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# Discrete solution for the nonlinear parabolic equations with diffusion terms in Museilak–spaces

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In this paper, a class of nonlinear evolution equations with damping arising in fluid dynamics and rheology is studied. The nonlinear term is monotone and possesses a convex potential but exhibits non-standard growth. The appropriate functional framework for such equations is the modularly Museilak-spaces. The existence and uniqueness of a weak solution are proved using an approximation approach by combining an internal approximation with the backward Euler scheme, also a priori error estimate for the temporal semi-discretization is given.

**Keywords:** *discrete solution, parabolic equation, weak solution, Museilak-spaces, non-standard growth, backward Euler scheme, intern approximations.*

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## 1. Introduction

There is considered the approximation of the initial boundary value problem for a non-linear parabolic equation that reads,

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,\nabla u) + K(u)) = f & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here, $\Omega$ is a bounded open set of $\mathbb{R}^d$ $(d \geqslant 2)$, $T > 0$, and $Q = \Omega \times (0,T)$. The stress $a\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to have the potential $\varphi\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^+$. Such an equation arises fluid dynamics and rheology (see [1,2]).

Throughout this paper, we assume that:
the field $b\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function, $b(0) = 0$, and there exists $b_0 \in \mathbb{R}$ such that

$$0 < b_0 < b'(s) < b_1 = 2b_0 \quad \text{for all} \quad s \in \mathbb{R}. \tag{2}$$

For any Musielak-function $\varphi$ (see definition below 2.1), the stress $a\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function such that for all $\xi, \xi^* \in \mathbb{R}^d$, for a.e. $x \in \Omega$

$$|a(x,\xi)| \leqslant \overline{\varphi}^{-1}\varphi(x,|\xi|), \tag{3}$$

$$(a(x,\xi) - a(x,\xi^*))(\xi - \xi^*) > 0, \tag{4}$$

$$a(x,\xi)\xi \geqslant \nu\varphi(x,|\xi|) \quad \text{for some} \quad \nu > 0, \tag{5}$$

 

The diffusion terms $K\colon \mathbb{R} \to \mathbb{R}^d$ is a continuous function such that

$$|K(s)| \leqslant \nu_0 \overline{\varphi}^{-1} \varphi\left(x, \frac{s}{\lambda}\right) \quad \text{for all } s \text{ in } \mathbb{R}, \text{ for some } \nu_0 > 0, \tag{6}$$

and

$$f \in L^1\big(0,T; L^2(\Omega)\big). \tag{7}$$

In classical Sobolev spaces, starting with the paper [3], the authors proved an existence result of a weak solution for the non coercive problem (1) in the stationary case $b(u) = 0$ using the symmetrization method. More later Di Nardo et al. [4] has shown the existence of renormalized solution for the parabolic version, more precisely in the linear case $b(u) = u$, and the uniqueness for such solutions in the paper [5], A. Aberqi et al. [6, 7] has proved the existence of a renormalized solution for (1) with more general parabolic terms $b(x, s)$.

In Orlicz spaces we refer to [8] where L. Aharouch, J. Bennouna have proved the existence and uniqueness of entropy solutions in the framework of Orlicz Sobolev spaces $W_0^1 L_\varphi(\Omega)$ assuming the $\triangle_2$-condition on the Orlicz-function $\varphi$. Recently, the uniqueness of renormalized solution of (1) in the general case has been proven by A. Aberqi et al. in [9] and by F. Kh. Mukminov in [10, 11] for the Cauchy problem for anisotropic parabolic equation using Kruzhkov's method of doubling the variable.

Concerning Musielak spaces, these are spaces that generalize Orlicz spaces, Lebesgue spaces with weight, and Lebesgue spaces with variable exponent, we refer to [2]. To our knowledge, articles dealing with this type of problem numerically, in these spaces, are rare. This prompt us to think about contributing to this study.

The difficulty encountered during the proof of the existence and uniqueness of discrete solution, is the fact that Musielak spaces $L_\varphi(Q)$ are not isometrically isomorphic to Musielak space $L_\varphi(0,T; L_\varphi(\Omega))$, the term $K$ does not satisfy the coercivity condition, and the nonlinearities are characterized by an Musielak–function $\varphi$, for which the $\Delta_2$-condition not imposed, and thus the spaces $L_\varphi(Q)$ and $W_0^{1,x} L_\varphi(Q)$ are not necessarily reflexive.

In this paper, we consider the weak formulation of (1) and propose a convergent full discretization combining a piecewise constant finite element approximation with the backward Euler scheme, also there is constructed an approximation solution sequence for problem (1) and establish a priori estimation. Our study is done on the isotropic case and generalizes [12] and [13] where the authors studied only the case $b(u) = u$ and $K = 0$.

The outline of this paper is structured as follows: in Section 2, we introduce the necessary notation, give a brief introduction to Musielak–Orlicz spaces. The description of the numerical method we employ, the construction of the Galerkin scheme, the proof of existence and uniqueness of the numerical solution, and the derivation of a priori estimates for the fully discrete solution and the discrete time derivative follow in Section 3. Finally, in Section 4, there is showed convergence towards and, thus, existence of an exact solution (weak solution of (1)), as well as its uniqueness. An error estimate for the temporal semidiscretization is contained in the Appendix.

## 2. Preliminaries and auxiliary results

### 2.1. Musielak function

Let $\Omega$ be an open subset of $\mathbb{R}^d$ $(d \geqslant 2)$ and let $\varphi$ be a real-valued function defined in $\Omega \times \mathbb{R}^+$. The function $\varphi$ is called a Musielak function if

- $\varphi(x, \cdot)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for $t > 0$, $\lim\limits_{t\to 0} \frac{\varphi(x,t)}{t} = 0$ and $\lim\limits_{t\to\infty} \frac{\varphi(x,t)}{t} = \infty$).
- $\varphi(\cdot, t)$ is a measurable function for all $t \geqslant 0$.

We put $\varphi_x(t)=\varphi(x,t)$ and associate its non-negative reciprocal function $\varphi_x^{-1}$ with respect to $t$, that is, $\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t$.

The Musielak function $\varphi$ is said to satisfy the $\Delta_2$-condition if for some $C>0$, and a non negative function $h$, integrable in $\Omega$,

$$\varphi(x,2t)\leqslant C\varphi(x,t)+h(x) \text{ for all } x\in\Omega \text{ and all } t\geqslant 0. \tag{8}$$

When (8) holds only for $t\geqslant t_0>0$, then $\varphi$ is said to satisfy the $\Delta_2$-condition near infinity.

Let $\varphi$ and $\gamma$ be two Musielak functions, we say that $\varphi$ dominates $\gamma$ denoting $\gamma\prec\varphi$ near infinity (respectively, globally) if there exist two positive constants $c$ and $t_0$ such that for a.e. $x\in\Omega$, $\gamma(x,t)\leqslant\varphi(x,ct)$ for all $t\geqslant t_0$ (respectively, for all $t\geqslant 0$). $\varphi$ and $\gamma$ are equivalents, denoting $\varphi\sim\gamma$ if $\varphi$ dominates $\gamma$ and $\gamma$ dominates $\varphi$. Finally, we say that $\gamma$ grows essentially less rapidly than $\varphi$ at 0 (respectively, near infinity), denoting $\gamma\ll\varphi$, if for every positive constant $c$, $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}=0$ (respectively, $\lim\limits_{t\to\infty}\sup\limits_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)})=0$).

**Proposition 1 (Ref. [14]).** Let $\gamma\ll\varphi$ near infinity and for all $t>0$, $\sup\limits_{x\in\Omega}\gamma(x,t)<\infty$, then for all $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\gamma(x,t)\leqslant\varphi(x,\varepsilon t)+C_\varepsilon,\quad \forall t>0. \tag{9}$$

### 2.2. Musielak space

Let $\varphi$ be a Musielak function and a measurable function $u\colon\Omega\to\mathbb{R}$, we define the functional $\varrho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)dx$. The set $K_\varphi(\Omega)=\{u\colon\Omega\to\mathbb{R} \text{ measurable}\colon \varrho_{\varphi,\Omega}(u)<\infty\}$ is called the Musielak class. The Musielak space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently,

$$L_\varphi(\Omega)=\left\{u\colon\Omega\to\mathbb{R} \text{ measurable}\colon \varrho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)<\infty, \text{ for some } \lambda>0\right\}.$$

On the other hand, we put $\overline{\varphi}(x,s)=\sup\limits_{t\geqslant 0}(st-\varphi(x,s))$. $\overline{\varphi}$ is called the Musielak function complementary to $\varphi$ (or conjugate of $\varphi$) in the sense of Young with respect to $s$. A sequence of functions $u_n\in L_\varphi(\Omega)$ is modular convergent to $u\in L_\varphi(\Omega)$ if there exists a constant $\lambda>0$ such that $\lim\limits_{n\to\infty}\varrho_{\varphi,\Omega}(\frac{u_n-u}{\lambda})=0$. This implies convergence for $\sigma(\prod L_\varphi,\prod L_{\overline{\varphi}})$ (see [16]).

In the space $L_\varphi(\Omega)$, there are defined two following norms:

$$\|u\|_\varphi=\inf\left\{\lambda>0\colon \int_\Omega\varphi\left(x,\frac{|u(x)|}{\lambda}\right)dx\leqslant 1\right\},$$

which are called the Luxemburg norm, and the so-called Musielak norm

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_{\overline{\varphi}}\leqslant 1}\int_\Omega|u(x)v(x)|dx,$$

where $\overline{\varphi}$ is the Musielak function complementary to $\varphi$. These two norms are equivalent [2]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$. $E_\varphi(\Omega)$ is defined as the subset of $L_\varphi(\Omega)$ of all measurable functions $u\colon\Omega\mapsto\mathbb{R}$ such that $\int_\Omega\varphi\big(x,\frac{|u(x)|}{\lambda}\big)dx<\infty$ for all $\lambda>0$. It is a separable space and $(E_\varphi(\Omega))^*=L_\varphi(\Omega)$ [15]. $E_\varphi(\Omega)=K_\varphi(\Omega)$ if and only if $\varphi$ satisfies the $\Delta_2$-condition for the large values of $t$ or for all values of $t$, according to whether $\Omega$ has finite measure or not.

For two complementary Musielak functions $\varphi$ and $\overline{\varphi}$, we have (see [16]) the Young inequality, $st \leqslant \varphi(x,s) + \overline{\varphi}(x,t)$ for all $s,t \geqslant 0$, $x \in \Omega$, the Hölder inequality, $\left| \int_\Omega u(x)v(x)dx \right| \leqslant \|u\|_{\varphi,\Omega} \||v|\|_{\overline{\varphi},\Omega}$, for all $u \in L_\varphi(\Omega)$, $v \in L_{\overline{\varphi}}(\Omega)$.

A sequence $u_n$ converges to $u$ for the modular convergence in $W^1L_\varphi(\Omega)$ or in $W_0^1L_\varphi(\Omega)$ if, for some $\lambda > 0$,

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

Let us define the following spaces of distributions:

$$W^{-1}L_{\overline{\varphi}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \colon f = \sum_{\alpha \leqslant 1} (-1)^\alpha D^\alpha f_\alpha, \text{ where } f_\alpha \in L_{\overline{\varphi}}(\Omega) \right\},$$

$$W^{-1}E_{\overline{\varphi}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \colon f = \sum_{\alpha \leqslant 1} (-1)^\alpha D^\alpha f_\alpha, \text{ where } f_\alpha \in E_{\overline{\varphi}}(\Omega) \right\}.$$

**Lemma 1 (Ref. [15]).** *(Approximation result) $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $\varphi$ and $\overline{\varphi}$ be two complementary Musielak functions satisfying the following conditions:*

- *there exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) > c$,*
- *there exist two constants $A, B > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leqslant \frac{1}{2}$,*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant A|t|^{\frac{B}{\log \frac{1}{|x-y|}}} \quad \textit{for all} \quad t \geqslant 1,$$

- *$\int_K \varphi(x,\lambda)dx < \infty$, for any constant $\lambda > 0$ and for every compact $K \subset \Omega$,*
- *there exists a constant $C > 0$ such that $\overline{\varphi}(y,t) \leqslant C$ a.e. in $\Omega$.*

Under these assumptions $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence. Consequently, the action of a distribution $S$ in $W^{-1}L_{\overline{\varphi}}(\Omega)$ on an element $u$ of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $\langle S, u\rangle$.

**Remark 1.** The second condition in Lemma 1 coincides with an alternative log-Hölder continuity condition for the variable exponent $p$, namely, there exists $A > 0$ such that for $x, y$ close enough and each $t \in \mathbb{R}^N$

$$|p(x) - p(y)| \leqslant \frac{A}{\log \frac{1}{|x-y|}}.$$

### 2.3. Inhomogeneous Musielak–Sobolev spaces

Let $\Omega$ be an bounded open subset $\mathbb{R}^d$ and let $Q = \Omega\times]0,T[$ with some given $T > 0$. Let $\varphi$ be an Musielak–function, for each $\alpha \in \mathbb{N}^d$, denote by $\nabla_x^\alpha$ the distributional derivative on $Q$ of order $\alpha$ with respect to the variable $x \in \mathbb{N}^d$. The inhomogeneous Musielak–Sobolev spaces are defined as follows,

$$W^{1,x}L_\varphi(Q) = \left\{ u \in L_\varphi(Q) \colon \nabla_x^\alpha u \in L_\varphi(Q), \forall \alpha \in \mathbb{N}^d, |\alpha| \leqslant 1 \right\}, \tag{10}$$

$$W^{1,x}E_\varphi(Q) = \left\{ u \in E_\varphi(Q) \colon \nabla_x^\alpha u \in E_\varphi(Q), \forall \alpha \in \mathbb{N}^d, |\alpha| \leqslant 1 \right\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leqslant m} \|\nabla_x^\alpha u\|_{\varphi,Q}.$$

One can easily show that they form a complementary system when $\Omega$ satisfies the Lipschitz domain [16]. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which have as many copies as there are $\alpha$-order derivatives, $|\alpha| \leqslant 1$. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q)$ then the function: $t \mapsto u(t) = u(t,\cdot)$ is defined on $(0,T)$ with values $W^1 L_\varphi(\Omega)$.

Furthermore the following imbedding holds:

$$W^{1,x}E_\varphi(Q) \subset L^1\big(0,T,W^{1,x}E_\varphi(\Omega)\big).$$

**Lemma 2 (Ref. [17]).** *Under the assumptions of Lemma 1, and by assuming that $\varphi(x,\cdot)$ decreases with respect to one of $x$-coordinate, there exists a constant $\lambda > 0$ which depends only on $\Omega$ such that*

$$\int_Q \varphi(x,|u|)dx\,dt \leqslant \int_Q \varphi(x,\lambda|\nabla u|)dx\,dt. \tag{11}$$

Section is ended by some useful lemmas

**Lemma 3.** *If $(u_n) \subset L^1(\Omega)$ with $u_n \to u$ a.e. in $\Omega$, $u_n, u \geqslant 0$ a.e. in $\Omega$ and $\int_\Omega u_n dx \to \int_\Omega u\,dx$, then $u_n \to u$ in $L^1(\Omega)$.*

**Lemma 4 (Ref. [18]).** *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(\Pi L_\varphi, \Pi L_{\overline{\varphi}})$.*

**Lemma 5 (Ref. [12]).** *Let $\{\xi_l\} \in \mathcal{L}_\varphi(Q)$ and there exists a positive constatnt $C$ such that $\int_Q \varphi(x,|\xi_l|)\,dx\,dt \leqslant C$ for all $l \in \mathbb{N}$. Then there exists $\xi \in \mathcal{L}_\varphi(Q)$ and a subsequence, denoted by $l'$, such that $\xi_{l'} \rightharpoonup \xi$ weakly in $L^1(Q)$ and*

$$\int_Q \varphi(x,|\xi|)\,dx\,dt \leqslant \liminf_{l'\to\infty} \int_Q \varphi(x,|\xi_{l'}|)\,dx\,dt.$$

Now, we denote by $\gamma_0 w$ the trace of $w\colon \overline{\Omega} \to \mathbb{R}$ such that $\gamma_0 w = w$ on $\partial\Omega$, for smooth $w$.

**Lemma 6 (Ref. [12]).** *Let $w \in \mathcal{W}$, where*

$$\mathcal{W} = \Big\{ w \in W^{1,1}(0,T;L^2(\Omega))\colon \nabla w \in (\mathcal{L}_\varphi(Q))^d, \gamma_0(w(\cdot,t)) = 0 \text{ for almost } t \in (0,T) \Big\}.$$

*For any $\varepsilon > 0$ there is then a smooth function $w_\varepsilon$ vanishing in $\partial\Omega \times (0,T)$ such that*

$$\|w_\varepsilon - w\|_{W^{1,1}(0,T;L^2(\Omega))} < \frac{\varepsilon}{2},$$

*and for all $\eta \in L_{\overline{\varphi}}(Q)$*

$$\left| \int_Q (\nabla(w_\varepsilon - w)\eta\,dx\,dt \right| < \frac{\varepsilon}{2}.$$

**Notation**. Let $X$ be a Banach space, $\mathcal{C}([0,T];X)$, denotes the usual space of the continuous functions $u\colon [0,T] \to X$, and $\mathcal{C}_w([0,T];X)$ denotes the space of demicontinuous functions (i.e., continuous with respect to the weak topology in X).

And let $T_k$, denotes the truncation function at level $k > 0$, defined on $\mathbb{R}$ by

$$T_k(r) = \max(-k, \min(k,r)).$$

## 3. Description of the numerical method

Section describes the numerical method we employ, the construction of the Galerkin scheme, the proof of existence and uniqueness of the numerical solution, and the derivation of a priori estimates for the fully discrete solution and the discrete time derivative.

### A full discretization:

1. **For the spatial descritization:** there is considered a generalized internal approximation $(V_m)_{m\in\mathbb{N}}$ of the space

$$V=\left\{v\in L^2(\Omega)\colon \nabla v\in (E_\varphi(\Omega))^d, \gamma_0 v=0\right\},\quad \|v\|_V=\|v\|_{2,\Omega}+\|\nabla v\|_{\varphi,\Omega},$$

and the restriction operators $R_m\colon V\to V_m$ such that for any sequence $\{m_l\}_{l\in\mathbb{N}}$ with $m_l\to\infty$ as $l\to\infty$ there holds

$$R_{m_l}v\to v \text{ in } V \text{ as } l\to+\infty \quad v\in V. \tag{12}$$

Since $V$ is separable Banach space, there exists a Galerkin basis and an internal approximation scheme for $V$ (for more details, see [12, 19]) and [20] for such construction of restriction operator.

2. **For the temporal descritization:** for $N\in\mathbb{N}$ $(N\geqslant 1)$, let $\tau=\frac{T}{N}$ and $t_n=n\tau$ $(n=0,1,\dots,N)$, according to Taylor–Young's formula

$$b(u(x,t_n))-b(u(x,t_{n-1}))=\partial_t b(u(x,t_{n-1}))\tau+\tau\varepsilon(\tau) \ \text{ such that } \ \lim_{\tau\to 0}\varepsilon(\tau)=0,$$

then the problem is to find a $\{u^n\}_{n=1}^N\subset V_m$ such that for $n=1,2,\dots,N$

$$\int_\Omega\left[\frac{b(u^n)-b(u^{n-1})}{\tau}v+a(\nabla u^n)\nabla v+K(u^n)\nabla v\right]dx=\int_\Omega f(\cdot,t_n)\,v\,dx,\ \forall v\in V_m. \tag{13}$$

Here, $u^0\in V_m$ denotes a suitable approximation of the initial value $u_0\in L^2(\Omega)$.

**Lemma 7.** *Fix $u^0\in V_m$ and assume that (3)–(6) hold true, then there exists a weak solution of*

$$\int_\Omega\left[\frac{b(u)-b(u^{n-1})}{\tau}v+a(\nabla u)\nabla v+K(u)\nabla v\right]dx=\int_\Omega f(\cdot,t_n)\,v\,dx,\quad \forall v\in V_m,$$

*and if $K$ satisfies the condition*

$$|K(s)-K(s')|\leqslant \nu_1|s-s'| \ \textit{ for all } \ s,s'\in\mathbb{R}, \ \textit{ and some } \ \nu_1>0, \tag{14}$$

*then the solution is unique.*

Indeed, for the existence of the weak solution to (13), we refer to [9, 21].
For the uniqueness, let $v$ and $w$ are two solutions of (13) and taking $\frac{1}{k}T_k(v-w)$ as a test function, thus

$$\frac{1}{k}\int_\Omega\frac{b(v)-b(w)}{\tau}T_k(v-w)dx=-\frac{1}{k}\int_\Omega(a(\nabla v)-a(\nabla w))\nabla T_k(v-w)dx \\ -\frac{1}{k}\int_\Omega(K(v)-K(w))\nabla T_k(v-w)dx.$$

We have

$$\lim_{k\to 0}\frac{1}{k}\int_\Omega\frac{b(v)-b(w)}{\tau}T_k(v-w)dx=\int_\Omega\frac{b(v)-b(w)}{\tau}\operatorname{sign}(v-w)dx\geqslant\frac{b_0}{\tau}\int_\Omega|v-w|dx.$$

By (4), $-\frac{1}{k}\int_\Omega(a(\nabla v)-a(\nabla w))\nabla T_k(v-w)dx\leqslant 0$. Using (14) one can get

$$\frac{1}{k}\int_{\Omega}(K(v)-K(w))\nabla T_k(v-w)dx \leqslant \frac{1}{k}\int_{|v-w|\leqslant k}\nu_1|v-w||\nabla T_k(v-w)|dx \leqslant \int_{|v-w|\leqslant k}\nu_1|\nabla(v-w)|dx.$$

Since $|\nabla(v-w)| \in L^1(\Omega)$ then $\lim\limits_{k\to 0}\frac{1}{k}\int_{\Omega}(K(v)-K(w))\nabla T_k(v-w)dx = 0$.
Thus

$$\frac{b_0}{\tau}\int_{\Omega}|v-w|dx \leqslant 0.$$

This implies that $v = w$. So the conclusion is that for any $u^0 \in V_m$ and $f \in L^1(0,T;L^2(\Omega))$, there exists a unique solution $\{u^n\}_{n=1}^N \subset V_m$ to (13).

3. **A priori estimates for the discrete solution:** Section presents some a priori estimates, being important to achieve the convergence of the numerical solution.

**Theorem 1.** *Let $u^0 \in V_m$ and $f \in L^1([0,T];L^2(\Omega))$. Let $\{u^n\}_{n=1}^N \subset V_m$ be the solution to (13) and $\tau \leqslant \tau_0 < 1$. Then there exist positive constants $c_1$, $c_2$, $c_3$ and $c_4$ depending on $p_0$, $p_1$, $\nu$, $T$ and $\tau_0$ such that for all $n = 1,2,\ldots,N$,*

$$\|b(u^n)\|_{2,\Omega}^2+c_1\sum_{j=1}^n\|b(u^j)-b(u^{j-1})\|_{2,\Omega}^2+c_2\sum_{j=1}^n\int_{\Omega}\varphi(x,|\nabla u^j|)dx \leqslant c_3\|f\|_{L^1([0,T];L^2(\Omega))}^2+c_1\|b(u^0)\|_{2,\Omega}^2, \tag{15}$$

*and*

$$\sum_{j=1}^n\|u^n-u^{n-1}\|_{2,\Omega} \leqslant c_4. \tag{16}$$

**Proof.** Taking $v = b(u^n)$ in (13) for the discrete time derivative,

$$\|b(u^n)\|_{2,\Omega}^2 - \|b(u^{n-1})\|_{2,\Omega}^2 + \|b(u^n)-b(u^{n-1})\|_{2,\Omega}^2 \\ + 2\tau\int_{\Omega}\left[b'(u^n)a(\nabla u^n)\nabla u^n + b'(u^n)K(\nabla u^n)\nabla u^n\right]dx \leqslant 2\tau\int_{\Omega}|f_n(x)||b(u^n)|dx.$$

Using the assumptions (11), (5), (6) and using the Young inequality, we obtain

$$\|b(u^n)\|_{2,\Omega}^2 - \|b(u^{n-1})\|_{2,\Omega}^2 + \|b(u^n)-b(u^{n-1})\|_{2,\Omega}^2 + 2\nu\tau b_0\int_{\Omega}\varphi(x,|\nabla u^n|)dx \\ \leqslant 2\tau\int_{\Omega}|f_n(x)||b(u^n)|dx + 2\tau b_1\int_{\Omega}\left[\varphi\left(x,\frac{u^n}{\lambda}\right)+\varphi(x,|\nabla u^n|)\right]dx \\ \leqslant 2\tau\int_{\Omega}|f_n(x)||b(u^n)|dx + 4\tau b_1\nu_0\int_{\Omega}\varphi(x,|\nabla u^n|)dx.$$

Choosing $\nu_0$ such that $\nu > \frac{2b_1}{b_0}\nu_0 = 4\nu_0$, and summation for all $(n = 1,2,\ldots,N)$ implies

$$\|b(u^n)\|_{2,\Omega}^2 + \sum_{j=1}^n\|b(u^j)-b(u^{j-1})\|_{2,\Omega}^2 + 2\tau(\nu b_0 - 2b_1\nu_0)\sum_{j=1}^n\int_{\Omega}\varphi(x,|\nabla u^j|)dx \\ \leqslant 2\tau\sum_{j=1}^n\int_{\Omega}|f_n(x)||b(u^n)|dx + \|b(u^0)\|_{2,\Omega}^2.$$

Taking $\|b(u^n)\|_{2,\Omega} = \max\limits_{j=1,\ldots,N}\|b(u^j)\|_{2,\Omega}$ and $\tau\sum_{j=1}^n\|f_n\|_{L^1([0,T];L^2(\Omega))}^2 \leqslant \|f\|_{L^1([0,T];L^2(\Omega))}^2$ and using Holder inequality,

$$\|b(u^n)\|_{2,\Omega}^2 + \sum_{j=1}^n\|b(u^j)-b(u^{j-1})\|_{2,\Omega}^2 + 2\tau(\nu b_0 - 2b_1\nu_0)\sum_{j=1}^n\int_{\Omega}\varphi(x,|\nabla u^j|)dx \\ \leqslant \frac{1}{\varepsilon^2}\|f\|_{L^1([0,T];L^2(\Omega))}^2 + \varepsilon^2\|b(u^n)\|_{2,\Omega}^2 + \|b(u^0)\|_{2,\Omega}^2,$$

for any $0 < \varepsilon < 1$. From where we have (15).

For the second inequality using only (2) to get $b_0|u^n - u^{n-1}| \leqslant |b(u^n) - b(u^{n-1})|$ and by (15), we deduce (16). ∎

## 4. Existence via convergence of approximate solutions

Let consider two sequences $\{m_l\}_{l\in\mathbb{N}}$ and $\{N_\ell\}_{\ell\in\mathbb{N}}$ such that $m_\ell \to \infty$, $N_\ell \to \infty$ as $\ell \to \infty$ and assume that $\tau_\ell \leqslant \tau_0 < 1$ for all $\ell \in \mathbb{N}$.

**Construction of approximate solutions** $(u_\ell)$**:** there is considered a sequence $\{u_\ell^0\}_{\ell\in\mathbb{N}}$ of approximation of the initial datum value such that $u_\ell^0 \in V_m$ and

$$u_\ell^0 \to u_0 \ \text{ in } \ L^2(\Omega) \ \text{ as } \ \ell \to \infty. \tag{17}$$

Approximate solutions are constructed on the whole time interval as follows:

- For the parabolic part, from the discrete solution $\{u^n\}_{n=1}^N$ corresponding to the time step size $\tau_\ell = \frac{T}{N_\ell}$.

  Let $\widehat{u}_l$ denote the linear spline interpolating $\left(t_0 = 0, b(u_\ell^0)\right), \left(t_1, b(u_\ell^1)\right), \ldots, \left(t_{N_\ell}, b(u_\ell^{N_\ell})\right)$, i.e.

$$\widehat{u}_\ell(t) = b(u_\ell^{n-1}) + \frac{b(u_\ell^n) - b(u_\ell^{n-1})}{\tau_\ell}(t - t_{n-1}) \ \text{ for } \ t \in [t_{n-1}, t_n], \ (n = 1, 2, \ldots, N_\ell).$$

- For the elliptic part and source data, let $u_\ell$ denote the piecewise constant function such that

$$u_\ell(.,t) = u_\ell^n \text{ if } t \in (t_{n-1}, t_n], \quad (n = 1, 2, \ldots, N_\ell), \quad u_\ell(\cdot, 0) = u_\ell^1.$$

  And also the piecewise constant in time approximation $f_\ell$ is defined by

$$f_\ell(\cdot, t) = f(\cdot, t_n) \text{ if } t \in (t_{n-1}, t_n], \quad (n = 1, 2, \ldots, N_\ell), \quad f_\ell(\cdot, 0) = f(\cdot, t_1).$$

### 4.1. Convergence of the numerical solution $(u_\ell)$

The first result of this paper can be summarized by the following Lemma.

**Lemma 8.** *Let $u_0 \in L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$. Consider the numerical solution of (1) by the scheme (13) on a sequence of finite dimensional subspaces such that (12) is satisfied, and time step sizes which tend to zero and are bounded away from one. For the approximation of the initial value, assume (17).*

*Then, there is a subsequence, denoted by $\ell'$, and element $u \in L^\infty(0,T;L^2(\Omega))$ with $\nabla u \in (\mathcal{L}_\varphi(Q))^d$ and $\gamma_0 u(\cdot,t) = 0$ for almost all $t \in (0,T)$, $z \in L^2(\Omega)$, $\alpha \in (\mathcal{L}_{\overline{\varphi}}(Q))^d$, such that, as $\ell \to \infty$,*

$$b(u_\ell) - \widehat{u}_\ell \to 0 \text{ in } L^2(Q); \quad b(u_{\ell'}), \widehat{u}_{\ell'} \rightharpoonup b(u) \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{18}$$

$$\widehat{u}_{\ell'}(\cdot,T) = b(u_{\ell'}(\cdot,T)) \rightharpoonup b(z) \text{ in } L^2(\Omega); \quad \nabla u_{\ell'} \rightharpoonup \nabla u \text{ in } (L_\varphi(Q))^d, \tag{19}$$

$$K(u_{\ell'}) \rightharpoonup K(u) \text{ weakly-* in } (L_{\overline{\varphi}}(Q))^d; \ a(\cdot, \nabla u_{\ell'}) \rightharpoonup \alpha \ \text{ weakly-* in } (L_{\overline{\varphi}}(Q))^d. \tag{20}$$

**Proof.**

1. By (17), the sequence $\{u_\ell^0\}$ is bounded in $L^2(\Omega)$. Using the definition of $u_\ell$ and $\widehat{u}_\ell$ we obtain

$$\|b(u_\ell) - \widehat{u}_\ell\|_{2,Q}^2 = \frac{\tau_\ell}{3}\sum_{n=1}^{N_\ell} \|b(u^n) - b(u^{n-1})\|_{2,\Omega}^2$$

   and by (15), $\lim\limits_{l\to\infty} \|b(u_\ell) - \widehat{u}_\ell\|_{2,Q}^2 = 0$.

Also the definition of the approximate solutions allows us to get

$$\|b(u_\ell)\|_{L^\infty(0,T;L^2(\Omega))} = \max_{n=1,2,\dots,N_\ell} \|b(u^n)\|_{2,\Omega},$$
$$\|\widehat{u}_\ell\|_{L^\infty(0,T;L^2(\Omega))} = \max_{n=1,2,\dots,N_\ell} \|b(u^n)\|_{2,\Omega},$$

and the inequality (15) shows the boundedness of $\{b(u_\ell)\}$ and $\{\widehat{u}_\ell\}$ in $L^\infty(0,T;L^2(\Omega))$. Thus weak* convergence of a subsequence in $L^\infty(0,T;L^2(\Omega))$ is stated.
Since the difference of both the sequences, tends to zero in $L^2(Q)$, their limits must coincide and denoted $\varpi$. The condition (2) allows us to write $\varpi$ in the form $\varpi = p(u)$. That is $b(u_{\ell'}), \widehat{u}_{\ell'} \rightharpoonup b(u)$ in $L^\infty(0,T;L^2(\Omega))$.

2. Since $\|\widehat{u}_\ell(\cdot,T)\|_{2,\Omega} = \|b(u_\ell)(\cdot,T)\|_{2,\Omega} = \|b(u^{N_\ell})\|_{2,\Omega}$, the a priori estimate in (15) proves the weak convergence of a subsequence of $\{\widehat{u}_\ell(\cdot,T)\}$ in $L^2(\Omega)$ and also its limit can be written as $b(z)$ where $z \in L^2(\Omega)$.
Likewise by definition (17)

$$\int_Q \varphi(x,|\nabla u_\ell|)\,dx\,dt = \tau_\ell \sum_{n=1}^{N_\ell} \int_\Omega \varphi(x,|\nabla u^n|)dx$$

is uniformly bounded (see (15))). However, from the boundedness of the modular boundedness of the Luxemburg norm follows. Therefore, $\{\nabla u_\ell\} \subset (\mathcal{L}_\varphi(Q))^d \subseteq (L_\varphi(Q))^d$ is bounded with respect to $\|\|_{\varphi,Q}$. Since $(L_\varphi(Q))^* = E_{\overline{\varphi}}(Q)$ is separable Banach space, we obtain weak* convergence of a subsequence in $(L_\varphi(Q))^d$ to an element $\xi \in (L_\varphi(Q))^d$ such that $\nabla u_{\ell'} \rightharpoonup^* \xi$.
On the other hand it is known that $\mathcal{C}_c^\infty(\Omega) \otimes \mathcal{C}_c^\infty(0;T) \subset E_{\overline{\varphi}}(Q)$, then for all $g \in \mathcal{C}_c^\infty(\Omega)$, $\psi \in \mathcal{C}_c^\infty(0;T)$ we get

$$\int_Q \xi g\,\psi\,dx\,dt = \lim_{\ell'\to\infty} \int_Q \nabla u_{\ell'} g\psi\,dx\,dt = \lim_{\ell'\to\infty} \int_Q u_{\ell'} \nabla g\psi\,dx\,dt = \int_Q u\nabla g\psi\,dx\,dt$$

since $u_{\ell'} \rightharpoonup u$ weakly in $L^\infty(0,T;L^2(\Omega))$, thus $\xi = \nabla u$ and in view of Lemma 5, we get $\nabla u \in (\mathcal{L}_\varphi(\Omega))^d$. Since the trace of $u_{\ell'}$ is zero and,

$$\int_Q \nabla u_{\ell'} z\,dx\,dt \to \int_Q \nabla u z\,dx\,dt, \qquad \int_Q u_{\ell'} \nabla z\,dx\,dt \to \int_Q u\nabla z\,dx\,dt$$

for all $z \in L^\infty(0,T;W^{1,q}(\Omega))$ with $q \geqslant 2$ as $\ell \to \infty$, also the limit $u$ must have vanishing trace for all $t \in (0,T)$.

3. Finally in view of (15) and (3)

$$\int_Q \overline{\varphi}(x,|a(x,|\nabla u_\ell|)|)\,dx\,dt \leqslant \int_Q \varphi(x,|\nabla u_\ell|)\,dx\,dt = \tau_\ell \sum_{n=1}^{N_\ell} \int_\Omega \varphi(x,|\nabla u_\ell^n|)dx$$

and by (6) and (11)

$$\int_Q \overline{\varphi}(x,|K(u_\ell)|)\,dx\,dt \leqslant \int_Q \varphi\left(x,\frac{|u_\ell|}{\lambda}\right)\,dx\,dt \leqslant \tau_\ell C \sum_{n=1}^{N_\ell} \int_\Omega \varphi(x,|\nabla u_\ell^n|)dx$$

are uniformly bounded. Consequently there exist $\alpha$ and $\beta$ in $(L_\varphi(Q))^d$ such that $a(\cdot,\nabla u_{\ell'}) \rightharpoonup \alpha$ and $K(u_{\ell'}) \rightharpoonup \beta$ weakly in $(L_\varphi(Q))^d$ for a subsequence. And by Lemma 5, $\alpha \in (\mathcal{L}_\varphi(Q))^d$ and $\beta \in (\mathcal{L}_\varphi(Q))^d$.
On the other hand using (14), $|K(u_\ell) - K(u)| \leqslant \nu_1|u_\ell - u|$ and as $u_\ell \to u$ a.e. in $Q$ and $K$ is continuous, we obtain $\beta = K(u)$. ■

**Remark 2.** We omit writing $\ell'$ for the sequence from Lemma 8 and $x$ from $a(x,\nabla u)$ for the sake of simplicity.

**Theorem 2.** *Convergence of approximate solutions. Let $u_0 \in L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$. Consider the numerical solution of (1) by the scheme (13) on a sequence of finite dimensional subspaces such that (12) is satisfied, and time step sizes which tend to zero and are bounded away from one. For the approximation of the initial value, assume (17).*

*Then there are subsequences denoted by $\{u_{\ell'}\}$ and $\{\widehat{u}_{\ell'}\}$ of piecewise constant in time and piecewise linear in time prolongations, respectively, of the numerical solutions converging weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ to an exact solution $u \in \mathcal{C}_w([0,T];L^2(\Omega))$ to (1) and to $p(u)$ respectively. Moreover, and*

$$b(u_{\ell'}(\cdot,T)) = \widehat{u}_{\ell'}(\cdot,T) \rightharpoonup b(u(\cdot,T)) \ \ \text{weakly in} \ \ L^2(\Omega), \tag{21}$$

$$\nabla u_{\ell'} \rightharpoonup \nabla u \ \text{weakly* in} \ (L_\varphi(Q))^d \ \ \text{and} \ \ \nabla u \in (\mathcal{L}_\varphi(Q))^d, \tag{22}$$

$$a(\nabla u_{\ell'}) \rightharpoonup a(\nabla u) \ \text{weakly* in} \ (L_{\overline{\varphi}}(Q))^d \ \ \text{and} \ \ a(\nabla u) \in (\mathcal{L}_{\overline{\varphi}}(Q))^d. \tag{23}$$

**Proof. Step 1:**
Remark that by definition of $\widehat{u}_\ell$, the numerical scheme (13) can be written as

$$\int_\Omega \Big[\partial_t \widehat{u}_\ell v + (a(\nabla u_\ell) + K(u_\ell))\nabla v\Big] dx = \int_\Omega f_\ell \, v \, dx \quad \text{for all} \quad v \in V_{m_\ell}, \tag{24}$$

this equation holds almost everywhere in $(0,T)$ as well as in the weak sense. This implies

$$\begin{aligned} -\int_Q \widehat{u}_\ell R_{m_l} v\, \psi' \, dx\, dt + \int_\Omega \widehat{u}_\ell(\cdot,T) R_{m_\ell} v\, \psi(T) dx - \int_\Omega \widehat{u}_\ell(\cdot,0) R_{m_\ell} v\, \psi(0) dx \\ + \int_Q (a(\nabla u_\ell) + K(u_\ell)) \nabla R_{m_\ell} v\, \psi \, dx\, dt = \int_Q f_\ell R_{m_\ell} v\, \psi\, dx\, dt \ \ \text{for all} \ \ v \in V, \psi \in \mathcal{C}^1([0,T]), \end{aligned}$$

where $\widehat{u}_\ell(\cdot,T) = b(u^{N_\ell})$ and $\widehat{u}_\ell(.,0) = b(u_\ell^0)$ and by (2) we have

$$|b(u^{N_\ell}) - b(z)| \leqslant b_1 |u^{N_\ell} - z| \ \text{and} \ |b(u_\ell^0) - b(u_0)| \leqslant b_1 |u_\ell^0 - u_0|,$$

then using Lemma 8 and (17) we obtain respectively $\widehat{u}_\ell(\cdot,T) \to b(z)$ and $\widehat{u}_\ell(\cdot,0) \to b(u_0)$ strongly in $L^2(\Omega)$ and also $f_\ell \to f$ strongly $L^1(0,T;L^2(\Omega))$.

Let now tends $\ell$ to $\infty$ for the others terms,

$$\begin{cases} R_{m_\ell} v\psi' \to v\psi' \quad \text{in} \quad L^1(0,T;L^2(\Omega)), \\ R_{m_\ell} v \to v \quad \text{in} \quad L^2(\Omega), \\ \nabla R_{m_\ell} v\psi \to \nabla v\psi \quad \text{in} \quad (E_\varphi(Q))^d, \\ R_{m_\ell} v\psi \to v\psi \quad \text{in} \quad L^1(0,T;L^2(\Omega)). \end{cases} \tag{25}$$

After applying Lemma 8 and (25), as $\ell \to \infty$ the next is true:

$$\begin{aligned} -\int_Q b(u) v\, \psi' \, dx\, dt + \int_\Omega b(z) v\, \psi(T) dx - \int_\Omega b(u_0) v\, \psi(0) dx \\ + \int_Q (\alpha + K(u)) \nabla v\, \psi \, dx\, dt = \int_Q f\, v\, \psi\, dx\, dt \quad \text{for all} \quad v \in V, \psi \in \mathcal{C}^1([0,T]). \end{aligned} \tag{26}$$

This follows from (12) and the definition of the norm in $V$. Observe that $\|\nabla R_{m_\ell} v\psi - \nabla v\psi\|_{\varphi,Q} \leqslant \|\psi\|_{\infty,[0,T]} \max(1,T) \|\nabla R_{m_\ell} v - \nabla v\|_{\varphi,\Omega}$ and $V \hookrightarrow W^{1,1}(\Omega) \cap L^2(\Omega)$. Relation (26) implies, by density arguments,

$$\begin{aligned} -\int_Q b(u) \partial_t w \, dx\, dt + \int_\Omega b(z) w(\cdot,T) dx - \int_\Omega b(u_0) w(\cdot,0) dx \\ + \int_Q (\alpha + K(u)) \nabla w \, dx\, dt = \int_Q f\, w\, dx\, dt \ \text{for all} \ w \in \mathcal{W}. \end{aligned} \tag{27}$$

Now, remark that the tensor product $V \otimes \mathcal{C}^1([0,T]) \subset \mathcal{W}$, which shows that (26) is a particular case of (27). The function $w_\varepsilon$ that exists in view of Lemma 6 for any $w \in \mathcal{W}$ can be approximated, with respect to the strong convergence in $\mathcal{C}^1(\overline{Q})$, by a polynomial vanishing at $\partial\Omega \times [0,T]$, which possesses a tensor structure and thus belongs to $V \otimes \mathcal{C}^1([0,T])$. For any $u \in L^\infty(0,T;L^2(\Omega))$, $z, u_0 \in L^2(\Omega)$, $\alpha \in (\mathcal{L}_{\overline{\varphi}}(Q))^d$, $f \in \mathcal{C}(0,T;L^2(\Omega))$, any $\varepsilon > 0$ $w \in \mathcal{W}$, there is hence (recalling also the continuous embedding of $W^{1,1}(0,T;L^2(\Omega))$ into $\mathcal{C}(0,T;L^2(\Omega))$ an element $w_\varepsilon \in V \otimes \mathcal{C}^1([0,T])$ such that

$$\left|\int_Q b(u)\partial_t(w_\varepsilon - w)dx\,dt\right| + \left|\int_\Omega b(z)(w_\varepsilon(\cdot,T) - w(\cdot,T))dx\right| + \left|\int_\Omega b(u_0)(w_\varepsilon(\cdot,0) - w(\cdot,0))dx\right| + \left|\int_Q (\alpha + K(u))\nabla(w_\varepsilon - w)dx\,dt\right| + \left|\int_Q f(w_\varepsilon - w)dx\,dt\right| < \varepsilon.$$

On the other hand, since $u \in L^\infty(0,T;L^2(\Omega))$, with $\nabla u \in (\mathcal{L}_\varphi(Q))^d \subset L^1(0,T;L_\varphi(\Omega))$, $\alpha \in (\mathcal{L}_{\overline{\varphi}}(Q))^d \subset L^1(0,T;L_{\overline{\varphi}}(\Omega))$, and $f \in L^1(0,T;L^2(\Omega))$, we see that for any $v \in V$ the functions

$$t \mapsto \int_\Omega b(u(\cdot,t))v\,dx, \quad t \mapsto \int_\Omega \alpha(\cdot,t)\nabla v\,dx, \quad t \mapsto \int_\Omega K(u)(\cdot,t)\nabla v\,dx, \quad t \mapsto \int_\Omega f(\cdot,t)v\,dx$$

are in $L^1(0,T)$ and with (26),

$$\frac{d}{dt}\int_\Omega b(u(\cdot,t))v\,dx = \int_\Omega (f(\cdot,t)v\,dx - \int_\Omega (\alpha(\cdot,t) + K(u)(\cdot,t))\nabla v)dx \tag{28}$$

holds true in the weak sense.

Thus, the function $t \mapsto \int_\Omega b(u(\cdot,t))v\,dx$ is absolutely continuous and since $V$ is dense in $L^2(\Omega)$ with respect to the strong convergence in $L^2(\Omega)$, we obtain $b(u) \in \mathcal{C}_w(0,T;L^2(\Omega))$.

**Step 2: Initial and final values**

- We now prove $b(u(\cdot,0)) = b(u_0) \in L^2(\Omega)$. For any $v \in V$, we have with (24)

$$\begin{aligned}\int_\Omega \widehat{u}_l^0 R_{m_\ell} v\,dx &= \left[\int_\Omega \widehat{u}_\ell(\cdot,t)R_{m_\ell}v\,dx\frac{t-T}{T}\right]_{t=0}^T \\ &= \int_0^T \left(\int_\Omega \partial_t\widehat{u}_\ell R_{m_\ell}v\,dx\frac{t-T}{T} + \int_\Omega \widehat{u}_\ell R_{m_\ell}v\,dx\frac{1}{T}\right)dt \\ &= \int_0^T \left\{\left[\int_\Omega f_\ell R_{m_\ell}vdx - \int_\Omega (a(\nabla u_\ell) + K(u_\ell))\nabla R_{m_\ell}v\,dx\right]\frac{t-T}{T} + \int_\Omega \widehat{u}_\ell R_{m_\ell}vdx\frac{1}{T}\right\}dt.\end{aligned}$$

Pass to the limit, integrating by parts and using (28)

$$\begin{aligned}\int_\Omega b(u_0)v\,dx &= \int_0^T \left(\left[\int_\Omega f\,v\,dx - \int_\Omega (\alpha + K(u))\nabla v\,dx\right]\frac{t-T}{T} + \int_\Omega p(u)\,v\,dx\frac{1}{T}\right)dt \\ &= \left[\int_\Omega b(u)v\,dx\frac{t-T}{T}\right]_{t=0}^T = \int_\Omega b(u(\cdot,0))v\,dx.\end{aligned}$$

- Changing now the function $t \to \frac{t-T}{T}$ by $t \to \frac{t}{T}$ and using the same argumentation as above provides that the limit $p(z)$ of $\widehat{u}_\ell(\cdot,T) = b(u_\ell(\cdot,T))$ weakly in $L^2(\Omega)$, is exactly $b(u(\cdot,T))$,

$$\widehat{u}_\ell(\cdot,T) \to b(z) = b(u(\cdot,T)) \in L^2(\Omega) \quad \text{as} \quad \ell \to \infty.$$

**Step 3:**

- For the convergence of the term $\int_Q K(u_\ell)\nabla b(u_\ell)\,dx\,dt$, we use (14) and Lemma 8, to deduce that $K(u_\ell)$ converges to $K(u)$ in $(L_{\overline{\varphi}}(Q))^d$ and

$$\int_Q K(u_\ell)\nabla b(u_\ell)\,dx\,dt \to \int_Q K(u)\nabla b(u)\,dx\,dt.$$

- For the source term, we know that

$$\int_Q f_\ell b(u_\ell)\,dx\,dt \to \int_Q f\,b(u)\,dx\,dt \quad \text{as} \quad \ell\to\infty.$$

- Let show that $\alpha = a(\nabla u)$. To do so, we employ a variant of Minty's monotony trick. Using $(a-b)a \geqslant \frac{1}{2}(a^2-b^2)$,

$$\begin{aligned}\int_Q \partial_t\widehat{u}b(u_\ell)\,dx\,dt &= \sum_{n=1}^{N_\ell}\int_\Omega \left(b(u^n)-b(u^{n-1})\right)b(u^n)dx\\ &\geqslant \frac{1}{2}\left(\|b(u^{N_\ell})\|_{2,\Omega}^2 - \|b(u_\ell^0)\|_{2,\Omega}^2\right)\\ &= \frac{1}{2}\left(\|b(u_\ell(\cdot,T))\|_{2,\Omega}^2 - \|b(u_\ell^0)\|_{2,\Omega}^2\right),\end{aligned}$$

which implies, because of the weak lower semi-continuity of the norm, the weak convergence of $u_\ell(\cdot,T)$ to $z=u(T)$ in $L^2(\Omega)$ and the strong convergence (17),

$$\frac{1}{2}\left(\|b(u(\cdot,T))\|_{2,\Omega}^2 - \|b(u^0)\|_{2,\Omega}^2\right) \leqslant \liminf_{\ell\to\infty}\int_Q \partial_t\widehat{u}b(u_\ell)\,dx\,dt. \tag{29}$$

For all $\eta\in(L^\infty(Q))^d$ with (2) and (4),

$$\int_Q a(\nabla u_\ell)\nabla b(u_\ell)\,dx\,dt \geqslant b_0\int_Q a(\nabla u_\ell)\eta\,dx\,dt + b_0\int_Q a(\eta)(\nabla u_\ell-\eta)\,dx\,dt.$$

Remark that $a(\eta)\in(E_{\overline{\varphi}}(Q))^d$ since $\eta\in(L^\infty(Q))^d$ and $a$ is continuous one. In the limit, we thus obtain (see again Lemma 8)

$$\liminf_{\ell\to\infty}\int_Q a(\nabla u_\ell)\nabla u_\ell\,dx\,dt \geqslant b_0\int_Q \alpha\,\eta\,dx\,dt + b_0\int_Q a(\eta)(\nabla u-\eta)\,dx\,dt. \tag{30}$$

Now taking $v=u_\ell(\cdot,t)\in V_{m_\ell}$ in (26) and using (29) and (30),

$$\begin{aligned}\frac{1}{2}\left(\|b(u(T))\|_{2,\Omega} - \|b(u^0)\|_{2,\Omega}\right) + b_0\int_Q \alpha\eta\,dx\,dt + b_0\int_Q a(\eta)(\nabla u_\ell-\eta)\,dx\,dt&\\ + \int_Q K(u)\nabla b(u)\,dx\,dt \leqslant \int_Q f\,b(u)\,dx\,dt.&\end{aligned} \tag{31}$$

Therefore the centered Steklov average of $u$, given by is considered

$$(S_hu)(\cdot,t) = \frac{1}{2h}\int_{t-h}^{t+h} b(u(\cdot,s))ds, \quad t\in[0,T],$$

where $h>0$ and where $u$ is extended by zero outside $[0,T]$. The properties of $b$ and $u$ imply that $S_hu\in\mathcal{W}$. It is known that

$$\lim_{h\to 0}\int_Q f S_hu\,dx\,dt = \int_Q f\,b(u)\,dx\,dt.$$

On the other hand, one can find with (27)

$$\begin{aligned}\int_Q fS_hu\,dx\,dt = -\int_Q b(u)\partial_tS_hu\,dx\,dt + \int_\Omega b(u(\cdot,T))S_hu(\cdot,T)\,dx&\\ -\int_\Omega b(u(\cdot,0))S_hu(\cdot,0)dx + \int_Q(\alpha+K(u))\nabla S_hu\,dx\,dt,&\end{aligned}$$

where $\partial_tS_hu(\cdot,t) = \frac{b(u(\cdot,t+h))-b(u(\cdot,t-h))}{2h}$, and thus

$$\int_Q b(u)\partial_t S_h u\,dx\,dt = \frac{1}{2h}\int_0^T\int_\Omega b(u(\cdot,t))(b(u(\cdot,t+h)) - b(u(\cdot,t-h)))\,dx\,dt$$
$$= \frac{1}{2h}\int_0^{T-h}\int_\Omega b(u(\cdot,t))b(u(\cdot,t+h))\,dx\,dt - \frac{1}{2h}\int_h^T\int_\Omega b(u(\cdot,t))b(u(\cdot,t-h))\,dx\,dt = 0. \quad (32)$$

Moreover,

$$\int_\Omega b(u(\cdot,T))S_h u(\cdot,T)dx = \frac{1}{2h}\int_{T-h}^T\int_\Omega b(u(\cdot,T))b(u(\cdot,s))\,dx\,ds$$
$$\to \frac{1}{2}\int_\Omega b(u(\cdot,T))^2dx = \frac{1}{2}\|b(u(\cdot,T))\|_{2,\Omega}^2 \quad \text{as} \quad h\to 0. \quad (33)$$

Recall here that $u \in \mathcal{C}_w([0,T];L^2(\Omega))$ and thus $s = T$ is a Lebesgue's point of the mapping $[0,T] \ni s \mapsto \int_\Omega b(u(\cdot,T))b(u(\cdot,s))dx$.
Similarly, we have

$$\int_\Omega b(u(\cdot,0))S_h u(\cdot,0)dx \to \frac{1}{2}\|b(u_0)\|_{2,\Omega}^2 \quad \text{as} \quad h\to 0.$$

Finally, we observe that

$$\int_Q \alpha\nabla S_h u\,dx\,dt - \int_Q \alpha\nabla b(u)\,dx\,dt = \frac{1}{2}\int_0^T\int_{t-h}^{t+h}\int_\Omega \alpha(\cdot,t)\nabla(b(u(\cdot,s)) - b(u(\cdot,t)))\,dx\,ds\,dt$$
$$= \frac{1}{2}\int_{-1}^1\int_0^T\int_\Omega \alpha(\cdot,t)\nabla(b(u(\cdot,t+rh)) - b(u(\cdot,t)))\,dx\,dr\,dt \to 0 \quad \text{as} \quad h\to 0. \quad (34)$$

Also, in analogy,

$$\int_Q K(u)\nabla S_h u\,dx\,dt - \int_Q K(u)\nabla b(u)\,dx\,dt \to 0 \quad \text{as} \quad h\to 0. \quad (35)$$

Since the translation of a function in the Musielak space $L_\varphi(Q)$ is continuous with respect to the weak convergence in $E_\varphi(Q)$ (see [22]).
Finally, as $h\to 0$, then

$$\frac{1}{2}\left(\|b(u(T))\|_{2,\Omega} - \|b(u^0)\|_{2,\Omega}\right) + b_0\int_Q \alpha\nabla b(u)\,dx\,dt + \int_Q K(u)\nabla b(u)\,dx\,dt = \int_Q f\,b(u)\,dx\,dt, \quad (36)$$

and from (31), (36) and (2), for all $\eta \in (L^\infty(Q))^d$

$$0 \leqslant \int_Q [b_0 a(\eta) - (b_1 - b_0)\alpha](\eta - \nabla u)\,dx\,dt = b_0\int_Q (a(\eta) - \alpha)(\eta - \nabla u)\,dx\,dt.$$

Following the modification of Minty's trick in [1], we set $Q_k = \{(x,t)\colon |\nabla u(x,t)| > k\}$ for any $k\in\mathbb{N}$. For arbitrary $i,j\in\mathbb{N}$ with $j<i$, arbitrary $\lambda>0$, and arbitrary $\zeta \in (L^\infty(Q))^d$, we take

$$\eta = \nabla u\chi_{Q\backslash Q_i} + \lambda\zeta\chi_{Q\backslash Q_j} = \begin{cases} 0 & \text{in} \quad Q_i, \\ \nabla u & \text{in} \quad Q_j\backslash Q_i, \\ \nabla u + \lambda\zeta & \text{in} \quad Q\backslash Q_j. \end{cases}$$

Thus

$$0 \leqslant -\int_{Q_i}(a(0) - \alpha)\nabla u\,dx\,dt + \lambda\int_{Q\backslash Q_j}(a(\nabla u + \lambda\zeta) - \alpha)\zeta\,dx\,dt.$$

Since $(a(0) - \alpha)\nabla u \in L^1(Q)$, we have

$$\int_{Q_i}(a(0) - \alpha)\nabla u\,dx\,dt \to 0 \quad \text{as} \quad i\to\infty,$$

then

$$0 \leqslant \lambda \int_{Q\backslash Q_j} (a(\nabla u + \lambda\zeta) - \alpha)\zeta \, dx \, dt.$$

On the other hand since $a$ is monotone, for $\lambda \in [0,1]$ it yields

$$a(\nabla u + \lambda\zeta)\zeta \leqslant a(\nabla u + \zeta)\zeta \in L^1(Q),$$

then by Dominated Convergence Theorem,

$$\int_{Q\backslash Q_j} (a(\nabla u + \lambda\zeta) - \alpha)\zeta \, dx \, dt \to \int_{Q\backslash Q_j} (a(\nabla u) - \alpha)\zeta \, dx \, dt \quad \text{as} \quad \lambda \to 0,$$

and thus

$$0 \leqslant \int_{Q\backslash Q_j} (a(\nabla u) - \alpha)\zeta \, dx \, dt \quad \text{for any } j \in \mathbb{N} \text{ and any } \xi \in (L^\infty(Q))^d.$$

The choice $\begin{cases} \zeta = -\frac{a(\nabla u) - \alpha}{|a(\nabla u) - \alpha|} & \text{if} \quad a(\nabla u) \neq \alpha \\ \zeta = 0, \quad \text{otherwise} \end{cases}$ allows us to get

$$\int_{Q\backslash Q_j} |a(\nabla u) - \alpha| \, dx \, dt \leqslant 0,$$

and thus $\alpha = a(\nabla u)$ almost everywhere in $Q\backslash Q_j$, since $j$ was arbitrary, this proves the equality almost everywhere in $Q$, which complete the proof.

**Step 4: Uniqueness**

Let $u$ and $v$ be two solutions to the problem with the same data $(u_0, f)$. From the proof above,

$$\int_Q (b(u) - b(v))\partial_t w \, dx \, dt + \int_\Omega (b(u(\cdot,T)) - b(v(\cdot,T)))w(\cdot,T)dx$$
$$+ \int_Q (a(\nabla u) - a(\nabla v))\nabla w \, dx \, dt + \int_Q (K(u) - K(v))\nabla w \, dx \, dt = 0 \ \text{ for all } \ w \in \mathcal{W}.$$

Thus

$$\int_Q \partial_t(b(u) - b(v))w \, dx \, dt \leqslant -\int_Q (a(\nabla u) - a(\nabla v))\nabla w \, dx \, dt - \int_Q (K(u) - K(v))\nabla w \, dx \, dt. \tag{37}$$

For all $\bar{t} \in [0;T]$, taking $w = \frac{1}{k} T_k(b(u) - b(v)) w_{\varepsilon,\bar{t}}$ where

$$w_{\varepsilon,\bar{t}}(t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant t \leqslant \bar{t} - \varepsilon, \\ \frac{\bar{t} - t}{\varepsilon} & \text{if} \quad \bar{t} - \varepsilon < t \leqslant \bar{t}, \\ 0 & \text{otherwise.} \end{cases}$$

In the same way as the above

$$\lim_{k \to 0} \int_Q (K(u) - K(v))\nabla w \, dx \, dt = 0,$$

and by the monotonicity of $a$,

$$\int_Q \partial_t(b(u) - b(v)) \operatorname{sign}(b(u) - b(v)) w_{\varepsilon,\bar{t}} \, dx \, dt = \lim_{k \to 0} \int_Q \partial_t(b(u) - b(v)) w \, dx \, dt \leqslant 0.$$

On the other hand,

$$\int_Q \partial_t(b(u) - b(v)) \operatorname{sign}(b(u) - b(v)) w_{\varepsilon,\bar{t}} \, dx \, dt$$

$$= \int_0^{\bar{t}-\varepsilon}\int_\Omega \partial_t(b(u)-b(v))\operatorname{sign}(b(u)-b(v))\,dx\,dt + \int_{\bar{t}-\varepsilon}^{\bar{t}} \frac{\bar{t}-t}{\varepsilon}\int_\Omega \partial_t(b(u)-b(v))\operatorname{sign}(b(u)-b(v))\,dx\,dt.$$

Employing Dominated Convergence Theorem, as $\varepsilon \to 0$ we get

$$0 \geqslant \int_0^{\bar{t}}\int_\Omega \partial_t(b(u)-b(v))\operatorname{sign}(b(u)-b(v))\,dx\,dt = \int_0^{\bar{t}} \frac{d}{dt}\|b(u)-b(v)\|_{L^1(\Omega)}\,dt,$$

and

$$\|b(u(\cdot,\bar{t}))-b(v(\cdot,\bar{t}))\|_{L^1(\Omega)} = 0 \quad \text{for all} \quad \bar{t}\in(0;T],$$

and thus the uniqueness is proved. ■

## Appendix

**Error estimate for the temporal semi-discretization**: we just give the error estimate in the temporal semi-discretization case since in the complete discretization is far from being easy to make. Let $e_n = u(\cdot,t_n) - u_n$ be the error between the exact solution and the numerical solution.

**Theorem 3.** *Let $u_0, u^0 \in L^2(\Omega)$, $f \in L^1(0,T;L^2(\Omega))$, $u, \partial_t u, \partial^2_{tt} u \in L^1(0,T;L^2(\Omega))$ with $u(\cdot,t) \in \mathcal{V} = \{v \in L^2(\Omega)\colon \nabla v \in (\mathcal{L}_\varphi(\Omega))^d, \gamma_0 u = 0\}$, $a(\nabla u(\cdot,t)), K(u(\cdot,t)) \in (\mathcal{L}_{\overline{\varphi}}(\Omega))^d$ for all $t\in[0,T]$. Let $u^n \in \mathcal{V}$ with $a(\nabla u^n), K(u^n) \in (\mathcal{L}_{\overline{\varphi}}(\Omega))^d$ be an approximation of $u(.,t_n)$ such that $n = 1,2,\dots,N$,*

$$\int_\Omega \left[\frac{b(u^n)-b(u^{n-1})}{\tau} v + a(\nabla u^n)\nabla v + K(u^n)\nabla v\right] dx = \int_\Omega f(\cdot,t_n)\, v\,dx \quad \text{for all} \quad v\in\mathcal{V}.$$

*Then for $n = 1,2,\dots,N$,*

$$\|e^n\|_{L^1(\Omega)} \leqslant C\left(\left\|u_0 - u^0\right\|_{L^1(\Omega)} + \tau\left\|\partial^2_{tt} u\right\|_{L^1(0,T;L^2(\Omega))} + \left\|\overline{f} - f\right\|_{L^1(\Omega)}\right), \tag{38}$$

*where $\overline{f}$ denotes the piecewise constant in time interpolation of $f$ with respect to $(t_n)_{n=1}^N$.*

**Proof.** Let $R^n = b(u(\cdot,t_n)) - b(u^n)$ then

$$\int_\Omega \frac{R^n - R^{n-1}}{\tau} v\,dx = \int_\Omega \frac{b(u(\cdot,t_n)) - b(u(\cdot,t_{n-1}))}{\tau} v\,dx - \int_\Omega \frac{b(u^n)-b(u^{n-1})}{\tau} v\,dx.$$

Integration par parts and(13) results as

$$\begin{aligned}
\int_\Omega \frac{R^n - R^{n-1}}{\tau}\, v\,dx &= \int_\Omega \partial_t b(u(\cdot,t_n))\, v\,dx - \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_\Omega (t-t_{n-1})\partial_{tt} b(u(\cdot,t))\, v\,dx\,dt \\
&\quad - \int_\Omega \frac{b(u^n)-b(u^{n-1})}{\tau}\, v\,dx \\
&= \int_\Omega (f(\cdot,t_n) - f_n)\, v\,dx - \int_\Omega (a(\nabla u(.,t_n)) - a(\nabla u^n))\, v\,dx \\
&\quad - \int_\Omega \left(K(u(\cdot,t_n)) - K(u^n)\right)\, v\,dx - \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_\Omega (t-t_{n-1})\partial_{tt} b(u(\cdot,t))\, v\,dx\,dt.
\end{aligned} \tag{39}$$

Taking $v_k = \frac{1}{k}T_k(R^n)$ and using (4),

$$\begin{aligned}
\left|\int_\Omega (f(\cdot,t_n) - f_n)\, v\,dx\right| &\leqslant \|f(\cdot,t_n) - f_n\|_{L^1(\Omega)} - \int_\Omega (a(\nabla u(\cdot,t_n)) - a(\nabla u^n))\nabla v\,dx \\
&\leqslant -p_0 \int_{\{|R^n|\leqslant k\}} \left(a(\nabla u(.,t_n)) - a(\nabla u^n)\right)\nabla(u(.,t_n) - u^n)\,dx \leqslant 0,
\end{aligned}$$

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_\Omega (t-t_{n-1})\partial^2_{tt} b(u(\cdot,t))\, v\,dx\,dt \leqslant \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_\Omega (t-t_{n-1})|\partial^2_{tt} b(u(\cdot,t))|\,dx\,dt,$$

and

$$\left|\int_\Omega (K(u(\cdot,t_n)) - K(u^n))\, v\, dx\right| \leqslant \frac{p_0\nu_1}{k}\int_{\{|R^n|\leqslant k\}} |u(\cdot,t_n) - u^n|\, |\nabla u(\cdot,t_n) - \nabla u^n|\, dx$$
$$\leqslant \frac{p_0\nu_1}{k}\int_{\{|R^n|\leqslant k\}} |\nabla u(\cdot,t_n) - \nabla u^n|\, dx.$$

Since $|\nabla u(\cdot,t_n)) - \nabla u^n| \in L^1(\Omega)$, we pass to the limit as $k \to +\infty$, and

$$\lim_{k\to+\infty}\int_\Omega (K(u(\cdot,t_n)) - K(u^n))\, v\, dx = 0,$$

and also

$$\lim_{k\to+\infty}\int_\Omega \frac{R^n - R^{n-1}}{\tau}\, v_k\, dx = \int_\Omega \frac{R^n - R^{n-1}}{\tau}\,\mathrm{sign}(R^n)\, dx.$$

Recalling (39),

$$\int_\Omega \frac{|R^n|}{\tau}\, dx \leqslant \|f(\cdot,t_n) - f_n\|_{L^1(\Omega)} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_\Omega (t - t_{n-1})|\partial^2_{tt} b(u(\cdot,t))|\, dx\, dt + \int_\Omega \frac{|R^{n-1}|}{\tau}\, dx.$$

Thus summation $n = 1,\ldots,N$ and using (2)

$$b_0\|e^n\|_{L^1(\Omega)} \leqslant \tau\sum_{n=1}^{N}\|f(\cdot,t_n) - f_n\|_{L^1(\Omega)} + 2\tau\int_0^T |\partial^2_{tt} b(u(\cdot,t))|_{L^1(\Omega)} dt + p_1\|e^0\|_{L^1(\Omega)}.$$

Thus, together with the estimate $\tau\sum_{n=1}^{N}\|f(\cdot,t_n) - f_n\|_{L^1(\Omega)} \leqslant c\|\overline{f} - f\|_{L^1(\Omega)}$, and $L^1(0,T;L^2(\Omega)) \subset L^1(0,T;L^1(\Omega))$, we deduce (38). ■

**Comments**: the lack of regularity results for weak solutions in Museilak spaces makes the task of showing convergence results for such regular solutions very difficult.


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## Дискретний розв'язок нелінійних параболічних рівнянь із дифузійними членами в просторах Мусейлака


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У цій статті досліджується клас нелінійних еволюційних рівнянь зі загасанням, що виникають у гідродинаміці та реології. Нелінійний член монотонний і має опуклий потенціал, але нестандартно зростає. Відповідним функціональним каркасом для таких рівнянь є модульні простори Музейлака. Доведено існування та єдиність слабкого розв'язку, використовуючи наближений підхід та комбінуючи внутрішнє наближення зі зворотною схемою Ейлера, а також дано апріорну оцінку похибки часової напівдискретизації.



**Ключові слова:** *дискретний розв'язок, параболічне рівняння, слабкий розв'язок, простори Мусейлака, нестандартне зростання, зворотня схема Ейлера, внутрішнє наближення.*