multivariate statistical theory. New York, Wiley (1982).

Властивості бета коефіцієнта портфеля з найменшою дисперсією

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Стаття присвячена дослідженню статистичних властивостей вибіркової оцінки бета коефіцієнта у випадку, коли ваги еталонного портфеля є постійні, а цільовим є портфель з найменшою дисперсією. Знайдено асимптотичний розподіл вибіркової оцінки бета коефіцієнта за припущення, що вектор дохідностей активів має багатовимірний нормальний розподіл. На основі асимптотичного розподілу побудовано довірчий інтервал для бета коефіцієнта. Використовуючи щоденні дохідності акцій, включених до індексу DAX за період з 01.01.2018 по 30.09.2019, порівняно емпіричні та асимптотичні середні, дисперсії та щільності стандартизованої вибіркової оцінки бета коефіцієнта. Зауважено, що для великої кількості активів у портфелі зміщення стандартизованої вибіркової оцінки бета коефіцієнта збігається до нуля дуже повільно. Представлено скориговану оцінку бета коефіцієнта, для якої збіжність емпіричних дисперсій до асимптотичних не є значно повільнішою, ніж для вибіркової оцінки, але зміщення скоригованої оцінки є істотно меншим.

Ключові слова: *портфель з найменшою дисперсією, бета коефіцієнт, теорія тестування, асимптотичний розподіл, невизначеність параметрів.*