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ANGLES BETWEEN MATRICES AND BETWEEN POLYNOMIALS IN FRACTIONAL LINEAR SYSTEMS AND ELECTRICAL CIRCUITS

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Abstract: The notions of angles between matrices and between polynomials of fractional linear systems and electrical circuits are proposed. In analysis of angles between state matrices of fractional linear systems the Hadamard product of two matrices is applied. The angles between matrices and their functions are also addressed. The angles between symmetrical and asymmetrical part of matrices are investigated. The angles between polynomials of transfer matrices of fractional linear systems are analyzed and some new properties are established.

Key words: angle between matrices, polynomials, matrix function, linear, electrical circuit.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [1, 3, 8, 12]. Variety of models having positive behavior can be found in engineering, especially in electrical circuits [15], economics, social sciences, biology and medicine, etc. [3, 12].

The positive electrical circuits have been analyzed in [5–7, 9–11, 15]. A new class of normal positive linear electrical circuits has been introduced in [7]. Positive fractional linear electrical circuits have been investigated in [10, 15]. Stability of continuous-time and discrete-time linear systems with inverse state matrices has been analyzed in [14] and the transfer matrices with positive coefficients of standard and fractional positive systems in [11, 16]. The angles between state matrices and between polynomials of transfer matrices of linear electrical circuits have been investigated in [6]. Some recent results in fractional systems theory have been given in [2, 17–19].

In this paper the notions of angles between matrices and polynomials of fractional linear systems will be introduced and their basic properties will be investigated.

The paper is organized as follows. In section 2 the basic definitions and properties of fractional positive linear systems are recalled. The angles between matrices of fractional linear systems and electrical circuits are introduced and their properties are analyzed in section 3. The angles between matrices and their functions are addressed in section 4. The angles between two polynomials are introduced and their properties are investigated in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. Positive fractional linear systems

Consider the fractional linear continuous-time system described by the state equations

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \tag{2.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{2.1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$,

$$ {}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot f(\tau)}{(t-\tau)^\alpha} d\tau, \tag{2.1c}$$

$0<\alpha<1$

where $\dot f(\tau) = \frac{df(\tau)}{d\tau}$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $\mathrm{Re}(x)>0$ is the Euler gamma function.

It is well-known [15] that in fractional electrical circuits as the state variables $x_1(t),\dots, x_n(t)$ (the components of the state vector $x(t)$) the currents in the coils and voltages on the capacitors are chosen.

Definition 2.1. [3, 8] The fractional linear system (2.1) is called (internally) positive if $x(t)\in\Re_+^n$ and

$y = y(t) \in \Re_+^p$, $t \in [0,+\infty]$ for any $x_0 = x(0) \in \Re_+^n$ and every $u(t) \in \Re_+^m$, $t \in [0,+\infty]$.

Theorem 2.1. [3, 8] The fractional linear system (2.1) is positive if and only if

$$A \in M_n,\ B \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}. \quad (2.2)$$

Definition 2.2. [3, 8] The positive fractional linear system is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for any } x_0 \in \Re_+^n. \quad (2.3)$$

Theorem 2.2. [3, 8] The positive fractional linear system is asymptotically stable if and only if:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \quad (2.4)$$

are positive, i.e. $a_k > 0$ for $k = 0,1,\dots,n-1$.

2) There exists strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\dots,n$ such that

$$A\lambda < 0 \quad (2.5)$$

3. Angles between state matrices of fractional linear systems

In this section the angle between two matrices will be defined.

To any given matrix $A = [a_{ij}] \in R^{n\times m}$ the following two corresponding vectors can be defined

$$\overline{A} = [a_{11}\dots a_{1m} \ a_{21}\dots a_{2m} \ a_{31}\dots a_{nm}]^T \in \Re^{nm} \quad (3.1a)$$

and

$$\hat{A} = [a_{11}\dots a_{n1} \ a_{21}\dots a_{n2} \ a_{31}\dots a_{nn}]^T \in \Re^{nm} \quad (3.1b)$$

T denotes the transpose.

Using the vectors of the matrices $A \in \Re^{n\times m}$ and $B = [b_{ij}] \in \Re^{n\times m}$ we may defined the following scalar product of the two matrices.

Definition 3.1. The scalar

$$(\overline{A},\overline{B}) = (\hat{A},\hat{B}) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij} \quad (3.2)$$

is called the scalar product of the matrices A and B.

In particular case if $A = B$ then

$$(\overline{A},\overline{A}) = (\hat{A},\hat{A}) = |\overline{A}|^2 = |\hat{A}|^2 = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2 > 0 \quad (3.3)$$

for any nonzero matrix $A \in \Re^{n\times m}$.

Using (3.2) and (3.3) we may define the angle φ between two given matrices A and B of the same dimensions.

Definition 3.2. The angle defined by

$$\phi = \phi_{A,B} = \arccos\frac{(\overline{A},\overline{B})}{|\overline{A}||\overline{B}|} = \arccos\frac{(\hat{A},\hat{B})}{|\hat{A}||\hat{B}|}, \quad (3.4a)$$

$$0 < \phi < \pi,$$

is called the angle ϕ between the matrices A and B.

The relation (3.4a) can be equivalently written in the form

$$\cos\phi = \cos\phi_{A,B} = \frac{(\overline{A},\overline{B})}{|\overline{A}||\overline{B}|} = \frac{(\hat{A},\hat{B})}{|\hat{A}||\hat{B}|}. \quad (3.4b)$$

From (3.4b) it follows the following conclusion.

Conclusion 3.1.

$$\cos\phi_{A,B} = \cos\phi_{B,A} \text{ and } \cos\phi_{-A,-B} = \cos\phi_{B,A}. \quad (3.5)$$

In particular case if $\overline{B} = \overline{A}$ then from (3.4b) we have $\cos\phi = 1$ and $\phi = 0$.

Example 3.1. Find the $\cos\varphi$ between the following matrices

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}. \quad (3.6)$$

In this case

$$\overline{A} = [1\ -2\ 0\ 1\ 2\ 3]^T,\ \overline{B} = [0\ 2\ 1\ 0\ -1\ 1]^T \quad (3.7a)$$

and

$$\hat{A} = [1\ 0\ 2\ -2\ 1\ 3]^T,\ \hat{B} = [0\ 1\ -1\ 2\ 0\ 1]^T. \quad (3.7b)$$

Using (3.2), (3.3), (3.4b) and (3.7) we obtain

$$(\overline{A},\overline{B}) = (\hat{A},\hat{B}) = -3,\ |\overline{A}|^2 = |\hat{A}|^2 = 19, \quad |\overline{B}|^2 = |\hat{B}|^2 = 7 \quad (3.8a)$$

and

$$\cos\phi = \cos\phi_{A,B} = = \frac{(\overline{A},\overline{B})}{|\overline{A}||\overline{B}|} = \frac{(\hat{A},\hat{B})}{|\hat{A}||\hat{B}|} = -\frac{3}{\sqrt{19}\sqrt{7}} = -0.260. \quad (3.8b)$$

Consider the following two matrices of the same dimensions

$$A = [a_{ij}] \in \Re^{n\times m},\ B = [b_{ij}] \in \Re^{n\times m} \quad (3.9)$$

Definition 3.3. The matrix defined by

$$A \circ B = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{1m}b_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1}b_{n1} & \cdots & a_{nm}b_{nm} \end{bmatrix} \in \Re^{n\times m} \quad (3.10)$$

is called the Hadamard product of the matrices (3.9) [13].

Theorem 3.1. If the Hadamard product (3.10) of the matrices (3.9) is zero matrix then the angle φ between the matrices (3.9) is equal to $\frac{\pi}{2}$.

Proof. From Definitions 3.1 and 3.3 it follows that $A \circ B = 0$ implies

$$(\overline{A},\overline{B}) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij} = 0 . \qquad (3.11)$$

In this case from (3.4b) we obtain $\cos\varphi = 0$ and $\phi = \frac{\pi}{2}$. □

Example 3.2. Using (3.10) for the matrices

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \qquad (3.12)$$

we obtain

$$A \circ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3.13)$$

and

$$(\overline{A},\overline{B}) = \sum_{i=1}^{2}\sum_{j=1}^{3} a_{ij}b_{ij} = 0 . \qquad (3.14)$$

Therefore, by Theorem 3.1 the angle between the matrices (3.12) is equal $\frac{\pi}{2}$.

Theorem 3.2. The angle φ between the matrices $A = [a_{ij}] \in \Re^{n\times m}$, $B = [b_{ij}] \in \Re^{n\times m}$ satisfies the condition $\cos\varphi \ge 0$ if and only if

$$(\overline{A},\overline{B}) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij} \ge 0 \qquad (3.15a)$$

and $\cos\varphi < 0$ if and only if

$$(\overline{A},\overline{B}) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij} < 0 . \qquad (3.15b)$$

Proof. Note that if (3.15a) is satisfied then from (3.4b) it follows that $\cos\varphi \ge 0$ since $|\overline{A}| > 0$ and $|\overline{B}| > 0$.

Proof of (3.15b) is similar. □

By Theorem 2.2 the Metzler matrix is asymptotically stable (Hurwitz) if and only if there exists a strictly positive vector $\lambda = [\lambda_1,...,\lambda_n]$, $\lambda_k > 0,\ k = 1...,n$ such that the condition (2.5) is satisfied.

Examples of electrical circuits with Metzler state matrix A are given in [15].

Theorem 3.3. The angle φ between two asymptotically stable Metzler matrices $A = [a_{ij}] \in M_n$, $B = [b_{ij}] \in M_n$ satisfies the condition $0 < \varphi < \frac{\pi}{2}$.

Proof. From (2.5) it follows that the diagonal entries a_{ii} and b_{ii} for $i = 1...,n$ of asymptotically stable Metzler matrices A and B are negative. In this case the condition (3.15a) is satisfied and $0 < \varphi < \frac{\pi}{2}$. □

Example 3.3. Consider the following two asymptotically stable Metzler matrices

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}. \qquad (3.16)$$

Using (3.2), (3.4b) and (3.16) we obtain

$$\overline{A} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix},\ \overline{B} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, (\overline{A},\overline{B}) = 6, \qquad (3.17)$$
$$|\overline{A}|^2 = 14, |\overline{B}|^2 = 3$$

and

$$\cos\phi = \frac{(\overline{A},\overline{B})}{|\overline{A}||\overline{B}|} = \frac{6}{\sqrt{14}\sqrt{3}} = 0.926 \qquad (3.18)$$

This confirms the thesis of Theorem 3.3

Example 3.4. Find the $\cos\phi$ between asymptotically stable Metzler matrix A given by (3.16) and the unstable Metzler matrix

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad (3.19)$$

In this case we obtain

$$\overline{A} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix},\ \overline{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix},\ (\overline{A},\overline{B}) = -7, \qquad (3.20)$$
$$|\overline{A}|^2 = 14,\ |\overline{B}|^2 = 6$$

and

$$\cos\phi = \frac{(\overline{A},\overline{B})}{|\overline{A}||\overline{B}|} = \frac{-7}{\sqrt{14}\sqrt{6}} = -0.764 ,\ \frac{\pi}{2} \le \phi \le \pi \qquad (3.21)$$

The angles between two state matrices corresponding to different choice of the state vectors in linear electrical circuits have been investigated in [6].

Theorem 3.4

Let $A \in \Re_+^{n\times n}$ and $B \in \Re_+^{n\times n}$ be $n \times n$ real matrices with nonnegative entries and at least one positive entry:

1) The angle $\varphi_{A,B}$ between the matrices A and B satisfies the condition

$$0 < \phi_{A,B} < \frac{\pi}{2} \qquad (3.22a)$$

2) The angle φ_{A,A^k} between the matrices A, A^k, $k=1,2,...$ satisfies the condition

$$0<\phi_{A,A^k}<\frac{\pi}{2} \text{ for } k=0,1,... \quad (3.22b)$$

Proof. Note that the Hadamard product of the matrices $A=[a_{ij}]\in\Re_+^{n\times n}$ and $B=[b_{ij}]\in\Re_+^{n\times n}$ is positive

$$A\circ B=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}b_{ij}>0 \quad (3.23)$$

and from (3.4) we have

$$\cos\varphi_{A,B}>0 \text{ and } 0<\varphi_{A,B}<\frac{\pi}{2} \quad (3.24)$$

since $|A||B|>0$.

The proof of (3.22b) is similar since $A^k\in\Re_+^{n\times n}$ for $k=0,1,...$ □

Remark 3.1. If $B=A\in\Re_+^{n\times n}$ then $\varphi_{A,A}=0$.

Theorem 3.5. The angle φ between the Metzler Hurwitz matrix $A\in M_n$ and its inverse A^{-1} satisfies the condition

$$0<\phi<\frac{\pi}{2} \quad (3.25)$$

Proof. From (3.4b) we have

$$\cos\phi=\frac{(\overline{A},\overline{A}^{-1})}{|\overline{A}||\overline{A}^{-1}|}. \quad (3.26)$$

By assumption the matrix $A\in M_n$ is Hurwitz and it satisfies the condition (2.5). The strictly positive vector λ can be chosen as $\lambda=A^{-1}c$ for $c\in\Re_+^n$ strictly positive. Taking into account that for the Metzler Hurwitz matrix A we have $-A^{-1}\in\Re_+^{n\times n}$ and from (3.26) and (2.5) we obtain (3.25) since $\cos\varphi>0$. □

Example 3.5. Consider the Metzler matrix

$$A=\begin{bmatrix}-2 & 1 & 0\\ 2 & -3 & 1\\ 0 & 1 & -3\end{bmatrix} \quad (3.27)$$

which is Hutwitz since the condition (3.25) is satisfied for

$$\lambda=A^{-1}c=\frac{1}{10}\begin{bmatrix}8 & 3 & 1\\ 6 & 6 & 2\\ 2 & 2 & 4\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}6\\7\\4\end{bmatrix}, \quad (3.28)$$
$$c=[1\quad 1\quad 1]^T.$$

Using (3.27), (3.26) and (3.28) we obtain

$$\begin{aligned}&\overline{A}=[-2\ 1\ 0\ 2\ -3\ 1\ 0\ 1\ -3]^T\\ &\overline{A}^{-1}=-\frac{1}{10}[8\ 3\ 1\ 6\ 6\ 2\ 2\ 2\ 4]^T\\ &(\overline{A},\overline{A}^{-1})=-\frac{1}{10}\cdot\\ &\cdot(-16+3+12-18+2+2-12)=\frac{25}{10}>0\end{aligned}, \quad (3.29)$$

and from (3.29) $\cos\varphi>0$ and $0<\varphi<\frac{\pi}{2}$ since $|\overline{A}||\overline{A}^{-1}|>0$.

Example 3.6. Consider the fractional linear electrical circuit shown in Fig. 3.3 with given resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2.

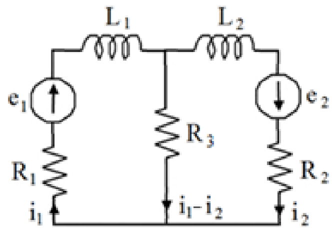

Fig. 3.3. Electrical circuit.

Using the Kirchhoff's laws we may write the equations

$$\begin{aligned}e_1&=(R_1+R_3)i_1-R_3i_2+L_1\frac{d^\alpha i_1}{dt^\alpha}\\ e_2&=(R_2+R_3)i_2-R_3i_1+L_2\frac{d^\alpha i_2}{dt^\alpha}\end{aligned} \quad (3.30)$$

The equations (3.30) can be written in the form

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix}i_1\\ i_2\end{bmatrix}=A\begin{bmatrix}i_1\\ i_2\end{bmatrix}+B\begin{bmatrix}e_1\\ e_2\end{bmatrix} \quad (3.31a)$$

where

$$A=\begin{bmatrix}-\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1}\\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2}\end{bmatrix},\ B=\begin{bmatrix}\frac{1}{L_1} & 0\\ 0 & \frac{1}{L_2}\end{bmatrix} \quad (3.31b)$$

For $R_k>0$, $k=1,2$ and $L_i>0$, $i=1,2$ the matrix A is Metzler Hurwitz matrix and its inverse has the form

$$A^{-1}=-\frac{1}{R}\begin{bmatrix}(R_2+R_3)L_1 & R_3L_2\\ R_3L_1 & (R_1+R_3)L_2\end{bmatrix}, \quad (3.32)$$
$$R=R_1(R_2+R_3)+R_2R_3.$$

The angle $\phi_{A,A^{-1}}$ between the matrices A and A^{-1} is given by

$$\phi_{A,A^{-1}} = \arccos\frac{(\bar{A},\bar{A}^{-1})}{|\bar{A}||\bar{A}^{-1}|} =$$
$$\arccos\left\{\left[2\frac{(R_1+R_3)(R_2+R_3)}{R} - \right.\right. \tag{3.33a}$$
$$\left.\left. -\frac{R_3^2}{R}\left(\frac{L_1}{L_2}+\frac{L_2}{L_1}\right)\right]\frac{1}{|\bar{A}||\bar{A}^{-1}|}\right\}$$

where

$$|\bar{A}|^2 = \left(\frac{R_1+R_3}{L_1}\right)^2 + \left(\frac{R_3}{L_1}+\frac{R_3}{L_2}\right)^2 + \left(\frac{R_2+R_3}{L_2}\right)^2$$
$$|\bar{A}^{-1}|^2 = \left(\frac{(R_2+R_3)L_1}{R}\right)^2 + R_3^2\left(\frac{L_1^2+L_2^2}{R}\right)^2 + \tag{3.33b}$$
$$+\left(\frac{(R_1+R_3)L_2}{R}\right)^2$$

In particular case when $L_1 = L_2$ we obtain

$$\phi_{A,A^{-1}} = \arccos\frac{2}{|\bar{A}||\bar{A}^{-1}|} \tag{3.34}$$

Let the matrix $A \in \Re^{n\times n}$ be nonsingular i.e. $\det A \neq 0$ and A_{ad} be its adjoint matrix, then

$$A^{-1} = \frac{A_{ad}}{\det A} \tag{3.35}$$

Theorem 3.6. The angle φ between the matrix A and its inverse matrix A^{-1} is equal to the angle φ between the matrix A and its adjoint matrix A_{ad}

$$\cos\phi_{A,A^{-1}} = \cos\phi_{A,A_{ad}}. \tag{3.36}$$

Proof. Applying Definition 3.2 to the matrices (A, A^{-1}) and (A, A_{ad}) and using (3.35a) we obtain

$$\cos\phi_{A,A^{-1}} = \frac{(\bar{A},\bar{A}^{-1})}{|\bar{A}||\bar{A}^{-1}|} = \frac{\left(\bar{A},\frac{A_{ad}}{\det A}\right)}{|\bar{A}|\left|\frac{A_{ad}}{\det A}\right|} = \tag{3.37}$$
$$= \frac{(\bar{A},\bar{A}_{ad})}{|\bar{A}||\bar{A}_{ad}|} = \cos\phi_{A,A_{ad}}$$

Therefore, the angle between the matrices A and A^{-1} is equal to the angle between the matrices A and A_{ad}. □

Example 3.8. The inverse matrix of the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \tag{3.38}$$

has the form

$$A^{-1} = \frac{A_{ad}}{\det A} = -\frac{1}{4}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}. \tag{3.39}$$

Taking into account that

$$\begin{aligned} \bar{A} &= [-2 \quad 1 \quad 2 \quad -3]^T, \\ \bar{A}_{ad} &= [-3 \quad -1 \quad -2 \quad -2]^T, \\ \bar{A}^{-1} &= \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T \end{aligned} \tag{3.40}$$

and using (3.4a) we obtain

$$\cos\phi_{A,A_{ad}} = \frac{(\bar{A},\bar{A}_{ad})}{|\bar{A}||\bar{A}_{ad}|} = \frac{7}{18} \tag{3.41a}$$

and

$$\cos\phi_{A,A^{-1}} = \frac{(\bar{A},\bar{A}^{-1})}{|\bar{A}||\bar{A}^{-1}|} = \frac{7}{18}. \tag{3.41b}$$

Therefore, we have $\cos\phi_{A,A^{-1}} = \cos\phi_{A,A_{ad}}$ and this confirms Theorem 3.6.

It is well-known that any matrix $A \in \Re^{n\times n}$ can be decomposed into the symmetrical part

$$A_s = \frac{A + A^T}{2} \in \Re^{n\times n} \tag{3.42}$$

and the asymmetrical (antisymmetrical) part

$$A_a = \frac{A - A^T}{2} \in \Re^{n\times n} \tag{3.43}$$

such that

$$A_s + A_a = A. \tag{3.44}$$

Theorem 3.7. The angle

$$\phi_{A_s,A_a} = \arccos\frac{(\bar{A}_s,\bar{A}_a)}{|\bar{A}_s||\bar{A}_a|} \tag{3.45}$$

between the symmetrical part A_s and the asymmetrical part A_a of the matrix A is equal to zero.

Proof. For the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{3.46}$$

the symmetrical part has the form

$$A_s = \begin{bmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \dots & \frac{a_{1n}+a_{n1}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} & \dots & \frac{a_{2n}+a_{n2}}{2} \\ \vdots & \vdots & \dots & \vdots \\ \frac{a_{1n}+a_{n1}}{2} & \frac{a_{2n}+a_{n2}}{2} & \dots & a_{nn} \end{bmatrix} \tag{3.47}$$

and the asymmetrical part

$$A_a = \begin{bmatrix} 0 & \frac{a_{12}-a_{21}}{2} & \dots & \frac{a_{1n}-a_{n1}}{2} \\ \frac{a_{21}-a_{12}}{2} & 0 & \dots & \frac{a_{2n}-a_{n2}}{2} \\ \vdots & \vdots & \dots & \vdots \\ \frac{a_{n1}-a_{1n}}{2} & \frac{a_{n2}-a_{2n}}{2} & \dots & 0 \end{bmatrix}. \tag{3.48}$$

Using (3.1), (3.2), (3.47) and (3.48) we obtain

$$\begin{aligned} &\left(\frac{a_{12}+a_{21}}{2}\right)\left(\frac{a_{12}-a_{21}}{2}\right)+\dots \\ &+\left(\frac{a_{1n}+a_{n1}}{2}\right)\left(\frac{a_{1n}-a_{n1}}{2}\right)+ \\ &+\left(\frac{a_{21}+a_{12}}{2}\right)\left(\frac{a_{21}-a_{12}}{2}\right)+\dots \\ &+\left(\frac{a_{n,n-1}+a_{n-1,n}}{2}\right)\left(\frac{a_{n,n-1}-a_{n-1,n}}{2}\right)=0 \end{aligned} \tag{3.49}$$

Therefore, the angle between the symmetrical and asymmetrical parts of the matrix (3.46) is equal to zero.

Example 3.9. The symmetrical part of the matrix (3.38) has the form

$$A_s = \frac{A+A^T}{2} = \begin{bmatrix} -2 & 1.5 \\ 1.5 & -3 \end{bmatrix} \tag{3.50}$$

and the asymmetrical part

$$A_a = \frac{A-A^T}{2} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}. \tag{3.51}$$

Using (3.50) and (3.51) we obtain $\overline{A}_s = [-2 \quad 1.5 \quad 1.5 \quad -3]$, $\overline{A}_a = [0 \quad -0.5 \quad 0.5 \quad 0]$ and

$$\phi_{A_s,A_a} = \arccos\frac{(\overline{A}_s, \overline{A}_a)}{|\overline{A}_s||\overline{A}_a|} = 0 \tag{3.52}$$

since $(\overline{A}_s, \overline{A}_a) = 0$.

This confirms Theorem 3.7.

4. Angles between matrices and their functions

Let $f(\lambda)$ be a scalar function well defined on the spectrum of the matrix A, i.e. $f(\lambda_k)$ has finite values for $k = 1,\dots,n$.

If the eigenvalues are distinct the matrix function $f(A) \in \Re^{n\times n}$ can be written in the form [4, 13]

$$\sum_{k=1}^{n} Z_k f(\lambda_k) \tag{4.1}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i\neq k}}^{n} \frac{A - I_n\lambda_k}{\lambda_k - \lambda_i}. \tag{4.2}$$

For general case the formula (4.1) is given in [4,13].

Example 4.1. The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \tag{4.3}$$

has the form

$$\det[I_2\lambda - A] = \begin{vmatrix} \lambda+2 & -1 \\ -2 & \lambda+3 \end{vmatrix} = \lambda^2 + 5\lambda + 4 \tag{4.4}$$

and its zeros are $\lambda_1 = -1$, $\lambda_2 = -4$. Therefore, the spectrum of the matrix (4.3) is {-1, -4}.

Using (4.2) in particular case for $f(A) = e^{At}$ and (4.3) we obtain

$$\begin{aligned} Z_1 &= \frac{A - I_2\lambda_2}{\lambda_1 - \lambda_2} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \\ Z_2 &= \frac{A - I_2\lambda_1}{\lambda_2 - \lambda_1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}' \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} e^{At} &= \exp\begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} = Z_1 e^{\lambda_1 t} + Z_2 e^{\lambda_2 t} = \\ &= \begin{bmatrix} \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} & \frac{1}{3}(e^{-t} - e^{-4t}) \\ \frac{2}{3}(e^{-t} - e^{-4t}) & \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} \end{bmatrix}. \end{aligned} \tag{4.6}$$

Using (3.4a) we may define the angle ϕ between the matrices A and $f(A)$ as follows

Definition 4.1.The angle defined by

$$\phi = \arccos\frac{(\overline{A}, \overline{f}(A))}{|\overline{A}||\overline{f}(A)|} \tag{4.7}$$

is called the angle between the matrices A and $f(A)$.

In particular case for $f(A) = A^{-1}$ we have

$$\varphi = \arccos\frac{(\overline{A}, \overline{A}^{-1})}{|\overline{A}||\overline{A}^{-1}|}. \tag{4.8}$$

Example 4.2. (Continuation of Example 4.1). Find the angle between the matrices A^{-1} and A, e^{At} for (4.3).

In the first case taking into account that for (4.3)

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}^{-1} = -\frac{1}{4}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \tag{4.9}$$

and using (3.4b) we obtain

$$\overline{A} = [-2,1,2,-3] \text{ and } \overline{A}^{-1} = -\left[\frac{3}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2}\right]^T \tag{4.10}$$

and

$$\cos\phi = \frac{(\overline{A}, \overline{A}^{-1})}{\sqrt{|\overline{A}|^2}\sqrt{|\overline{A}^{-1}|^2}} = \frac{\frac{7}{4}}{\sqrt{18}\sqrt{\frac{5}{8}}} = 0.39, \quad , \tag{4.11}$$

$$\phi = 67.1°$$

In the second case taking into account (4.8) and (4.7) we obtain

$$\begin{aligned} e^{\overline{A}t} = \Big[& \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}, \frac{1}{3}(e^{-t} - e^{-4t}), \\ & \frac{2}{3}(e^{-t} - e^{-4t}), \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} \Big]^T \end{aligned} \tag{4.12}$$

and

$$\cos\phi = \frac{(\overline{A}, e^{\overline{A}t})}{\sqrt{|\overline{A}|^2}\sqrt{|e^{\overline{A}t}|^2}} =$$

$$= \frac{-\frac{2}{3}e^{-t} - \frac{13}{3}e^{-4t}}{\sqrt{18}\sqrt{\frac{1}{3}(10e^{-2t} - 2e^{-5t} + 10e^{-8t})}} < 0, \tag{4.13}$$

$$t \ge 0, \ \frac{\pi}{2} < \phi < \pi.$$

These considerations can be extended to[4,13]:

1) any matrix functions well-defined on the spectrum of the matrix A ,

2) any two matrix functions well-defined on the spectrum of the matrix A,

3) any two matrix functions well-defined on the spectrum of the matrix A and B, respectively.

5. Angles between polynomials

Consider fhe fractional linear system (2.1) with zero initial conditions. Applying the Laplace transfom $\mathcal{L}$ to (2.1) and taking into account that $\mathcal{L}\left[\frac{d^\alpha x(t)}{dt^\alpha}\right] = s^\alpha X(s)$ we obtain

$$T(\lambda) = C[I_n\lambda - A]^{-1}B + D, \ \lambda = s^\alpha \tag{5.1a}$$

where

$$X(s) = \mathcal{L}\left[\frac{d^\alpha x(t)}{dt^\alpha}\right] = \int_0^\infty \frac{d^\alpha x(t)}{dt^\alpha}e^{-st}dt. \tag{5.1b}$$

All. nonzero entries of (5.1a) are rational function of $\lambda = s^\alpha$.

In this section the angles between two polynomials of fractional linear systems will be defined and their basic properties will be established.

Consider the polynomials in variable s

$$\begin{aligned} p(s) &= p_n(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \\ q(s) &= q_m(s) = b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0 \end{aligned} \tag{5.2}$$

with constant coefficients a_i, i = 0,1,…,n and b_j, j = 0,1,…,m.

Definition 5.1.The scalar

$$\big(p(s), q(s)\big) = \int_a^b p(s)q(s)ds \tag{5.3}$$

where a, b are given real numbers, is called the scalar product of the polynomials (5.2).

In particular case if $p(s) = q(s)$ then

$$\big(p(s), p(s)\big) = \int_a^b [p(s)]^2 ds \tag{5.4}$$

and

$$|p(s)| = \sqrt{(p(s), p(s))} \tag{5.5}$$

is the module of the polynomial $p(s)$.

Using (5.3) and (5.5) we may define the angle φ between the polynomials (5.2)

Definition 5.2.The angle defined by

$$\phi = \phi_{p,q} = \arccos \frac{p(s), q(s)}{|p(s)||q(s)|} \quad 0 < \phi < \pi \tag{5.6}$$

is called the angle φ between the polynomials (5.2).

The formula (5.6) can be equivalently written in the form

$$\cos\phi = \cos\phi_{p,q} = \frac{(p(s), q(s))}{|p(s)||q(s)|} \tag{5.7}$$

In particular case if $p(s) = q(s)$ then from (5.7) we have $\cos\phi = 1$ and $\phi = 0$.

Example 5.1.Find the $\cos\varphi$ between the following polynomials

$$p(s) = s^2 + 2s + 3 \text{ and } q(s) = 2s + 1 \tag{5.8}$$

for a = 1 and b = 2.

Using (5.3), (5.4), (5.5) and (5.8) we obtain

$$\begin{aligned}(p(s), q(s)) &= \int_1^2 p(s)q(s)ds = \\ &= \int_1^2 (s^2 + 2s + 3)(2s + 1)ds = \\ &= \int_1^2 (2s^3 + 5s^2 + 8s + 3)ds = \\ &= \frac{1}{2}s^4 + \frac{5}{3}s^3 + 4s^2 + 3s \Big|_1^2 = 34.167\end{aligned} \tag{5.9}$$

$$\begin{aligned}|p(s)|^2 &= \int_1^2 [p(s)]^2 ds = \int_1^2 (s^2 + 2s + 3)^2 ds = \\ &= \frac{1}{5}s^5 + s^4 + \frac{10}{3}s^3 + 6s^2 + 9s \Big|_1^2 = 71.533 \\ |q(s)|^2 &= \int_1^2 [q(s)]^2 ds = \int_1^2 (4s^2 + 4s + 1)ds = \\ &= \frac{4}{3}s^3 + 2s^2 + s \Big|_1^2 = 16.333\end{aligned} \tag{5.10}$$

and

$$\cos\phi = \frac{(p(s), q(s))}{|p(s)||q(s)|} = \frac{34,167}{\sqrt{34,182}} = 0.995. \tag{5.11}$$

From (5.7) it follows that

$$\cos\phi_{p,q} = \cos\phi_{q,p} \tag{5.12}$$

and

$$\cos\phi_{-p,-q} = \cos\phi_{p,q} \tag{5.13}$$

Let us consider the transfer function of fractional linear system of the form

$$T(s) = \frac{n(s)}{d(s)} \tag{5.14}$$

where

$$n(s) = b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0 \tag{5.15}$$

$$d(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{5.16}$$

The inverse transfer function of (5.14) has the from

$$T^{-1}(s) = \frac{d(s)}{n(s)} \tag{5.17}$$

From (5.6) applied to (5.14) and (5.17) we have the following conclusion.

Conclusion 5.1.The angle $\phi_{n,d}$ between the polynomials (5.15) and (5.16) and the angle φ_T of (4.14) and $\phi_{T^{-1}}$ of (4.17) satisfy the equalities

1) $$\phi_{n,d} = \phi_{d,n} \tag{5.18}$$

2) $$\phi_T = \phi_{T^{-1}} \tag{5.19}$$

In particular case for

$$T(s) = \frac{1}{d(s)} \tag{5.20}$$

we have

$$\phi_T = \phi_{T^{-1}} = \phi_{1,d} = \phi_{d,1}. \tag{5.21}$$

Remark 5.1. Note that if zeros of the polynomials (5.15) and (5.16) have negative real parts (the polynomials are asymptotically stable) when the angle $\phi_{d,n}$ between the polynomials $d(s)$ and $n(s)$ has the same sign for all nonnegative $a \geq 0$ and $b \geq 0$ in (5.3) and (5.4). In this case we may assume for example a = 0 and b = 1.

Example 5.1.Find the angle of the transfer function

$$T(s) = \frac{n(s)}{d(s)} = \frac{2s + 4}{s^2 + 4s + 3} \tag{5.22}$$

with the poles $s_1 = -1, s_2 = -3$ and zero $z_1 = -2$.

Using (5.4), (5.5) and (5.22) we obtain

$$\begin{aligned}(n(s), d(s)) &= \int_0^1 (2s + 4)(s^2 + 4s + 3)ds = \\ &= \int_0^1 (2s^3 + 12s^2 + 22s + 12)ds = 27.5 \\ |n(s)|^2 &= \int_0^1 (2s + 4)^2 ds = \\ &= \int_0^1 (4s^2 + 16s + 16)ds = 25.333 \\ |d(s)|^2 &= \int_0^1 (s^2 + 4s + 3)^2 ds = \\ &= \int_0^1 (s^4 + 8s^3 + 22s^2 + 24s + 9)ds = 30.533\end{aligned} \tag{5.23}$$

and

$$\cos\phi_{n,d} = \frac{(n(s),d(s))}{|n(s)||d(s)|} = \frac{27,5}{\sqrt{773,509}} \approx 0.988 . \quad (5.24)$$

Example 5.2. (Continuation of Example 3.6) Consider the fractional electrical circuit shown in Fig. 3.4 for $R_1 = R_2 = 1$, $R_3 = 2$ and $L_1 = L_2 = 1$.

In this case the matrices (3.40b) have the forms

$$A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (5.25)$$

As the output y of the electrical circuit we choose

$$y = i_1 + i_2 = C\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad (5.26)$$

The transfer matrix $T(s)$ of the electrical circuit has the from

$$\begin{aligned} T(s) &= C[I_2 s - A]^{-1} B = \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+3 & -2 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \\ &= \frac{1}{s^2+6s+5}\begin{bmatrix} s+5 & s+5 \end{bmatrix} \end{aligned} \quad (5.27)$$

The angle ϕ_{T_1} of the transfer function

$$T_1(s) = \frac{s+5}{s^2+6s+5} \quad (5.28)$$

is equal to

$$\phi_{T_1} = \arccos T_1(s) = \arccos\frac{s+5}{s^2+6s+5} = 10,63^\circ \quad (5.29)$$

since

$$\begin{aligned} (n(s),d(s)) &= \int_0^1 (s+5)(s^2+6s+5)ds = 46.417 \\ |n(s)|^2 &= \int_0^1 (s+5)^2 ds = 30.333 \\ |d(s)|^2 &= \int_0^1 (s^2+6s+5)^2 ds = 73.533 \end{aligned} \quad (5.30)$$

6. Concluding remarks

The notions of angles between matrices and between polynomials of fractional linear systems and electrical circuits have been introduced and investigated. In analysis of angles between state matrices of fractional linear systems the Hadamard product of two matrices has been applied and some basic properties of the angles between matrices of fractional linear systems have been established (Theorems 3.1-3.6). The angles between symmetrical and asymmetrical parts of the state matrices have been defined and have been analyzed (Theorem 3.8). Next the angles between matrices and their functions have been also introduced (Definition 4.1). The angles between two polynomials are defined (Definition 5.2). Some basic properties of transfer functions of fractional linear systems are analyzed and some new properties have been also established. The considerations are illustrated by examples of fractional linear systems and linear electrical circuits. The considerations can be extended to fractional descriptor linear continuous-time and discrete-time systems.

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КУТИ МІЖ МАТРИЦЯМИ ТА МІЖ ПОЛІНОМАМИ В ДРОБОВИХ ЛІНІЙНИХ СИСТЕМАХ ТА ЕЛЕКТРИЧНИХ КОЛАХ

Тадеуш Качорек

Сформульовано поняття кутів між матрицями та між поліномами дробових лінійних систем та електричних кіл. Під час аналізу кутів між матрицями стану дробових лінійних систем засновано добуток Адамара з двох матриць. Розглянуто також кути між матрицями та їх функції. Досліджено кути між симетричною й несиметричною частинами матриць. Проаналізовано кути між поліномами передатних матриць дробових лінійних систем та встановлено їхні деякі нові властивості.

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