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THE FORMULAS FOR SUM OF PRODUCTS OF SEQUENCES ASSOCIATED WITH THE METALLIC MEANS

In this paper, the regularities of convolution of sequences c of Fibonacci numbers {F_n} generated by metallic means and the sum of products of two statistically independent sequences {F_i} and $J_n=j\cdot\sin(0.5\pi(n-j))$ are investigated. It is shown that the known closed forms of sums for convolution $\sum_{j=1}^{n} F_iF_{n-i}$ and product $\sum_{j=1}^{n-1} jF_j\cos\frac{(n-j-1)\pi}{2}$ are similar. Attention to the study of the convolution of two sequences of discrete data is associated with the use of this method for statistical signal processing. This problem involves calculating finite sums as discrete analogs of definite integrals. Such a problem is considered solved if the formula for the sum is expressed in a closed form as a function of its members and their number.

Key words: Golden ratio, metallic means, Fibonacci sequences, the roots of the quadratic equation

Introduction

Attention to the study of the convolution of two sequences of discrete data is associated with the use of this method for statistical signal processing [1-2]. This problem involves calculating finite sums as discrete analogs of definite integrals. Such a problem is considered solved if the formula for the sum is expressed in a closed form as a function of its members and their number. The convolution of two F_iF_{n-i} recurrent Fibonacci-Luca sequences was studied in [3-5].

Problem statement

In this paper, the regularities of convolution of sequences c of Fibonacci numbers {F_n} generated by metallic means [6] and the sum of products of two statistically independent sequences {F_n} and $J_n=j\cdot\sin(0.5\pi(n-j))$ $J_n = j\times\sin\frac{(n-j)\pi}{2}$ [7] are investigated. It is shown that the known closed forms of sums for convolution $\sum_{i=0}^{n} F_iF_{n-i}$ and product $\sum_{j=1}^{n-1} jF_j\cos\frac{(n-j-1)\pi}{2}$ are similar.

Theoretical results

Let be a two-dimensional system of Cartesian rectangular coordinates $p0q$ divided into four quadrants $p>0, q>0$ (first), $p>0, q<0$ (second), $p<0, q<0$ (third) i $p<0, q>0$ (fourth). In the third quadrant, there is a phase direction of points with coordinates $p\geq 1$, $q=-1$, which has the following positive solutions*:*

$$\varphi_p = \frac{p+\sqrt{p^2+4}}{2} \tag{1}$$

for the following quadratic equation

$$\varphi_p^2 - p\varphi_p - 1 = 0\,, \tag{2}$$

known as the (quadratic) metallic means. So $\varphi_{p=2} = 1+\sqrt{2}$ is known as the silver mean, $\varphi_{p=3} = \frac{1}{2}(1+\sqrt{3})$ is called the bronze means.

Consider the decomposition of a square trinomial (2) in the form

$$\varphi_p^{\,n} = \alpha_n \varphi_p + \beta_n \tag{3}$$

for positive roots $\varphi_p(\pm p,\ q=-1) > 0$ in second (φ_{II}) and the third (φ_{III}) quadrants:

$$\begin{cases} \varphi_{II} = \dfrac{-p+\sqrt{p^2+4}}{2} > 0,\ \ \varphi_{III} = \dfrac{p+\sqrt{p^2+4}}{2} > 0, \\ \varphi_{II,III}^{\,0} = 0\cdot\varphi_{II,III} + 1, \\ \varphi_{II,III}^{\,1} = 1\cdot\varphi_{II,III} + 0, \\ \varphi_{II,III}^{\,2} = \mp p\varphi_{II,III} + 1, \\ \varphi_{II,III}^{\,3} = (p^2+1)\varphi_{II,III} \mp p, \\ \varphi_{II,III}^{\,4} = \mp p(p^2+2)\varphi_{II,III} + (p^2+1), \\ \varphi_{II,III}^{\,5} = [p^2(p^2+2)+(p^2+1)]\varphi_{II,III} \mp p(p^2+2), \\ \varphi_{II,III}^{\,6} = \mp[p(p^2+3)(p^2+1)]\varphi_{II,III} + p^2(p^2+2)+(p^2+1), \\ \qquad\qquad ... \end{cases} \Rightarrow \begin{cases} \alpha_0 = 0, \beta_0 = 1, \\ \alpha_1 = 1, \beta_1 = 0, \\ \varphi_{II,III}^{\,n} = \mp\alpha_n \varphi_{II,III} \mp \beta_n \end{cases} \tag{4}$$

For the points $p=\pm 2, q=-1$ the sequences $\{\alpha_n\}$ have the following form:

$$\begin{cases} n & 0\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \alpha_n, & 0, 1, & \mp 2, & +5, & \mp 12, & +29, & \mp 70, & +169, ... \\ \beta_n & 1, 0, & +1, & \mp 2, & +5, & \mp 12, & +29, & \mp 70,\ \ ,.. \end{cases} \tag{5}$$

Discussion

In the second quadrant, the sequence $\{\alpha_n = \alpha 2_n\}$ is alternating, it oscillates with the increasing amplitude around the equilibrium value $\alpha = 0$ (Fig.1a). The Members of the sequence $\{\alpha_n\}$ are calculated by a recurrent formula

$$\alpha_n = p\cdot\alpha_{n-1} + \alpha_{n-2}, \quad n \geq 2\,. \tag{6}$$

In the third quadrant, the sequence $\{\alpha_n = \alpha 3_n\}$ changes monotonically as the envelope of oscillations $\{\alpha_n = \alpha 2_n\}$ (Fig..1). The limit value for members of both neighboring sequences varies and goes to the root values (Fig.1)

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} \cong \mp 2.414\,. \tag{7}$$

In general, $\dfrac{\alpha_n}{\alpha_{n-1}}$ i $\dfrac{\alpha_n}{\alpha_{n-2}}$ adjacent sequence values $\{\alpha_n(\alpha_0,\alpha_1;p,q)\}$ in points p,q along the first root lines are expressed employing chain fractions with the corresponding limits

$$\begin{cases} \dfrac{\alpha_n}{\alpha_{n-1}} = p + \dfrac{q}{\alpha_{n-1}/\alpha_{n-2}} = p + \dfrac{q}{p + \dfrac{q}{\alpha_{n-2}/\alpha_{n-3}}} = p + \dfrac{q}{p + \dfrac{q}{p + \dfrac{q}{\dots}}} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n-1}} = \max(x_\pm), \\ \dfrac{\alpha_n}{\alpha_{n-2}} = q + \dfrac{p}{\alpha_{n-1}/\alpha_{n-2}} = q + \dfrac{p}{q + \dfrac{p}{\alpha_{n-2}/\alpha_{n-3}}} = q + \dfrac{p}{q + \dfrac{p}{q + \dfrac{p}{\dots}}} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n-1}} = \max(x_\pm)^2, \end{cases} \tag{8}$$

oscillating or changing monotonically in the second quadrant under arbitrary initial conditions. The sequence $\{\alpha_n = \alpha 3_n\}$ refers to the classical Fibonacci sequence

$$F_n = pF_{n-1} + F_{n-2},\ n \ge 2,\ F_0 = 0, F_1 = 1, p = 2, q = 1. \tag{9}$$

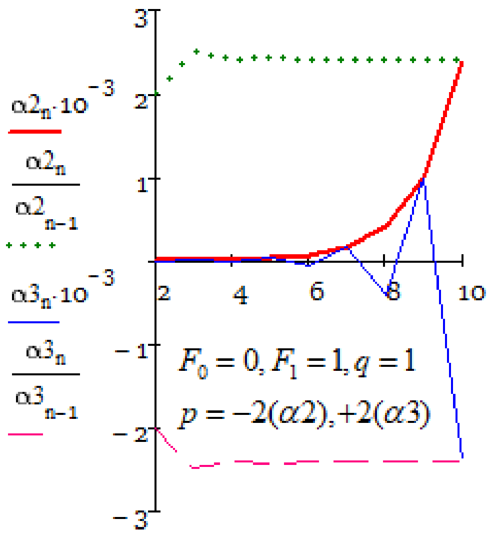

Fig. 1 Dynamics of changes in the recurrent sequence (9)
and the relations of its neighboring members

Consider the convolution of two sequences (9) generated in the second ($\alpha_n * \alpha_n$) and the third ($F_n * F_n$) quadrants, which are calculated as the sums $\sum_{i=0}^{n} \alpha_i \cdot \alpha_{n-i}$ (table 1) and $\sum_{i=0}^{n} F_i \cdot F_{n-i}$ (table 2) products of statistically dependent multipliers by the algorithm [9]. As follows from tables 1 i 2, the convolutions differ only in the sign of the summation result.

The Regularities of products $\alpha_i \cdot \alpha_{n-i}$, $\dfrac{\alpha_i}{\alpha_{i+1}} \cdot \dfrac{\alpha_{n-i}}{\alpha_{n-i+1}}$ and $F_i \cdot F_{n-i}$, $\dfrac{F_i}{F_{i+1}} \cdot \dfrac{F_{n-i}}{F_{n-i+1}}$ are similar to each other, taking into account the sign. The Regularities of the formation of sums $\sum_{i=0}^{n} \alpha_i \cdot \alpha_{n-i}$ and $\sum_{i=0}^{n} F_i \cdot F_{n-i}$ statistically dependent multipliers of products are shown in Figure 2.

Table 1

Convolution the two sequences $\{\alpha_n\}$ for n=7

j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Sum
α_i	0	1	-2	5	-12	29	-70	169	
α_{7-i}	169	-70	29	-12	5	-2	1	0	
$\sum_{i=0}^{n}\alpha_i\cdot\alpha_{n-i}$	0	-70	-58	-60	-60	-58	-70	0	-376

Table 2

Convolution the two sequences $\{F_n\}$ for n=7

j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Sum
F_i	0	1	2	5	12	29	70	169	
F_{7-i}	169	70	29	12	5	2	1	0	
$\sum_{i=0}^{n}F_i\cdot F_{n-i}$	0	70	58	60	60	58	70	0	376

For the recurrent sequences (9), a closed-form of the sum $\sum_{i=0}^{n}F_iF_{n-i}$ with arbitrary values of the coefficients p,q was first found in [7].

$$\sum_{i=0}^{n}F_iF_{n-i}=\frac{(n-1)pF_n+2nqF_{n-1}}{p^2+4q}=\frac{(n-1)pF_n+2nF_{n-1}}{p^2+4}=\Sigma(n,p,q,F)\,. \tag{10}$$

For $p=1,q=1$ and $F_0=0,F_1=1$ formula (13) has the form

$$\sum_{i=1}^{n}F_iF_{n-i}=\frac{nL_{n+1}+2F_{n-1}}{5}\,. \tag{11}$$

As can be seen from Fig. 2, the formula for the sum in closed form (10) for the sequence $\{\alpha_n\}$ does not come true but comes true only at the vertices of oscillations.

The following formula was found for the sum [7]

$$\sum_{j=1}^{n-1}F_j\cdot F_{n-j}=\sum_{j=1}^{n-1}F_j\cdot J_j=\sum_{j=0}^{n}F_j\cdot j\cdot\cos\frac{(n-j-1)\pi}{2}\,. \tag{12}$$

In the formula (12), ~~a~~ multipliers F_j i J_j *are* statistically independent, so the sum $\sum_{j=1}^{n-1}F_j\cdot J_j$ is valid only for the sequence $\{F_n\}$(Fig.2). Finally, we present other formulations of the formula (12) :

$$\sum_{i=0}^{n}F_iF_{n-i}=\sum_{j=0}^{n}F_j\cdot(n-j)\sin\frac{j\pi}{2}=\sum_{j=0}^{n}F_{n-j}\cdot j\sin\frac{(n-j)\pi}{2}\,. \tag{13}$$

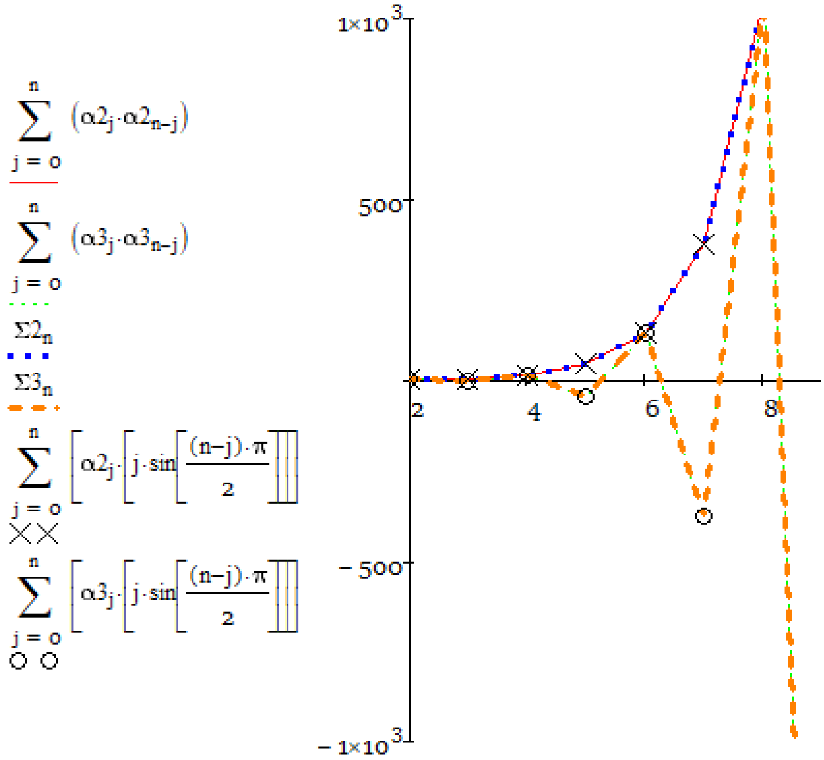

Fig. 2. Comparative analysis of convolution of the sequence terms and (12) – (13)

We formulate the next two propositions:

Proposition 1.

$$\sum_{i=0}^{n} F_i F_{n-i} = \sum_{j=0}^{n} F_j \cdot (n-j)\sin\frac{j\pi}{2} = \sum_{j=0}^{n} F_{n-j} \cdot j\sin\frac{(n-j)\pi}{2} \qquad (14)$$

Proposition 2.

$$\sum_{i=0}^{n} (n-j)F_{n-j}\cos\frac{(j-1)\pi}{2} = n\sum_{i=0}^{n} F_{n-j}\cos\frac{(j-1)\pi}{2} - \sum_{i=0}^{n} jF_{n-j}\cos\frac{(j-1)\pi}{2} = n\sum_{i=0}^{n} F_{n-j}\sin j\frac{\pi}{2} - \sum_{i=0}^{n} jF_{n-j}\sin j\frac{\pi}{2} \qquad (15)$$

Since for initial condition $F_0 = 0, F_1 = 1$ satisfy equal -

$$\sum_{i=0}^{n} (n-j)F_{n-j}\sin j\frac{\pi}{2} = \sum_{i=0}^{n} jF_j \sin j\frac{\pi}{2}, \qquad (16)$$

then

$$\sum_{i=0}^{n} (n-j)F_{n-j}\sin j\frac{\pi}{2} = \sum_{i=0}^{n} jF_j\cos\frac{(n-j-1)\pi}{2} = \sum_{i=0}^{n} jF_j \sin j\frac{\pi}{2} \qquad (17)$$

Conclusion and future work

In this paper, the regularities of convolution of sequences c of Fibonacci numbers $\{F_n\}$ generated by metallic means and the sum of products of two statistically independent sequences $\{F_i\}$ and $J_n=j\cdot\sin(0.5\pi(n-j))$ are investigated. It is shown that the known closed forms of sums for convolution $\sum_{i=0}^{n} F_i F_{n-i}$ and product $\sum_{j=1}^{n-1} jF_j\cos\frac{(n-j-1)\pi}{2}$ are similar.

To further our developing we are planning to expand the idea of using explored parameters in the algorithm of modeling systems with the different parameters

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ФОРМУЛИ СУМ ДОБУТКІВ ПОСЛІДОВНОСТЕЙ, ЗВ’ЯЗАНИХ З МЕТАЛІЧНИМИ СЕРЕДНІМИ

У цій роботі досліджуються закономірності згортки послідовностей сум чисел Фібоначчі {F_n}, породжених металевими середніми, та суми добутків двох статистично незалежних послідовностей {F_n} та $J_n=j\cdot\sin(0.5\pi(n-j))$. Показано, що відомі закриті форми сум для згортки $\sum_{j=1}^{n} F_i F_{n-i}$ та добутків $\sum_{j=1}^{n-1} jF_j \cos\frac{(n-j-1)\pi}{2}$ є подібними. Така увага до вивчення згортки двох послідовностей дискретних даних пов'язана із застосуванням цього методу для статистичної обробки сигналів. Ця задача передбачає обчислення скінченних сум як дискретних аналогів певних інтегралів. Така проблема вважається вирішеною, якщо формула суми виражається у закритому вигляді як функція її членів та їх кількості.

Ключові слова: Золоте січення, металеві середні, послідовності Фібоначчі, корені квадратного рівняння