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A Frequency criterion for analysis of stability of systems with fractional-order derivatives

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Based on an analysis of the influence of the roots of a characteristic polynomial on the increment of the argument of the frequency characteristic of the system, the frequency criterion of stability of a system with fractional-order derivatives has been suggested. The boundaries of the zone of location of the roots of the characteristic polynomial of a stable system have been determined in a complex plane when the index α of the basis of the characteristic polynomial changes.

Keywords: *fractional-order derivative, stability criterion, stability boundaries.*

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1. Introduction

In modern approaches to the creation of models of processes and of the synthesis of control actions in dynamic systems, there are used fractional derivatives and integrals, which describe so-called fractal peculiarities of such processes [1, 2]. In particular, the design of some electromechanical systems using fractional-order regulators significantly advances the possibilities of optimization of such systems [3–6]. The problems of analysis and synthesis of dynamic systems which are described by equations with derivatives of fractional orders have recently received considerable attention [7, 8]. Despite considerable interest [9, 10], the problem of creation of a quite simple criterion of stability, similarly to the systems with integer derivatives, for the systems with fractional-order derivatives still remains unsolved. Analogously to the linear invariant systems of integer order, the stability of the linear systems of fractional order, as it is known, depends on the location of the poles of the system in a complex plane. On this basis, for the fractional-order systems the root stability criteria have been proposed, the most famous of which is the Matignon's stability theorem [11] and the Routh–Hurwitz criterion [12]. However, in the general case, the analysis of the location of the poles remains a rather difficult task. Along with this, a significant number of papers are devoted to the creation of a frequency criterion of stability. In particular, in [13] for the fractional-order systems, it is proposed to use the Mikhailov criterion [14] after the transition to the integer polynomial with the basis $s^{1/\alpha}$. With this, to eliminate the problem of high degrees of polynomials, it is proposed to analyze the argument of the modified characteristic polynomial of the system of the form $\frac{\sum_{k=0}^{m} a_{m-k}\cdot(s^{1/\alpha})^k}{a_0\cdot(s+\lambda)^{m/\alpha}}$, where λ is a natural number. The system will be stable if the argument of the modified polynomial is equal to zero. In [7], there is demonstrated the application of this approach to the subsystems with different bases of fractional-order derivatives, to the positively defined systems, and to the systems with time delay. In [15], a stability criterion is proposed for linear fractional-order systems with $s^{\frac{1}{\alpha}}$, where α satisfies $1 < 1/\alpha < 2$, being formed on the basis of the analysis of the intersection of the real and imaginary axes by the hodograph of the system. In [16], the application of the Lyapunov method for the analysis of stability of systems with fractional derivatives and the problems in the formation of the Lyapunov function are shown. In [17], the possibility of using the quadratic Lyapunov function to analyze the stability of the system is proved. The application of the apparatus of matrix inequalities for the formation of the stability criterion of a system is shown in [18].

Given the conducted analysis of the use by many authors of the argument of the characteristic polynomial function to determine the domain of the roots of a stable system, it is expedient, in our opinion, to use the frequency domain to analyze the stability of systems with fractional-order derivatives.

2. The problem formulation

A characteristic polynomial of a closed dynamic system with integer derivatives has the form:

$$H(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n. \tag{1a}$$

In fractional-order systems, when the Riemann–Liouville or Caputo formulae are used to describe a derivative of the fractional order, we pass to such a form of the polynomial by applying the Laplace transform by finding the common denominator α of all terms of the fractional order:

$$H(d) = b_0 d^m + b_1 d^{m-1} + b_2 d^{m-2} + \ldots + b_{m-1} d + b_m. \tag{1b}$$

where $d = s^{1/\alpha}$. With this, the degree of the polynomial increases significantly. Thus, in this case, when $\alpha = 16$, then to obtain an analogue of an integer polynomial of the n-th order $m = 16n$.

According to the basic theorem of algebra, any polynomial can be written as follows:

$$H(s) = a_0 (s - p_1)(s - p_2)(s - p_3) \ldots (s - p_n). \tag{2a}$$

where $p_1, p_2, p_3, \ldots, p_n$ are polynomial roots. Accordingly, for a polynomial of fractional order, the form of the polynomial is as follows:

$$\begin{aligned} H(s^{1/\alpha}) &= b_0 (d - p_1)(d - p_2)(d - p_3) \ldots (d - p_m) \\ &= b_0 (s^{1/\alpha} - p_1)(s^{1/\alpha} - p_2)(s^{1/\alpha} - p_3) \ldots (s^{1/\alpha} - p_m). \end{aligned} \tag{2b}$$

When passing to the frequency domain by substituting $s \to j\omega$ for an integer polynomial we obtain:

$$\begin{aligned} H(j\omega) &= a_0 (j\omega)^n + a_1 (j\omega)^{n-1} + a_2 (j\omega)^{n-2} + \ldots + a_{n-1}(j\omega) + a_n \\ &= a_0 (j\omega - p_1)(j\omega - p_2)(j\omega - p_3) \ldots (j\omega - p_n). \end{aligned} \tag{3a}$$

and, accordingly, for a polynomial with fractional coefficients:

$$\begin{aligned} H\big((j\omega)^{1/\alpha}\big) &= b_0 \big((j\omega)^{1/\alpha}\big)^m + b_1 \big((j\omega)^{1/\alpha}\big)^{m-1} + b_2 \big((j\omega)^{1/\alpha}\big)^{m-2} + \ldots + b_{m-1} \big(j\omega\big)^{1/\alpha} + b_m \\ &= b_0 \big((j\omega)^{1/\alpha} - p_1\big)\big((j\omega)^{1/\alpha} - p_2\big)\big((j\omega)^{1/\alpha} - p_3\big) \ldots \big((j\omega)^{1/\alpha} - p_m\big) \\ &= b_0 \big(j^{1/\alpha}\omega^* - p_1\big)\big(j^{1/\alpha}\omega^* - p_2\big)\big(j^{1/\alpha}\omega^* - p_3\big) \ldots \big(j^{1/\alpha}\omega^* - p_m\big). \end{aligned} \tag{3b}$$

where $\omega^* = \omega^{1/\alpha} = \sqrt[\alpha]{\omega}$.

In the general case, the trigonometric form of a complex number $z = a + j\,b$ is written as:

$$z = r\,(\cos\varphi + j\,\sin\varphi) = r\,e^{j\varphi},$$

where $r = \sqrt{a^2 + b^2}$, and $\varphi = \arctan(a/b)$.

Thus, for an integer polynomial:

$$H(j\omega) = a_0 \prod_{i=1}^{n} \big(r_i e^{j\,\varphi_i}\big) = a_0 \prod_{i=1}^{n} r_i e^{j\sum_{i=1}^{n}\varphi_i},$$

where the argument of the function $H(j\,\omega)$ is equal to $\varphi = \sum_{i=1}^{n} \varphi_i$, is determined as a sum of the arguments of all multipliers.

Let us analyze the influence of the roots on the argument of the function $H(j^{1/\alpha}\omega^*)$ in the case of a fractional-order system. Consider two cases: the first case, when any number α is within the range $\alpha \in [1;\infty]$ and the second one, when $\alpha \in]0;1[$ which includes the range $1 < 1/\alpha \leqslant 2$, which is used to analyze fractional-order systems.

When using de Moivre's formula for natural α:

$$\sqrt[\alpha]{z} = \sqrt[\alpha]{|z|}\left(\cos\frac{\arg(z)+2k\pi}{\alpha} + j\sin\frac{\arg(z)+2k\pi}{\alpha}\right)$$

we obtain:

$$\sqrt[\alpha]{j} = \sqrt[\alpha]{1}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{\alpha} + j\sin\frac{\frac{\pi}{2}+2k\pi}{\alpha}\right)$$

and for $k=0$: $\sqrt[\alpha]{j} = \cos\frac{\pi}{2\alpha} + j\sin\frac{\pi}{2\alpha}$. In the case of fractional $\alpha = \lambda/\beta$ for $k=0$ we obtain $j^{1/\alpha} = j^{\frac{\beta}{\lambda}} = \sqrt[\lambda]{j^\beta} = \cos\frac{\pi\beta}{2\lambda} + j\sin\frac{\pi\beta}{2\lambda}$

For the first case $\alpha \in [1;\infty]$, let us demonstrate the influence of roots on the change of the argument of the function $H(j^{1/\alpha}\omega^*)$.

a) The case of the real root in the left half-plane (Fig. 1).

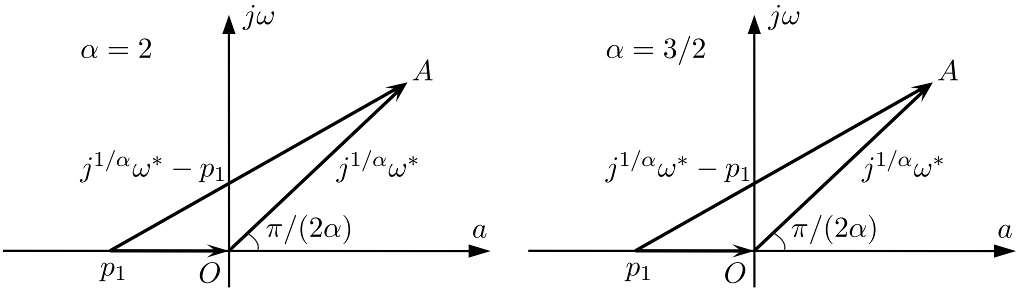


Fig. 1.

For the frequency $\omega^* = 0$, the vector $j^{1/2}\omega^* - p_1$ occupies a position $-p_1$. In the case $\alpha = 2$ with increasing frequency $\omega^* \to \infty$, the rotation angle of the vector $j^{1/2}\omega^* - p_1$ is equal to $\pi/4 = \pi/(2\cdot 2)$ counterclockwise. And in the case $\alpha = 3/2$ with increasing frequency $\omega^* \to \infty$, the rotation angle of the vector $j^{2/3}\omega^* - p_1$ is equal to $\pi/3 = \pi/(2\cdot 3/2)$ counterclockwise. Thus, the argument φ of the function $H(j^{1/\alpha}\omega^*)$ in the case of a real root being in the left half-plane increments by $\Delta\varphi = \pi/(2\alpha)$. If the polynomial (3b) has c real roots, the increment of the argument will be $\Delta\varphi = c\pi/(2\alpha)$.

b) The case of the real root in the right half-plane (Fig. 2).

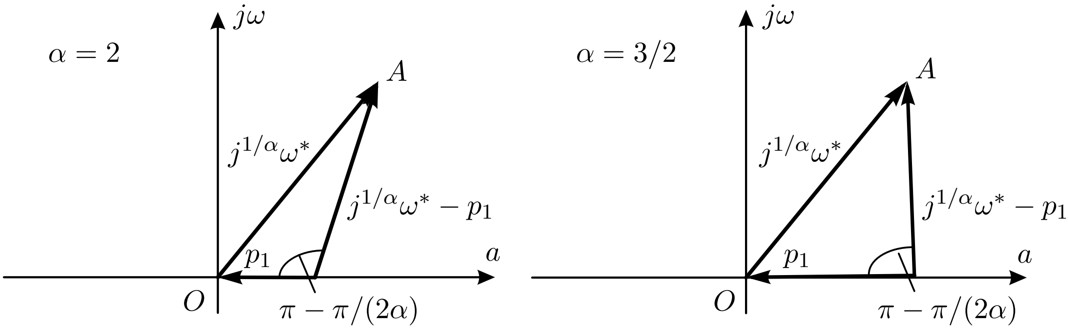
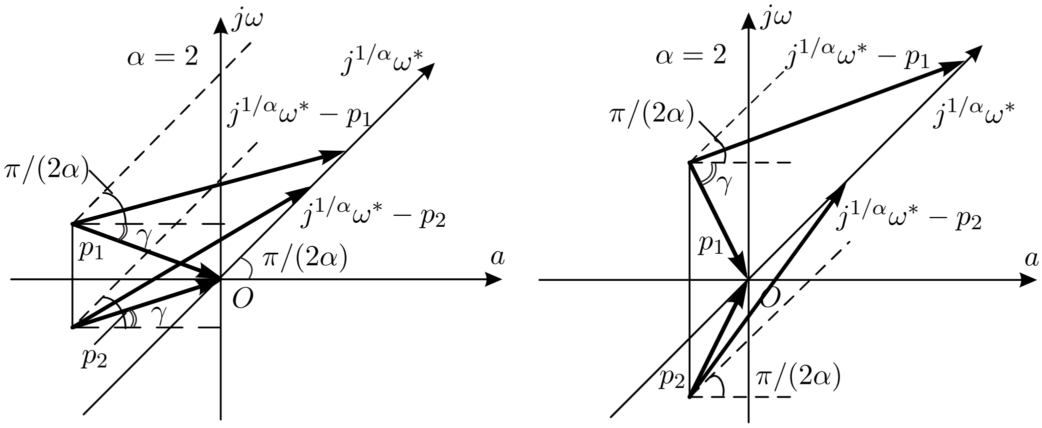


Fig. 2.

For the frequency $\omega^* = 0$ the vector $j^{1/\alpha}\omega^* - p_1$ occupies a position p_1. With increasing frequency $\omega^* \to \infty$, the rotation angle of the vector $j^{1/2}\omega^* - p_1$ is equal to $3\pi/4 = \pi - \pi/4$ clockwise, and the rotation angle of the vector $j^{2/3}\omega^* - p_1$ is equal to $2\pi/3 = \pi - \pi/3$ clockwise. Thus, the argument φ of the function $H(j^{1/\alpha}\omega^*)$ in the case of a real root being in the right half-plane increments by $\Delta\varphi = -(\pi - \pi/(2\alpha))$.

c) Consider the case of a pair of complex-conjugate roots in the left half-plane (Fig. 3) without losing generality when $\alpha = 2$.



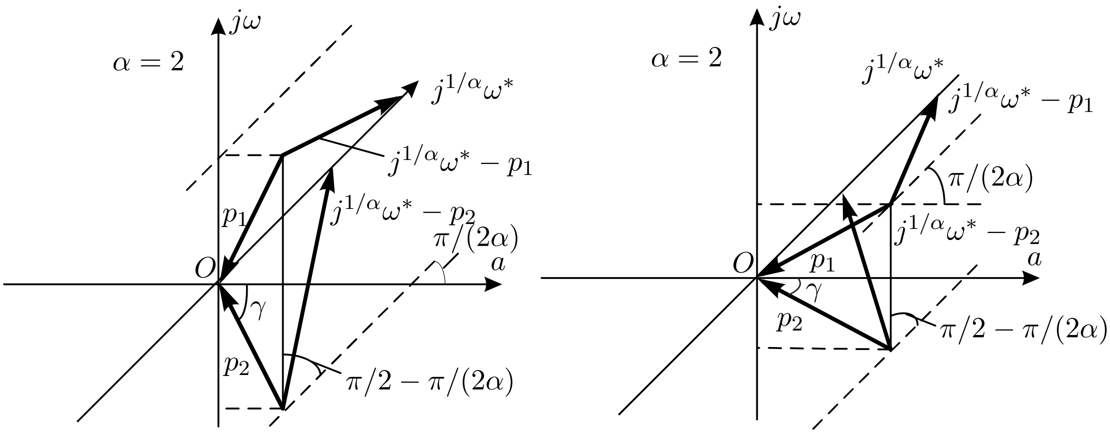
a ***b***

Fig. 3.

In the case of location of roots which corresponds to Fig. 3a, when changing the frequency from $\omega^* = 0$ till $\omega^* \to \infty$, the rotation angle of the vector $j^{1/2}\omega^* - p_1$ is equal to $\pi/4 + \gamma$ counterclockwise, the rotation angle of the vector $j^{1/2}\omega^* - p_2$ is equal to $\pi/4 - \gamma$ counterclockwise too. Thus, the argument φ of the function $H(j^{1/2}\omega^*)$ changes by the value $\pi/4 + \gamma + \pi/4 - \gamma = 2\pi/4$. In the case of location of roots which corresponds to Fig. 3b, when changing the frequency from $\omega^* = 0$ till $\omega^* \to \infty$, the rotation angle of the vector $j^{1/2}\omega^* - p_1$ is equal to $\pi/4 + \gamma$ counterclockwise, and the rotation angle of the vector $j^{1/2}\omega^* - p_2$ is equal to $\gamma - \pi/4$ clockwise. The argument φ of the function $H(j^{1/2}\omega^*)$ changes by the value $\pi/4 + \gamma - (\gamma - \pi/4) = 2\pi/4$.

In the general case, each pair of complex-conjugate roots in the left half-plane changes the argument φ of the function $H(j^{1/\alpha}\omega^*)$ by the value $\Delta\varphi = 2\pi/(2\alpha)$.

d) Consider the case of a pair of complex-conjugate roots in the right half-plane (Fig. 4) without losing generality also when $\alpha = 2$.



a ***b***

Fig. 4.

In the case of location of roots which corresponds to Fig. 4a, when changing the frequency from $\omega^* = 0$ till $\omega^* \to \infty$, the rotation angle of the vector $j^{1/2}\omega^* - p_1$ is equal to $5\pi/4 - \gamma$ counterclockwise, and the rotation angle of the vector $j^{1/2}\omega^* - p_2$ is equal to $3\pi/4 - \gamma$ clockwise. Thus, the argument φ of the function $H(j^{1/2}\omega^*)$ changes by the value $5\pi/4 - \gamma - (3\pi/4 - \gamma) = 2\pi/4$. In the general case, each pair of complex-conjugate roots in the right half-plane located in accordance with Fig. 4a changes the argument φ of the function $H(j^{1/\alpha}\omega^*)$ by the value $\Delta\varphi = 2\pi/(2\alpha)$.

In the case of location of roots which corresponds to Fig. 4*b*, when the frequency is changing from $\omega^* = 0$ till $\omega^* \to \infty$, the rotation angle of the vector $j^{1/2}\omega^* - p_1$ is equal to $3\pi/4 + \gamma$ clockwise, and the rotation angle of the vector $j^{1/2} \cdot \omega^* - p_2$ is equal to $3\pi/4 - \gamma$ clockwise. The argument φ of the function $H(j^{1/2}\omega^*)$ changes by the value $-(3\pi/4 + \gamma) - (3\pi/4 - \gamma) = -2 \cdot 3\pi/4 = -2(\pi - \pi/4)$. Thus, each pair of complex-conjugate roots in the right half-plane located in accordance with Fig. 4*b* changes the argument φ of the function $H(j^{1/\alpha}\omega^*)$ by the value $\Delta\varphi = -2(\pi - \pi/(2\alpha))$.

In the case $\alpha = 1$, we transit to an integer system. The function $H(j^{1/\alpha}\omega^*) = H(j\omega)$, the axis $j^{1/\alpha}\omega^*$ coincides with the axis $j\omega$ and for $\omega \to \infty$ the argument of the function $\varphi = \sum_{i=1}^{n} \varphi_i = \frac{n \cdot \pi}{2}$ since each real root located in the left half-plane provides the increment $\Delta\varphi = \pi/(2\alpha) = \pi/2$, and each pair of complex-conjugate roots, which are located in the left half-plane, changes φ by the value $\Delta\varphi = 2\pi/(2\alpha) = 2\pi/2$. As it is known, the location of the roots of an integer system in the left half-plane automatically ensures the stability of such a system.

By analogy with an integer system, a necessary condition for the stability of a system with derivatives of fractional order is to ensure the equality of the argument φ of the function $H(j^{1/\alpha}\omega^*)$ to the value $m\pi/(2\alpha)$ (where m is the polynomial degree (3b) when $\omega^* \to \infty$. And accordingly, a polynomial of the form (3b) can be interpreted, by analogy with integer systems, as a characteristic polynomial of a system with derivatives of fractional order.

The conducted analysis of the influence of the location of the roots of the polynomial (3b) in the case $\alpha \in [1; \infty]$ makes it possible to determine the boundaries of the sector of unstable operation of the system in the right half-plane as $\pm\pi/(2\alpha)$.

Let us demonstrate the influence of roots on the change of the argument of the function $H(j^{1/\alpha}\omega^*)$ for the case $\alpha \in]0; 1[$.

a) The case of the real root in the left half-plane (Fig. 5).

The resulting rotation angle of the vector $j^{3/2}\omega^* - p_1$ when changing the frequency from 0 to ∞ is equal to $3\pi/4$ counterclockwise. Therefore, the argument φ of the function $H(j^{1/\alpha}\omega^*)$ in the case of a real root being in the left half-plane also increments by $\Delta\varphi = \pi/(2\alpha)$.

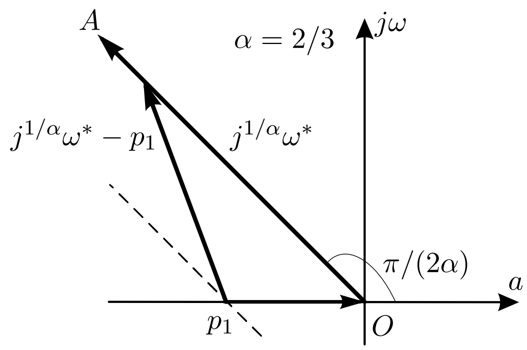


Fig. 5.

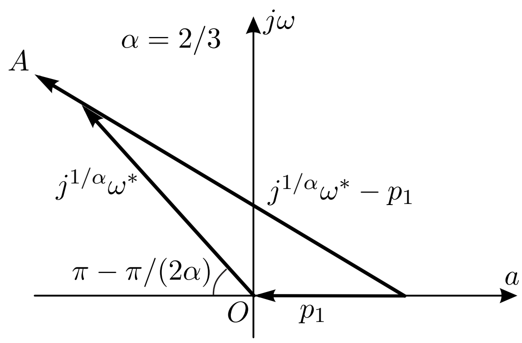


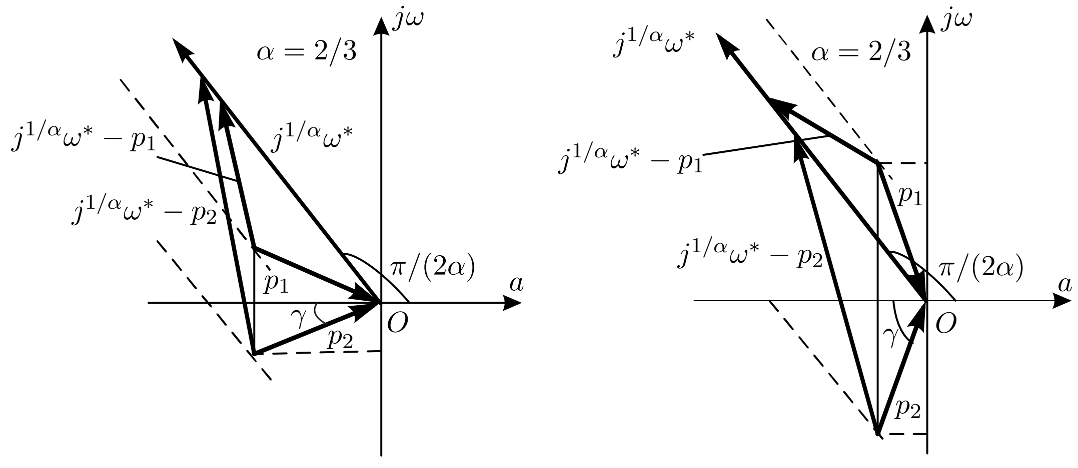
Fig. 6.

b) The case of the real root in the right half-plane (Fig. 6).

The resulting rotation angle of the vector $j^{3/2}\omega^* - p_1$ when changing the frequency from 0 to ∞ is equal to $\pi/4$ clockwise. Thus, the argument φ of the function $H(j^{1/\alpha}\omega^*)$ in the case of a real root being in the right half-plane increments by $\Delta\varphi = -(\pi - \pi/(2\alpha)$.

c) Consider the case of a pair of complex-conjugate roots in the left half-plane (Figs. 7) without losing generality when $\alpha = 2/3$.

In the case shown in Fig. 7*a* when changing the frequency from $\omega^* = 0$ to $\omega^* \to \infty$, the rotation angle of the vector $j^{3/2}\omega^* - p_1$ is equal to $3\pi/4 + \gamma$ counterclockwise, and that of the vector $j^{3/2}\omega^* - p_2$ is equal to $3\pi/4 - \gamma$ also counterclockwise. Thus, the total angle changes by the value $3\pi/4 + \gamma + 3\pi/4 - \gamma = 2 \cdot 3\pi/4$. In the general case, each pair of complex-conjugate roots in the left half-plane, located in accordance with Fig. 7*a*, changes the argument φ of the function $H(j^{1/\alpha}\omega^*)$ by the value $\Delta\varphi = 2\pi/(2\alpha)$.

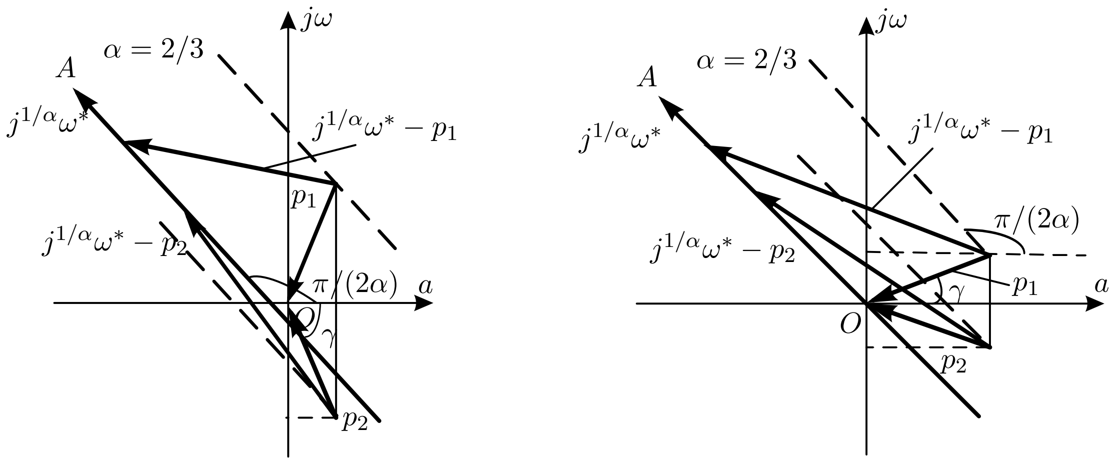


a ***b***

Fig. 7.

For the case of location of roots shown in Fig. 7b, when changing the frequency from $\omega^* = 0$ to $\omega^* \to \infty$, the rotation angle of the vector $j^{3/2}\omega^* - p_1$ is equal to $5\pi/4 - \gamma$ clockwise, and that of the vector $j^{3/2}\omega^* - p_2$ is equal to $3\pi/4 - \gamma$ counterclockwise. Then the total angle of rotation is equal to $-(5\pi/4 - \gamma) + 3\pi/4 - \gamma = -2\pi/4$. Thus, the argument φ of the function $H(j^{1/\alpha}\omega^*)$ in the case of a pair of complex-conjugate roots in the left half-plane, located in accordance with Fig. 7b, increments by $\Delta\varphi = -2\,(\pi - \pi/(2\alpha))$.

d) Consider the case of a pair of complex-conjugate roots in the right half-plane (Fig. 8) without losing generality also when $\alpha = 2/3$.



a ***b***

Fig. 8.

In the case shown in Fig. 8a when changing the frequency from $\omega^* = 0$ to $\omega^* \to \infty$, the rotation angle of the vector $j^{3/2}\omega^* - p_1$ is equal to $\pi/4 + \gamma$ clockwise, and the rotation angle of the vector $j^{3/2}\omega^* - p_2$ is equal to $\gamma - \pi/4$ counterclockwise. Thus, the total angle changes by the value $-(\pi/4 + \gamma) + \gamma - \pi/4 = -2\pi/4$. For the case of location of roots shown in Fig. 8b, when changing the frequency from $\omega^* = 0$ to $\omega^* \to \infty$, the rotation angle of the vector $j^{3/2}\omega^* - p_1$ is equal to $\pi/4 + \gamma$ clockwise, and the rotation angle of the vector $j^{3/2}\omega^* - p_2$ is equal to $\pi/4 - \gamma$ also clockwise. Then the total angle of rotation equals $-(\pi/4 + \gamma) - (\pi/4 - \gamma) = -2\pi/4$. Thus, in the general case, a pair of complex-conjugate roots in the right half-plane changes the argument φ of the function $H(j^{1/\alpha}\omega^*)$ by the increment $\Delta\varphi = -2(\pi - \pi/(2\alpha))$.

Thus, on the basis of the conducted analysis of the influence of the roots of the polynomial of the form (3b) taking into account the formulated criterion regarding the value of the argument φ of the function $H(j^{1/\alpha}\omega^*)$, the zone of stable operation of the system in the left half-plane has limits $\pm(\pi - \pi/(2\alpha))$ for $\alpha \in [0.5; 1[$. When $\alpha = 0.5$, the zone of stable operation becomes equal to 0. Thus, in the range $0 < \alpha \leqslant 0.5$ the system is unstable. It should be noted that when $\alpha = 0.5$ we are dealing with an integer characteristic polynomial in which there are only even polynomial orders and, accordingly, the necessary condition for the stability of the system, in accordance with the classical control theory, is not fulfilled.

A polynomial of the form (3b), using de Moivre's formula, can be changed to the form:

$$\begin{aligned} H(j^{1/\alpha}\omega^*) &= b_0\left(\left(\cos\frac{\pi}{2\alpha}+j\sin\frac{\pi}{2\alpha}\right)\omega^*-p_1\right)\left(\left(\cos\frac{\pi}{2\alpha}+j\sin\frac{\pi}{2\alpha}\right)\omega^*-p_2\right)\\ &\quad\times\left(\left(\cos\frac{\pi}{2\alpha}+j\sin\frac{\pi}{2\alpha}\right)\omega^*-p_3\right)\ldots\left(\left(\cos\frac{\pi}{2\alpha}+j\sin\frac{\pi}{2\alpha}\right)\omega^*-p_m\right)\\ &= U(\omega^*)+j\,V(\omega^*). \end{aligned}$$

Then, taking into account the above analysis of the influence of roots, the criterion of stability for a system with fractional-order derivatives can be formulated as follows: *for the stability of a system, it is necessary that the hodograph starting at $\omega = 0$ on the positive side of the real axis, with increasing ω to ∞ passes successively through the $m\pi/(2\alpha)$ sectors of a complex plane, where m is the order of the polynomial.*

When $\alpha = 1$ (polynomial of integer order), the proposed criterion corresponds to the classical Mikhailov criterion.

Consider the application of the proposed criterion on examples of systems with fractional-order derivatives. Let the system be described by a characteristic polynomial:

$$H\big(j^{1/2}\omega^*\big) = a_0\big(j^{1/2}\omega^*\big)^3 + a_1\big(j^{1/2}\omega^*\big)^2 + a_2\,j^{1/2}\omega^* + a_3.$$

We pass to the polynomial with integer j using the transform $j^{1/\alpha} = \cos\frac{\pi}{2\alpha} + j\sin\frac{\pi}{2\alpha}$ when $\alpha = 2$. Thus, after the substitution $j^{1/2} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+j)$, the characteristic polynomial of the system has the form:

$$H\big(j\omega^*\big) = a_0\,j\,\frac{\sqrt{2}}{2}(1+j)(\omega^*)^3 + a_1\,j\,(\omega^*)^2 + a_2\frac{\sqrt{2}}{2}(1+j)\,\omega^* + a_3$$

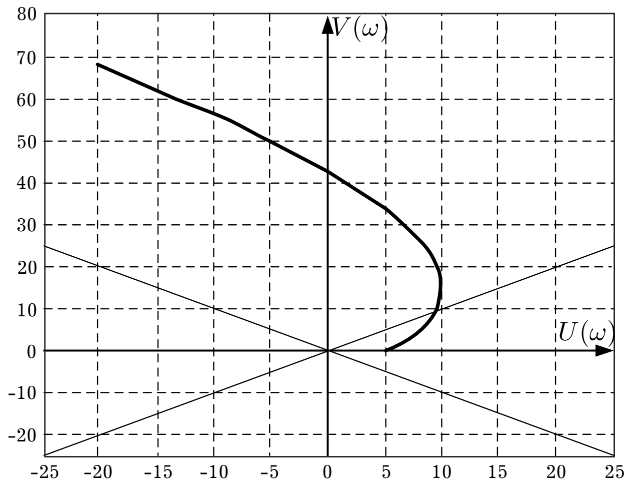
and respectively:

$$\begin{aligned} U(\omega^*) &= -a_0\,\frac{\sqrt{2}}{2}\,(\omega^*)^3 + a_2\,\frac{\sqrt{2}}{2}\,\omega^* + a_3,\\ V(\omega^*) &= a_0\,\frac{\sqrt{2}}{2}\,(\omega^*)^3 + a_1\,(\omega^*)^2 + a_2\,\frac{\sqrt{2}}{2}\,\omega^*. \end{aligned}$$

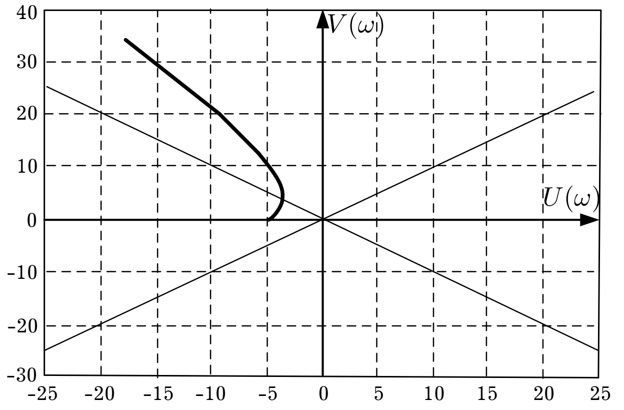
Then, when $\omega^* = 0$: $U(\omega^*) = a_3$, $V(\omega^*) = 0$, i.e. when $a_3 > 0$, the hodograph starts on the positive side of the real half-axis. For $\omega^* \to \infty$, the resulting angle of rotation is found from the expression:

$$\tan(\varphi) = \frac{a_0\,(\omega^*)^3\left(\frac{\sqrt{2}}{2} + \frac{a_1}{a_0\omega^*} + \frac{\sqrt{2}a_2}{2a_0(\omega^*)^2}\right)}{a_0\,(\omega^*)^3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}a_2}{2a_0(\omega^*)^2} + \frac{a_3}{a_0(\omega^*)^3}\right)}$$

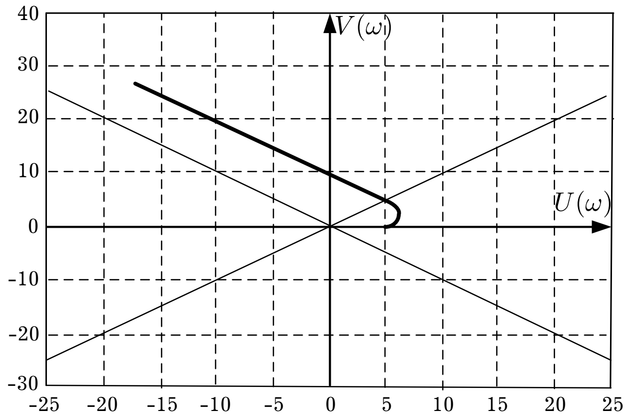
and after the evaluation of ambiguity $\tan(\varphi) \to -1$, and hence, when $a_0 > 0$, the angle $\varphi \to 3\pi/4$. Therefore, the stability criterion holds with respect to the resulting angle of rotation. The sequence of passing the sectors will ensure the fulfillment of the condition $\omega_1^* > \omega_2^*$, where ω_1^* is the frequency determined from the equation $U(\omega^*) = 0$, and the frequency ω_2^* is determined from the equation $U(\omega^*) = V(\omega^*)$.



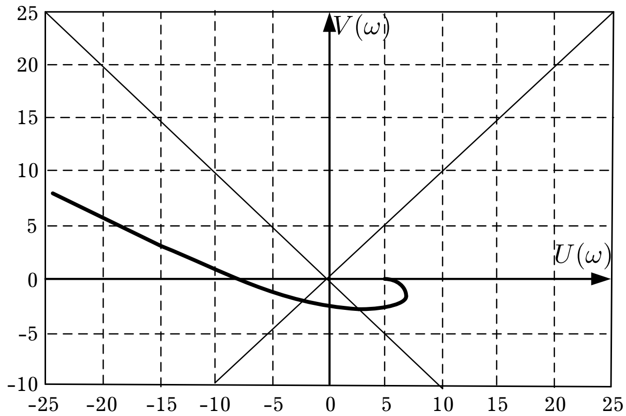
a A pair of complex-conjugate roots in the left half-plane and a real root in the left half-plane.



b A pair of complex-conjugate roots in the left half-plane and a real root in the right side.



c A pair of complex-conjugate roots in the right half-plane, which are located in the zone of stable work, and a real root in the left half-plane.



d A pair of complex-conjugate roots in the right half-plane, which are located in the zone of unstable work, and a real root in the left side.

Fig. 9. Hodographs of the third-order system at different locations of roots in a complex plane.

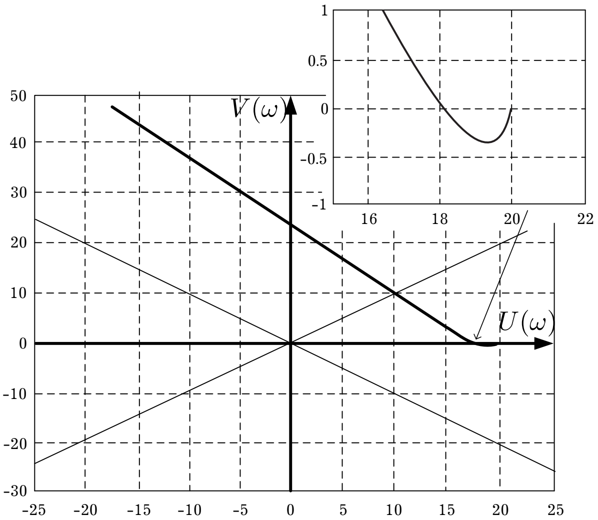
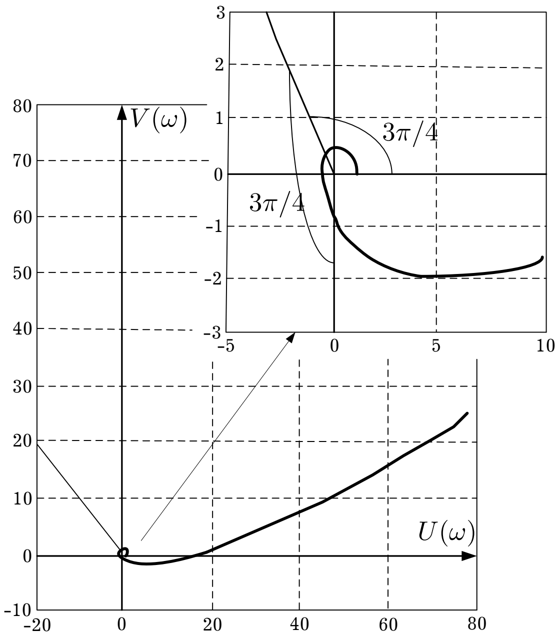


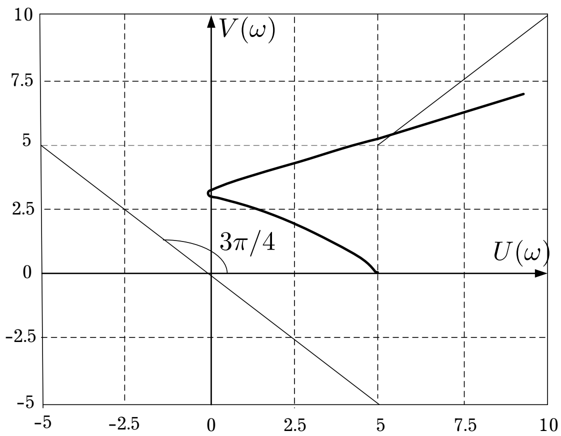
Fig. 10. Hodographs of the third-order system.

The hodographs obtained for the third-order system (Fig. 9) identify the stable and unstable systems for the characteristic third-order polynomial with the basis $s^{1/2}$. For the systems whose roots are located in the zone of stability, the hodograph successively passes through three sectors of a complex half-plane of $\pi/4$ size.

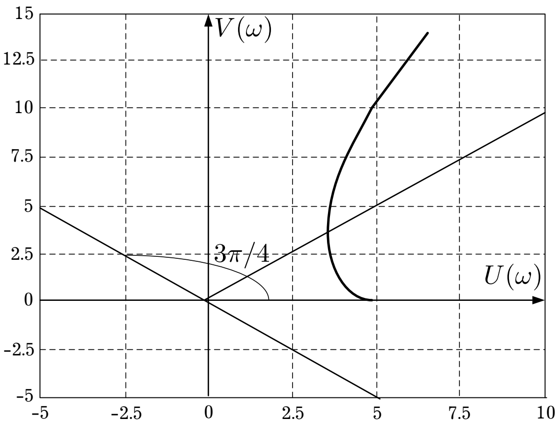
Only the behavior of the hodograph of the system with a pair of complex-conjugate roots in the right half-plane (located in the zone of stable work) and with a real root in the left half-plane requires separate consideration. In this case, under certain relations of the roots, the coefficient of the characteristic polynomial a_2 has a negative sign, and within the frequency range $\omega < 1$, the component $V(\omega^*) < 0$. The hodograph of the system has intersections with the axis of real numbers (Fig. 10), which is characteristic of an unstable system. Upon further change of ω, all conditions that define a stable system hold.



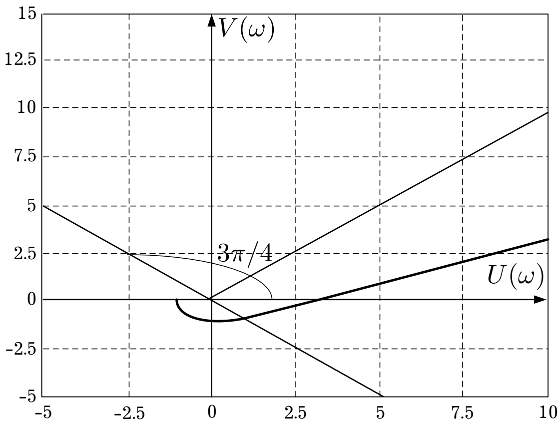
a A pair of complex-conjugate roots in the left half-plane, which are located in the zone of stable work, and a real root in the left half-plane.



b A pair of complex-conjugate roots in the left half-plane, which are located in the zone of unstable work, and a real root in the left half-plane.



c A pair of complex-conjugate roots in the right half-plane and a real root in the left half-plane.



d A pair of complex-conjugate roots in the left half-plane, which are located in the zone of stable work, and a real root in the left half-plane.

Fig. 11. Hodographs of the third-order system for different locations of roots in a complex plane.

In the case of a characteristic polynomial:

$$H\left(j^{3/2}\omega^*\right) = a_0\left(j^{3/2}\omega^*\right)^3 + a_1\left(j^{3/2}\omega^*\right)^2 + a_2\, j^{3/2}\omega^* + a_3$$

the components of the real and imaginary parts are determined by the expressions:

$$U(\omega^*) = a_0\frac{\sqrt{2}}{2}(\omega^*)^3 - a_2\frac{\sqrt{2}}{2}\omega^* + a_3,$$
$$V(\omega^*) = a_0\frac{\sqrt{2}}{2}(\omega^*)^3 - a_1(\omega^*)^2 + a_2\frac{\sqrt{2}}{2}\omega^*.$$

The hodographs obtained for the third-order system (Fig. 11) identify the stable and unstable systems for the characteristic third-order polynomial with the basis $s^{3/2}$. Only the hodograph in Fig. 11a successively passes through three sectors of a complex half-plane of $3\pi/4$ size.

Thus, the proposed criterion of stability provides an analysis of the stability of the systems with fractional-order derivatives when α is changing $\alpha \in]0.5; \infty[$.

3. Conclusions

1. In the system being described by a characteristic polynomial with a basis $s^{1/\alpha}$, when α changes $\alpha \in]0.5; \infty[$, the boundaries of the zone of stable operation in a complex plane are determined by the angle $\pm(\pi - \pi/(2\alpha))$ calculated clockwise from the real half-axis in the negative half-plane. For $\alpha \leqslant 0.5$, the system is unstable or is on the stability boundary.
2. A necessary condition for the stability of a system with fractional-order derivatives is ensuring the equality of the argument φ of the function $H(j^{1/\alpha}\omega^*)$ to the value $m\pi/(2\alpha)$ (where m is the order of integer polynomial with basis $s^{1/\alpha}$) when $\omega^* \to \infty$.
3. The application of de Moivre's formula makes it possible to transit from the polynomial $H(j^{1/\alpha}\omega^*)$ to the polynomial $H(j\omega^*)$ and to form a criterion for the stability of a system on the basis of a classical hodograph built in the coordinates of real $U(\omega^*)$ and imaginary $V(\omega^*)$ parts.
4. Some classical criteria of stability of systems with integer derivatives are obtained as a partial case of the used approach to the analysis of stability of systems with fractional-order derivatives when $\alpha = 0.5$ and $\alpha = 1$.

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Частотний критерій для аналізу стійкості систем з похідними дробового порядку

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На основі аналізу впливу коренів характеристичного полінома на приріст агумента частотної характеристики системи запропоновано частотний критерій стійкості системи з похідними дробового порядку. Визначено в комплексній площині межі зони розміщення коренів характеристичного полінома стійкої системи при зміні показника α основи характеристичного полінома.

Ключові слова: *похідна дробового порядку, критерій стійкості, границі стійкості.*