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INVESTIGATION OF FORMULAS DETERMINATION OF A POINT'S PLANE COORDINATES BY THE INVERS LINEAR-ANGULAR RESECTION

The aim. The study of formulas determination of the point coordinates by the inverse linear-angular intersection method. Previously, we investigated the possibility of using electronic total stations to control the geometric parameters of industrial buildings. The applied application of electronic total stations for high-precision measurements has been investigated as well. [Vivat, 2018]. The formula for optimal use of the device with certain accuracy characteristics relative to the measured basis is analytically proved and derived [Litynskyi, 2014]. Measurements on the basis of the II category are performed and theoretical calculations are confirmed. The possibility of achieving high accuracy in determining the segment by the method of linear-angular measurements is shown [Litynsky, 2015]. The influence of the angle value on the accuracy of determining the coordinates by the sine theorem is investigated and the possibility of optimizing the determination of coordinates by the method of inverse linear-angular serif by the formulas of cosines and sines is investigated [Litynskyi, 2019]. Method. Establishing a mathematical interconnection between measured values (distances and angles) with the required (flat coordinates of a point), differentiation and finding the minima of functions. Results.There were five formulas selected, of which six combinations had been created to calculate the increments of coordinates and to estimate their accuracy. Numerical experiments show that neither method has a significant advantage, which is supported by the results presented in the graphs and tables. It is worth noting one feature of the second method - in which it is possible to determine the increments of coordinates with an accuracy that exceeds the accuracy of measuring the sides. The possibility of optimizing the coordinate increments determination due to the choice of calculation formulas is considered. The possibility of increasing the accuracy of determination of the coordinates increments using different calculation formulas is researched. Consequently, it is suggested to optimize the choice of calculation formulas depending on the position of the desired point. The results of these studies can be used to create electronic total station or laser tracker application software in order to improve the accuracy of coordinate determination.

Key words: technical measurements, high-precision measurements, optimization of geodetic measurements, optimization of determination of coordinates by inverted linear-angular intersection.

Introduction

As usual, there are high requirements to determining the geometrical parameters of structures that exceed significantly the requirements of geodetic measurements instructions. For example, the accuracy of determining the parameters of metal parts cannot physically exceed 0.02–1.0 mm. [DSTU-N, 2009]. Mechanical and interference methods are used to achieve such accuracy. For such problems new methods and devices, mechanical hands, trackers, interferometers are being developed. Those structures whose geometrical parameters are to be determined (like beams, cranes, turbines, satellite antennas and etc.) are often mounted in inaccessible places and it is not always possible to make direct measurements of their structural elements. Therefore geodetic networks are built for measuring and marking the geometrical parameters of such units [Voytenko, 2010].

We have analyzed theoretical and practical researches of this problem and found out that during installation and verification of geometrical sizes of constructions inverted linear-angular intersection is used. Laser trackers, scanners and total stations are used for coordinate determination [Lienhart, 2017; Erol, 2010; Vivat, et al., 2018]. In [Gottwald, 2008; Novakovic et al., 2009] methods of control of electronic total stations and laser scanners for compliance with international standards of ISO metrological parameters are investigated.

Usually these measurements are dynamic, one needs to determine coordinates using minimum quantity of measurements. Interesting way of error determination of binding device to known basis is proposed in [Burak, 2011]. Author uses cotangent based formula from problem of stake outing of point to section line when section line is not visible.

[Litynskyi, et al., 2014] has analytically proved and deduced a formula for optimal arrangement of a device with certain accuracy characteristics with respect to the measured basis. In [Litynskyi, et al., 2015] measurements were made on the basis of the II category and the theoretical calculations were fully confirmed. The possibility of achieving higher accuracy of segment determination by linear-angular measurements method is shown. In [Gargula, 2009] the influence of the magnitude of the angle on the determination of coordinates by the sine theorem is researched. In [Litynskyi, et al., 2019] the possibility of optimizing the determination of coordinates by the method of inverse linear-angular intersection by the formulas of cosines and sines is researched.

Aim

To perform a more detailed research of formulas that determine the increments of coordinates that gives optimal value in inverted linear-angular geodetic intersection. We are aiming to propose and analyse calculational formulas to obtain minimal error without adjustments.

Methodology

Due to the fact that it is not always possible to directly measure structural elements, they often use the method of inverted linear-angular intersection. This method involves measuring the lines S_1, S_2, and the angle β between them (See Fig.1).

The accuracy of coordinate transfer is influenced not only by the shape of geometric constructions, but also by formulas for which the increments of coordinates are calculated, and that is confirmed by further research. These formulas are selecteddepending on the information received. In our case, since there are excess measurements of increments, we can calculate it using both by the sine theorem and Pythagorean theorem. As we have excess measurements we can ignore some of them as angular measurements errors sometimes are distorting result much more then linear measurements errors.

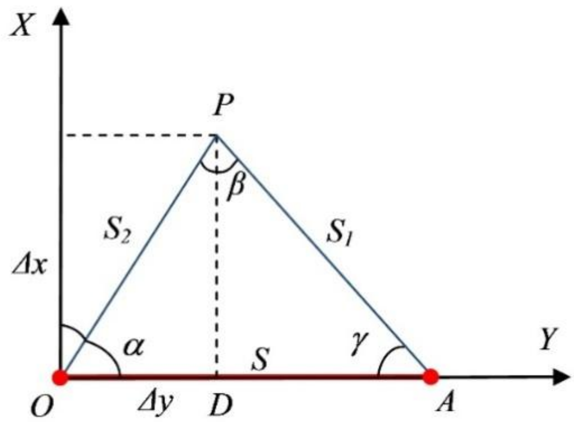

Fig. 1. Illustration of designations in coordinate determination by the method of planar linear-angular intersection

Mathematical research

The coordinates of the desired point ***P*** calculates as follows:

$$x_p = x_0 + \Delta x,$$
$$y_p = y_0 + \Delta y. \qquad (1)$$

Consider, at what values of the measured parameters S_1, S_2, β and the formulas of calculation the coordinate error will be minimal. Assume that the side (basis) of the formed triangle S is fixed. For a fixed basis and the measured two sides of a triangle and the angle between them, it is possible to calculate the increments of Δx and Δy coordinates by different methods, for example, by the formulas of the sine theorem:

$$\sin\alpha = \left(\frac{S_1}{S}\sin\beta\right), \quad \alpha = \arcsin\left(\frac{S_1}{S}\sin\beta\right), \qquad (2)$$

$$\sin\gamma = \left(\frac{S_2}{S}\sin\beta\right), \quad \gamma = \arcsin\left(\frac{S_2}{S}\sin\beta\right) \qquad (3)$$

Determine the increment of Δy coordinates by these formulas using (Fig. 1).

$$\Delta y_{TS_1} = S_2\cos\alpha = S_2\sqrt{1 - \frac{S_1^2\sin\beta}{S^2}} = \frac{S_2}{S}\sqrt{S^2 - S_1^2\sin^2\beta}, \qquad (4)$$

$$\Delta y_{TS_2} = S - S_1\cos\gamma = S - \frac{S_1}{S}\sqrt{S^2 - S_2^2\sin^2\beta}. \qquad (5)$$

We also express the increase Δy by Pythagoras' theorem:

$$\Delta y_{TP} = \frac{S_2^2 - S_1^2 + S^2}{2S}. \qquad (6)$$

The increment of the coordinate Δx, we express from the triangle OPA, using the sine theorem:

$$\Delta x_{TS} = S_2\sin\alpha = \frac{S_1 S_2\sin\beta}{S}, \qquad (7)$$

as well as from the triangle OPD, by Pythagorean theorem:

$$\Delta x_{TP} = \sqrt{S_1^2 - (\Delta x)^2} = \sqrt{S_1^2 - \left(\frac{S_1^2 - S_2^2 + S^2}{2S}\right)^2} = \frac{\sqrt{2S^2\left(S_1^2 + S_2^2\right) - \left(S^4 + \left(S_1^2 - S_2^2\right)^2\right)}}{2S}. \qquad (8)$$

The combination of formulas (4), (5), (6), (7) and (8) gives different ways of calculating the values of Δx, Δy and by differentiating them and performing simplifications, we obtain formulas for calculating the root mean square errors of the increments of coordinates. Let us introduce the following notation:

$m^2_{\Delta y(TS_1)}$ – is the mean square error of determining the increment Δy by the sine theorem across side S_2,

$m^2_{\Delta y(TS_2)}$ – the mean square error of determining the increment Δy by the theorem of sines across the side S_1,

$m^2_{\Delta y(TP)}$ – the mean square error of determining the increment Δy by the Pythagorean theorem,

$m^2_{\Delta x(TS)}$ – the mean square the error of determining the increment Δx by the sine theorem,

$m^2_{\Delta x(TP)}$ – the mean square error of determining the increment Δx by the Pythagorean theorem,

$$m^2_{\Delta y(TS_1)} = \frac{1}{S^2}\left\{(S^2 - S_1^2\sin^2\beta)m^2_{S_1} + S_2^2\frac{S_1^2\sin^4\beta\cdot m^2_{S_1} + S_1^4\sin^2\beta\left(\cos^2\beta\cdot m^2_\beta\right)}{S^2 - S_1^2\sin^2\beta}\right\}, \qquad (9)$$

$$m^2_{\Delta y(TS_2)} = \frac{1}{S^2}\left\{(S^2 - S_2^2\sin^2\beta)m^2_{S_1} + S_1^2\frac{S_2^2\sin^4\beta\cdot m^2_{S_1} + S_2^4\sin^2\beta\left(\cos^2\beta\cdot m^2_\beta\right)}{S^2 - S_2^2\sin^2\beta}\right\}, \qquad (10)$$

$$m^2_{\Delta y(TP)} = \frac{s_1^2 m^2_{s_1} + s_2^2 m^2_{s_2}}{S^2}, \qquad (11)$$

$$m^2_{\Delta x(TS)} = \frac{1}{S^2}(S_2^2\sin^2\beta\cdot m^2_{S_1} + S_1^2\sin^2\beta\cdot m^2_{S_2} + S_1^2S_2^2\cos^2\beta\cdot m^2_\beta), \qquad (12)$$

$$m^2_{\Delta x(TP)} = \frac{1}{4S^4(\Delta y)^2}\{S_1^2\left(S^2 - S_1^2 + S_2^2\right)^2 m^2_{s_1} + S_2^2\left(S^2 + S_1^2 - S_2^2\right)^2 m^2_{s_2}\}. \qquad (13)$$

Let us write down all possible combinations of an error calculation of the point P as the sum of the errors of these two increments of coordinates:

$$m_P^2 = \begin{cases} m^2_{\Delta y(TP)} + m^2_{\Delta x(TP)} & (I) \\ m^2_{\Delta y(TP)} + m^2_{\Delta x(TS)} & (II) \\ m^2_{\Delta y(TS_1)} + m^2_{\Delta x(TP)} & (III) \\ m^2_{\Delta y(TS_2)} + m^2_{\Delta x(TP)} & (IV) \\ m^2_{\Delta y(TS_1)} + m^2_{\Delta x(TS)} & (V) \\ m^2_{\Delta y(TS_2)} + m^2_{\Delta x(TS)} & (VI) \end{cases}. \qquad (14)$$

The exercise of finding the minimum value of the sum of errors Δx and Δy

$$m_P^2 = m^2_{\Delta x} + m^2_{\Delta y}, \qquad (15)$$

with the following condition

$$S^2 = S_1^2 + S_2^2 - 2S_1S_2\cos\beta, \qquad (16)$$

is formulated as finding the minimum of Lagrange function

$$F(S_1, S_2, \beta) = m_x^2 + m_y^2 + \lambda\,(S_1^2 + S_2^2 - 2S_1S_2\cos\beta - S^2).$$

The analysis of formulas (9), (10) for the calculation of errors reveals the following feature:

$$\lim m_y^2 \to \infty \text{, for}$$

$$\sin\beta \to \frac{S}{S_1} (S_1 \geq S, \sin\alpha \to 1,$$

$$\alpha \to \frac{\pi}{2}, \Delta y \to 0, S_1 \to S, m_y \to \infty)$$

In formula (9) and

$$\sin\beta \to \frac{S}{S_2} (S_2 \geq S, \sin\gamma \to 1, \gamma \to \frac{\pi}{2},$$

$$\Delta x \to 0, S_2 \to S, m_y \to \infty)$$

in formula (10).

Also, by tabulating functions (9) and (10), it is established that for certain *S₁*, *S₂*, fixed basis *S* and a measured angle *β*, there is no global extremum by cosine theorem [Litynskyi, et al., 2019]. The studies of the Lagrange functions for the best arrangement of the device were carried out in [Litynskyi, et al., 2019], where some partial cases are found that give optimal parameters. However, no definitive and clear answer received. Therefore, considering the expression $m^2(S_1, S_2, \beta)$ as a function of two variables S_1, S_2, since S_1, S_2, β are bounded by condition (16), we subsequently search for such values of S_1, S_2 that produces minimal error.

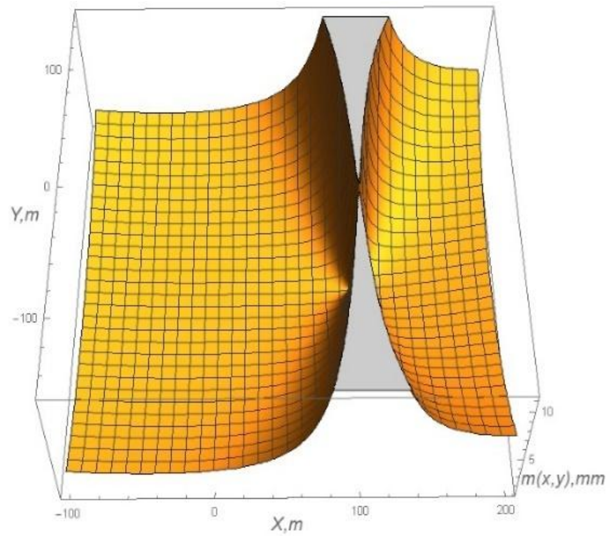

Fig. 2. The error behaviour of $m_{\Delta y}$ by formulas (9)–(10)

Results

A numerical experiment shows that extreme values can be only in degenerate cases for methods I, III-IV, and for method II the extreme can be reached for the real positions of the device. To calculate the position errors of the required points, we take a high-precision total station with the following characteristics: m_S = 1 mm, та m_β= 1".

We give the values of the errors for ten and hundred-meter basis (Table 2, Table 4) for some locations of the desired points from Table 1 and Table 3 (See Fig. 3-8).

Table 1

Characteristic points for the ten-meter basis

Point number	S_1, *m*	S_2, m	β^0	Coordinate values, m	
				y	*x*
P1	5	5	180	5	0
P2	10	10	60	5	8.66
P3	10	$10 \times \sqrt{2}$	45	10	10
P4	$10 \times \sqrt{2}$	10	45	0	10
P5	15	5	0	-5	0
P6	5	15	0	15	0

Table 2

The values of errors of characteristic points for the ten-meter basis

Point number	The method number for determining the error $m_{P,\ mm}$					
	I	II	III	IV	V	VI
P1	–	0.71	–	–	1.00	1.00
P2	1.63	1.87	1.78	1.78	2.00	2.00
P3	2.00	2.12	1.58	–	1.73	–
P4	2.00	2.12	–	1.58	–	1.73
P5	–	1.58	–	–	1.00	1.00
P6	–	1.58	–	–	1.00	1.00

Table 3

Characteristic points for the hundred-meter basis

Point number	S_1, m	S_2, m	b^0	Coordinate values, m	
				y	*x*
P1	50	50	180	50	0
P2	100	100	60	50	86.60
P3	100	100´ ´ $\sqrt{2}$	45	100	100
P4	100´ ´ $\sqrt{2}$	100	45	0	100
P5	150	50	0	-50	0
P6	50	150	0	150	0

Table 4

The error values of the characteristic points for the hundred-meter basis

Point number	*The* method of determining the error $m_{P,\ mm}$					
	I	II	III	IV	V	VI
P1	–	0.72	–	–	1.01	1.01
P2	1.63	1.89	1.83	1.83	2.06	2.06
P3	2	2.18	1.65	–	1.86	–
P4	2	2.18	–	1.65	–	1.86
P5	–	1.62	–	–	1.06	1.06
P6	–	1.62	–	–	1.06	1.06

Let us show these six characteristic points described in Tables 1–4 in the graphs. We plot graphs for any position of the points. Horizontals depicts the mean square error of coordinate determination (See Fig. 3–8), depending on the combination of formulas (14) by which the calculations are made. On Fig. 3–9 *OA* line shown in red is the basis. For example, if the location of the desired point is the same as P_1, then optimal calculations are performed using the formulas of the second method. None of the methods I–VI has a significant advantage alone, which is confirmed by the results presented in the graphs and tables. It is worth noting one feature of the method II ($m_P^2 = m^2_{Dy_{(TP)}} + m^2_{Dx_{(TS)}}$) – which is able to determine the increments of coordinates in the central part with an accuracy that exceeds the accuracy of the measurement of the sides.

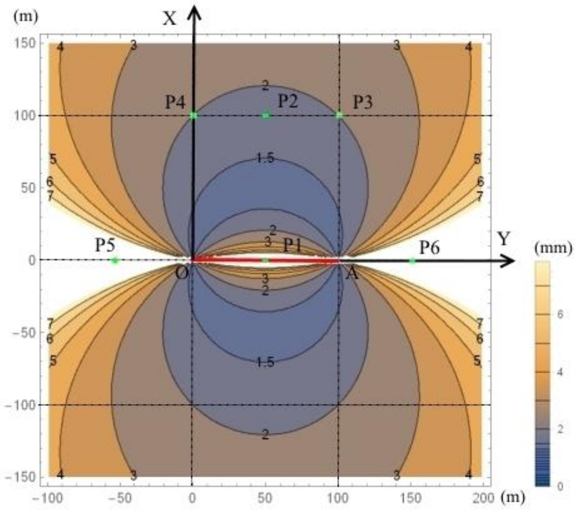

Fig. 3. Method I

$$(m_P = \sqrt{m^2_{Dy_{(TP)}} + m^2_{Dx_{(TP)}}})$$

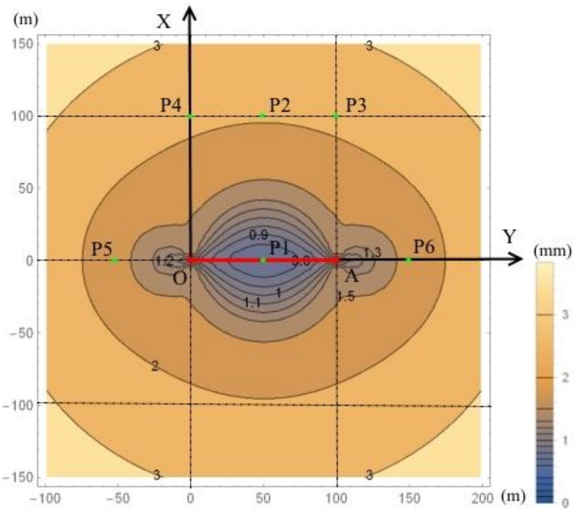

Fig. 4. Method II

$$(m_P = \sqrt{m^2_{Dy_{(TP)}} + m^2_{Dx_{(TS)}}})$$

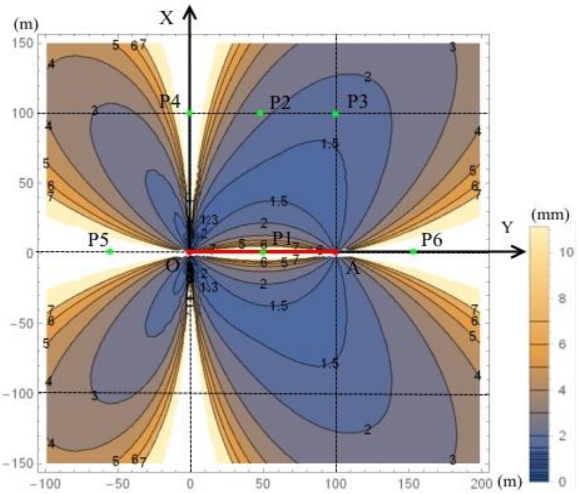

Fig. 5. Method III ($m_P = \sqrt{m^2_{Dy_{(TS_1)}} + m^2_{Dx_{(TP)}}}$)

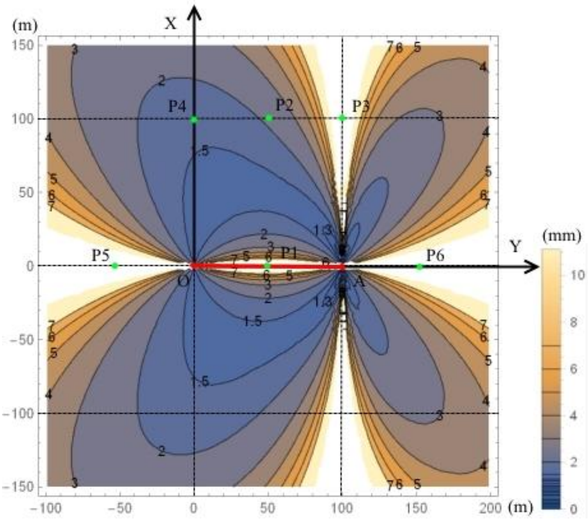

Fig. 6. Method IV ($m_P = \sqrt{m^2_{Dy_{(TS_2)}} + m^2_{Dx_{(TP)}}}$)

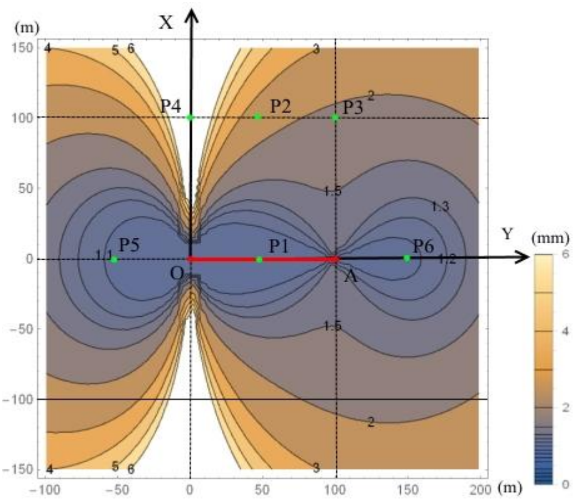

Fig. 7. Method V ($m_P = \sqrt{m^2_{Dy_{(TS_1)}} + m^2_{Dx_{(TS)}}}$)

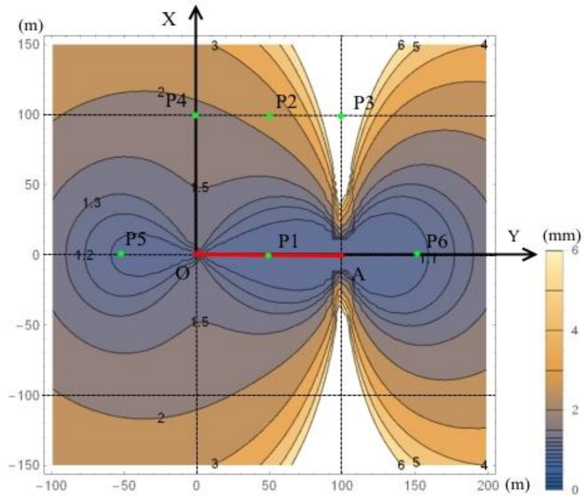

Fig. 8. Method VI ($m_P = \sqrt{m^2_{Dy_{(TS_2)}} + m^2_{Dx_{(TS)}}}$)

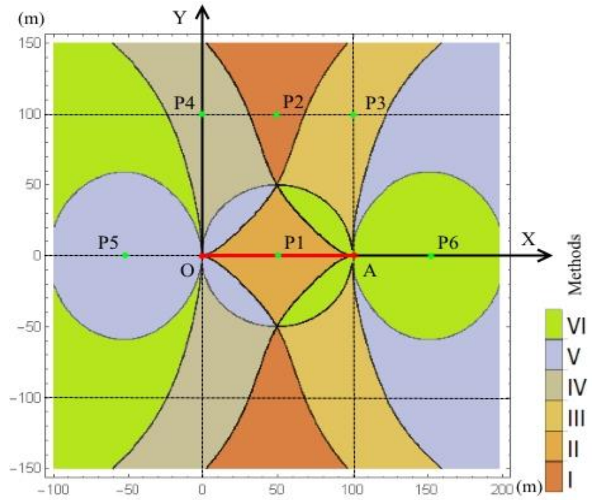

Fig. 9: The diagram of the choice of the optimal method for calculating the coordinates of a point for a basis of 100 m.

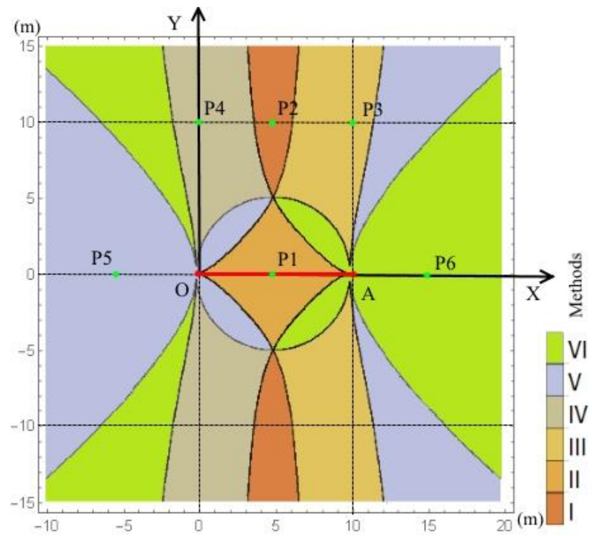

Fig. 10. The diagram of the choice of the optimal method of calculating the coordinates of the point for the 10 m

An example of the use of Fig. 9, 10 – if the approximate position of the desired point is close to the P_1, then for the most accurate calculation of coordinates one need to use the formulas of the second method.

Scientific novelty and practical significance

The possibility of increasing the accuracy of determining the increments of coordinates using different calculation formulas has been investigated. The choice of optimal calculation formulas depending on the position of the desired point is provided.

Conclusions

The analysis of the tables and graphs shows that the minimum error values are reached for degenerate cases, particularly, when the desired point is located on the basis (see formula 14 – II, V, VI) or perpendicular to it (see formula 14 – I, III, IV). If not taking into account the accuracy of the basis determination, the extreme value is reached (see formula 14 – II) in the case of the side`s perpendicularity. But, for all variants, the minimum error in calculations is obtained when the increments of coordinates are calculated by formulas 7 and 10. On the basis of the obtained calculations, plus taking into account the initial assumptions about the uniformity of the measurement`s error influence of the sides and the measurement`s error of the smaller influence to the angle and the error of determining the length of the basis, we can also assert:

1. When changing the size of the basis and maintaining the position of the point relative to the basis, its determination is practically unchanged (Table 2 and Table 4).

2. The smallest error of calculations of the value of a side is achieved by combination of formulas 6 and 7 obtained by the Pythagorean theorem.

3. In some cases, the accuracy of calculating line lengths is higher than the precision of measuring them, which is typical to near straight angles.

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ДОСЛІДЖЕННЯ ФОРМУЛ ВИЗНАЧЕННЯ ПЛОСКИХ КООРДИНАТ ТОЧКИ МЕТОДОМ ОБЕРНЕНОЇ ЛІНІЙНО-КУТОВОЇ ЗАСІЧКИ

Мета – виконати дослідження формул для визначення плоских координат точки методом оберненої лінійно-кутової засічки. У роботах [Vivat 2018, Novakovic 2009, Lienhart 2017, Erol 2010, Litynskyi 2014, Litynskyi 2015, Gargula 2009, Litynskyi 2019] досліджено можливість використання електронних тахеометрів для контролю геометричних параметрів промислових будівель. Розглянуто методи контролю електронних тахеометрів та лазерних сканерів на відповідність міжнародним стандартам метрологічних параметрів ISO. Досліджено прикладне застосування електронних тахеометрів для високоточних вимірювань та оптимізацію геодезичних мереж, які створюють для вимірювань витягнутих споруд. Аналітично доведено та виведено формулу для оптимального розміщення приладу з певними характеристиками точності відносно вимірюваного базиса, виконано вимірювання на базисі II розряду та підтверджено теоретичні розрахунки. Показано можливість досягнення вищої точності визначення відрізка методом лінійно-кутових вимірювань. Досліджено вплив величини кута на точність визначення координат за теоремою синусів та досліджено можливість оптимізації визначення координат методом оберненої лінійно-кутової засічки за формули косинусів та синусів. Методика. Встановлення математичного зв'язку вимірюваних величин (віддалей та кутів) із шуканими (плоскими координатами точки), диференціюванням та знаходженням мінімумів функцій. Результати. Приведено п'ять формул, з яких створено шість комбінацій для обчислення приростів координат та оцінки їхньої точності. Числові експерименти показують, що значної переваги жоден з методів не має, що підтверджується результатами, поданими на графіках та таблицях. Варто виокремити одну особливість другого методу, за яким є можливість визначити прирости координат з точністю, що перевищує точність вимірювання довжин сторін. Розглянута можливість оптимізації визначення приростів координат за рахунок вірогідного вибору формул обчислень. Досліджено можливість підвищення точності визначення приростів координат, використовуючи різні формули обчислень. Запропоновано оптимізацію вибору формул обчислень залежно від положення шуканої точки. Результати поданих досліджень можна використати у створенні прикладного програмного забезпечення електронного тахеометра чи лазерного трекера для підвищення точності визначення координат.

Ключові слова: оптимізація визначення координат оберненою лінійно-кутовою засічкою; технічні вимірювання; високоточні вимірювання; оптимізація геодезичних вимірювань.

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