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Mathematical modeling of wave propagation in viscoelastic media with the fractional Zener model

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The question of interest for the presented study is the mathematical modeling of wave propagation in dissipative media. The generalized fractional Zener model in the case of dimension d ($d = 1, 2, 3$) is considered. This work is devoted to the mathematical analysis of such model: existence and uniqueness of the strong and weak solution and energy decay result which guarantees the wave dissipation. The existence of the weak solution is shown using a priori estimates for solutions which are also presented.

Keywords: *fractional derivative, strong solution, weak solution, energy decay, plane waves, viscoelastic waves, Zener's model.*

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1. Introduction

Fractional calculus is used to model various problems in mechanics, physics, engineering and biology, etc. [1–6]. Fractional viscoelasticity is among the first areas where fractional operators have been applied. In addition, it has proven to be a tool important for the description of the memory property of many materials and processes since the fractional derivative of a function takes into account the entire history of the function and does not only reflect instantaneous characteristics as in the case of the elastic. It also fits naturally into the mechanical modeling of materials that retain the memory of past transformations [7, 8].

There are many different approaches to define fractional derivatives, the approaches that are frequently used in applications are Riemann–Liouville [1] and Caputo [7]. In our model, there is used Caputo approach, among its properties, the derivative of a constant equals zero and it is more suitable for numerical simulation [9–11].

We are interested in the analysis and mathematical study of the viscoelastic problem using the fractional Zener model in the general case. In particular, Moczo and Kristek [12] showed the full equivalence of the generalized Zener model and the generalized Maxwell model in definition by Emmerich & Korn [13]. On the other hand, in order to show the existence of the strong solution, Konjik et al. [14] used the fractional Zener model with the left derivative of Riemann–Liouville. Atanackovic et al. [7] used the fractional space-time distributional Zener model with the left Riemann–Liouville derivative and the symmetric Caputo derivative for the constraint. In [15] we used semi-group theory to study the existence and uniqueness of the strong solution in space-time for fractional Zener model with the derivative of Caputo. In addition, Atanackovic et al. [16] proposed a constitutive equation for a generalized Zener-type viscoelastic body that includes fractional derivatives of stress and strain of real and complex order. Also he presented a solution for the model studied in the form of a convolution based on the Laplace transform. In the present paper, the existence and uniqueness of both: strong and weak solutions for the generalized fractional Zener model are studied. Similar results are obtained in [17] in the case of integer derivative.

Introduction ends with a brief summary. In section 2, interesting results on the analysis of plane wave propagation in homogeneous media are presented. Section 3 is devoted to the study of heterogeneous media. First, we show the existence and uniqueness of the strong solution using the semi-group theory. Then, we present an energy decay, a result that guarantees the dissipation of the model. A priori estimates of the solution are also obtained. Finally, using this, the existence and uniqueness of the weak solution are shown. Conclusions are given in section 4.

2. Studies of homogeneous media by plane waves

In this section, we conduct a plane wave analysis in the case of a $3D$ isotropic viscoelastic medium. The isotropic problem in a homogeneous medium is considered (see [17])

$$\begin{cases} \rho\, \partial_{tt}^2 u - \operatorname{div}\sigma = 0, \\ \sigma + \tau_0 \partial_t^\alpha \sigma = \lambda \operatorname{Tr}(\varepsilon(u)) I + 2\mu\varepsilon(u) + \tau_0\big[\gamma_\lambda \operatorname{Tr}(\varepsilon(\partial_t^\alpha u)) I + 2\mu\gamma_\mu \varepsilon(\partial_t^\alpha u)\big]. \end{cases} \tag{1}$$

Form of the particular solution we are interested in is the next:

$$\begin{cases} u(\boldsymbol{x},t) = u_0 e^{i(\omega t - \boldsymbol{k}\boldsymbol{x})}\boldsymbol{d}, \\ \boldsymbol{\sigma} = \sigma_0 e^{i(\omega t - \boldsymbol{k}\boldsymbol{x})}\boldsymbol{D}. \end{cases} \tag{2}$$

with $\boldsymbol{k} = (k_1,k_2,k_3)^t$, $\boldsymbol{x} = (x_1,x_2,x_3)^t$, $\boldsymbol{d} = (d_1,d_2,d_3)^t$,

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}.$$

Let $\Psi = e^{i(\omega t - \boldsymbol{k}\boldsymbol{x})}$.Using expressions (1), one can get the equations:

$$\begin{cases} \partial_{tt}^2 u = -\omega^2 u_0 \Psi \boldsymbol{d}, \\ \operatorname{div}\boldsymbol{\sigma} = -i\sigma_0 \Psi \boldsymbol{D}\boldsymbol{k}, \\ \partial_t^\alpha \boldsymbol{\sigma} = (i\omega)^\alpha \sigma_0 \Psi \boldsymbol{D}, \\ \boldsymbol{\sigma} + \tau_0 \partial_t^\alpha \boldsymbol{\sigma} = (1 + (i\omega)^\alpha \tau_0)\sigma_0 \Psi \boldsymbol{D}, \\ \vartheta(u) = -iu_0\Psi \begin{bmatrix} k_1 d_1 & (k_2 d_1 + k_1 d_2)/2 & (k_3 d_1 + k_1 d_3)/2 \\ (k_1 d_2 + k_2 d_1)/2 & k_2 d_2 & (k_3 d_2 + k_2 d_3)/2 \\ (k_1 d_3 + k_3 d_1)/2 & (k_2 d_3 + k_3 d_2)/2 & k_3 d_3 \end{bmatrix}, \\ \varepsilon(\partial_t^\alpha u) = (i\omega)^\alpha \varepsilon(u), \\ \operatorname{Tr}(\varepsilon(u)) = -iu_0 \Psi \boldsymbol{k}\cdot\boldsymbol{d}, \\ \operatorname{Tr}(\varepsilon(\partial_t^\alpha u)) = (i\omega)^\alpha \operatorname{Tr}(\varepsilon(u)) = -i(i\omega)^\alpha u_0 \Psi \boldsymbol{k}\cdot\boldsymbol{d}, \end{cases} \tag{3}$$

Replacing these equations in (1), the dispersion relation is obtained :

$$\left(i^\alpha \omega^{\alpha+2} + \omega^2 - \left(\frac{\mu}{\rho} + (i\omega)^\alpha \tau_0 \frac{\mu\gamma_\mu}{\rho}\right)|\boldsymbol{k}|^2\right)\boldsymbol{d} = \left[\frac{\lambda+\mu}{\rho} + (i\omega)^\alpha \tau_0 \frac{\lambda\gamma_\lambda + \mu\gamma_\mu}{\rho}\right](\boldsymbol{k}\cdot\boldsymbol{d})\boldsymbol{k} \tag{4}$$

By introducing the velocity v_p and v_s and the relaxation times τ_p and τ_s, the equation (4) becomes:

$$\left(i^\alpha \omega^{\alpha+2} + \omega^2 - v_s^2\left(1 + (i\omega)^\alpha \tau_s\right)|\boldsymbol{k}|^2\right)\boldsymbol{d} = \left[v_p^2(1 + (i\omega)^\alpha \tau_p) - v_s^2(1 + (i\omega)^\alpha \tau_s)\right](\boldsymbol{k}\cdot\boldsymbol{d})\boldsymbol{k}, \tag{5}$$

with

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad v_s = \sqrt{\frac{\mu}{\rho}}, \quad \tau_p = \tau_0 \frac{\lambda\gamma_\lambda + 2\mu\gamma_\mu}{\lambda + 2\mu}, \quad \tau_s = \tau_0\gamma_\mu.$$

The scalar product of this relation by **k** implies in particular

$$\boldsymbol{k}\cdot\boldsymbol{d}\left[\omega^2\left((i\omega)^\alpha+1\right)-v_p^2(1+(i\omega)^\alpha\tau_p)|\boldsymbol{k}|^2\right]. \tag{6}$$

Two cases are then presented, depending on $\boldsymbol{k}\cdot\boldsymbol{d}$:

1. **The P waves** (compression waves): $\boldsymbol{k}\cdot\boldsymbol{d}\neq 0$.
   In this case, the equality (5) shows that $\boldsymbol{k}$ and $\boldsymbol{d}$ are collinear and (6) gives the dispersion relation of the P waves:
$$\omega^2\left((i\omega)^\alpha+1\right)=v_p^2|\boldsymbol{k}|^2(1+(i\omega)^\alpha\tau_p). \tag{7}$$
2. **The S waves** (shear waves): $\boldsymbol{k}\cdot\boldsymbol{d}=0$.
   The vectors $\boldsymbol{k}$ and $\boldsymbol{d}$ are therefore orthogonal and (5) gives the dispersion relation of the S waves:
$$\omega^2\left((i\omega)^\alpha+1\right)=v_s^2|\boldsymbol{k}|^2(1+(i\omega)^\alpha\tau_s). \tag{8}$$

We can notice that, if we change the variable $S=i\omega$, then the equations (7) and (8) are written in the form
$$\tau_0S^{\alpha+2}+S^2+\tau_jv_j^2|\boldsymbol{k}|^2S^\alpha+v_j^2|\boldsymbol{k}|^2=0,\quad \forall j=p,s, \tag{9}$$
with $0<\alpha<1$, this equation admits one real root and two conjugated complex roots, we are proving that numerically. Let us note that a real root $S=S^*$ corresponds to a non-propagative mode (purely damped mode) and two conjugated complex roots $S=\eta\pm i\omega^*$ corresponds to two propagative modes (see [17]). The properties of these modes:

— a purely amortized mode corresponding to $S=S^*$. The high frequencies are less damped (relaxation time τ_1) than the low frequencies (relaxation time $\tau_0<\tau_1$);
— two amortized propagative modes corresponding to $S=\eta\pm i\omega$. High frequencies are more damped than low frequencies;
— for propagative modes, high frequencies propagate faster but are more damped.

3. Study of heterogeneous media

3.1. Model Problem

The fractional Zener model for wave propagation is considered in the general case. Our goal is to determine the displacement $u\colon\mathbb{R}^d\times[0,T]\mapsto\mathbb{R}$ and the stress $\sigma\colon\mathbb{R}^d\times[0,T]\mapsto\mathbb{R}$ which verify:

$$\begin{cases}\rho(\boldsymbol{x})\dfrac{\partial^2u}{\partial t^2}-\operatorname{div}\boldsymbol{\sigma}=f(\boldsymbol{x},t), & (\boldsymbol{x},t)\in\mathbb{R}^d\times\,]0,T[\,,\\ \boldsymbol{\sigma}(\boldsymbol{x},t)+\tau_0(\boldsymbol{x})\dfrac{\partial^\alpha\boldsymbol{\sigma}}{\partial t^\alpha}=\boldsymbol{C}\boldsymbol{\varepsilon}(u)+\tau_0(\boldsymbol{x})\boldsymbol{D}\boldsymbol{\varepsilon}\left(\dfrac{\partial^\alpha u}{\partial t^\alpha}\right), & (\boldsymbol{x},t)\in\mathbb{R}^d\times\,]0,T[\,,\\ u(\boldsymbol{x},0)=u_0;\quad \dfrac{\partial u}{\partial t}(\boldsymbol{x},0)=u_1;\quad \boldsymbol{\sigma}(\boldsymbol{x},0)=\boldsymbol{C}\boldsymbol{\varepsilon}(u_0), & \boldsymbol{x}\in\mathbb{R}^d.\end{cases} \tag{10}$$

Where div is the divergence operator, $\rho(\boldsymbol{x})$ is the volume density, τ_0 is a relaxation time, f is the source density, and $\boldsymbol{C}$ and $\boldsymbol{D}$ are two tensors 4×4 that satisfy:

$$C_{ijkl}=C_{jilk}=C_{ijkl},\quad D_{ijkl}=D_{jilk}=D_{klij}.$$

We assume that there exist B_-, B_+ two positive constants, such as

$$0\leqslant B_-|\sigma|^2\leqslant C\sigma\colon\sigma\leqslant B_+|\sigma|^2,\quad \forall\sigma\in\mathcal{L}^{sym}(\mathbb{R}^d)\quad \text{a.e.}\quad \boldsymbol{x}\in\mathbb{R}^d, \tag{11}$$

$$0\leqslant B_-|\sigma|^2\leqslant D\sigma\colon\sigma\leqslant B_+|\sigma|^2,\quad \forall\sigma\in\mathcal{L}^{sym}(\mathbb{R}^d),\quad \text{a.e.}\quad \boldsymbol{x}\in\mathbb{R}^d. \tag{12}$$

Where the functional space $\mathcal{L}^{sym}(\mathbb{R}^d)$ is defined by:

$$\mathcal{L}^{sym}(\mathbb{R}^d) = \left\{\sigma \in \mathcal{L}^1(\mathbb{R}^d), \quad \sigma_{ij} = \sigma_{ji} \quad \forall i,j = 1,\ldots,d\right\}.$$

In addition, we will consider the following assumptions:

$$\begin{cases} 0 \leqslant \rho_- \leqslant \rho(x) \leqslant \rho_+ \leqslant +\infty & \text{a.e.} \quad \boldsymbol{x} \in \mathbb{R}^d, \\ 0 \leqslant \tau_- \leqslant \tau_0(x) \leqslant \tau_+ \leqslant +\infty & \text{a.e.} \quad \boldsymbol{x} \in \mathbb{R}^d. \end{cases}$$

3.2. Reformulation of the model problem

As in the monodimensional case (see [15]), we write the problem (10) as a first-order evolutionary system using the following auxiliary differential equation [18]:

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(\boldsymbol{x},t,\xi) = -\xi\,\psi(\boldsymbol{x},t,\xi) + \dfrac{\partial s}{\partial t}(\boldsymbol{x},t), \\ \psi(\boldsymbol{x},0,\xi) = 0. \end{cases} \tag{13}$$

We have the following proposition (see [15]).

Proposition 2. Suppose that ψ is the solution of the ordinary differential equation (13). Then

$$\partial_t^\alpha \boldsymbol{s}(\boldsymbol{x},t) = \frac{\partial^\alpha \boldsymbol{s}}{\partial t^\alpha}(\boldsymbol{x},t) = \int_0^{+\infty} \psi(\boldsymbol{x},t,\xi)\,dM_{1-\alpha}(\xi) = \int_0^{+\infty} \frac{\partial \varphi}{\partial t}(\boldsymbol{x},t,\xi)\,dM_{1-\alpha}(\xi), \tag{14}$$

with $dM_\alpha(\xi) = \dfrac{\sin(\alpha\pi)}{\pi}\xi^{-\alpha}\,d\xi$ and $\psi = \dfrac{\partial\varphi}{\partial t}$.

And by introducing the variables $\partial_t u = v$ and $\boldsymbol{s} = \boldsymbol{\sigma} - \boldsymbol{C}\boldsymbol{\varepsilon}(u)$ (the difference between viscoelastic and elastic stress) and by deriving the second equation from ∂_t^β with $\beta = 1-\alpha$, then we get:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v = 0, \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\rho}\operatorname{div}\left(\boldsymbol{C}\boldsymbol{\varepsilon}(u)\right) - \dfrac{1}{\rho}\operatorname{div}\boldsymbol{s} = \dfrac{f}{\rho}, \\ \dfrac{\partial^\beta \boldsymbol{s}}{\partial t^\beta} + \tau_0\dfrac{\partial \boldsymbol{s}}{\partial t} - \tau_0 \boldsymbol{Z}\boldsymbol{\varepsilon}(v) = 0, \\ u(\boldsymbol{x},0) = u_0, \quad v(\boldsymbol{x},0) = u_1, \quad \boldsymbol{s}(\boldsymbol{x},0) = 0. \end{cases}$$

With $\boldsymbol{Z} = \boldsymbol{D} - \boldsymbol{C}$ is the absorption condition is defined positive :

$$0 < M_-|\sigma|^2 \leqslant \boldsymbol{Z\sigma} : \boldsymbol{\sigma} \leqslant M_+|\sigma|^2, \quad \forall \boldsymbol{\sigma} \in \mathcal{L}^{sym}(\mathbb{R}^d) \quad \text{a.e.} \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{15}$$

By using the approach (14) of $\partial_t^\beta \boldsymbol{s}$, it follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v = 0, \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\rho}\operatorname{div}\left(\boldsymbol{C}\boldsymbol{\varepsilon}(u)\right) - \dfrac{1}{\rho}\operatorname{div}\boldsymbol{s} = \dfrac{f}{\rho}, \\ \dfrac{\partial s}{\partial t} + \dfrac{1}{\tau_0}\displaystyle\int_0^{+\infty} \dfrac{\partial\varphi}{\partial t}(\boldsymbol{x},t,\xi)\,dM_\alpha(\xi) - \boldsymbol{Z}\boldsymbol{\varepsilon}(v) = 0, \\ \dfrac{\partial \varphi}{\partial t} + \xi\varphi - s = 0, \\ u(\boldsymbol{x},0) = u_0, \quad v(\boldsymbol{x},0) = u_1, \quad \boldsymbol{s}(\boldsymbol{x},0) = 0, \quad \varphi(\boldsymbol{x},0,\xi) = 0. \end{cases} \tag{16}$$

Using the last equation and replacing $\partial_t \varphi$ in the third one, one can rewrite:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v = 0, \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\rho}\operatorname{div}(\boldsymbol{C}\boldsymbol{\varepsilon}(u)) - \dfrac{1}{\rho}\operatorname{div}\boldsymbol{s} = \dfrac{f}{\rho}, \\ \dfrac{\partial s}{\partial t} + \dfrac{1}{\tau_0}\displaystyle\int_0^{+\infty}(s - \xi\varphi)\,dM_\alpha(\xi) - \boldsymbol{Z}\boldsymbol{\varepsilon}\left(\dfrac{\partial u}{\partial t}\right) = 0, \\ \dfrac{\partial \varphi}{\partial t} + \xi\varphi - s = 0, \\ u(x,0) = u_0,\ v(x,0) = u_1,\ s(x,0) = 0, \varphi(x,0,\xi) = 0. \end{cases} \tag{17}$$

The problem (10) becomes as follows:

$$\begin{cases} \dfrac{d\boldsymbol{U}}{dt} + \boldsymbol{A}\boldsymbol{U} = \boldsymbol{F}, \\ \boldsymbol{U}(0) = \boldsymbol{U}_0, \end{cases}$$

with $\boldsymbol{U} = (u, v, s, \varphi)^t$, $\boldsymbol{U}_0 = (u_0, u_1, 0, 0)^t$, $\boldsymbol{F} = (0, f/\rho, 0, 0)^t$ and

$$\boldsymbol{A}\boldsymbol{U} = \begin{pmatrix} -v \\ -\dfrac{1}{\rho}\operatorname{div}(\boldsymbol{C}\boldsymbol{\varepsilon}(u)) - \dfrac{1}{\rho}\operatorname{div}\boldsymbol{s} \\ \dfrac{1}{\tau_0}\displaystyle\int_0^{+\infty}(s - \xi\varphi)\,dM_\alpha(\xi) - \boldsymbol{Z}\boldsymbol{\varepsilon}(v) \\ \xi\varphi - s \end{pmatrix}. \tag{18}$$

3.3. Existence and uniqueness of strong solutions

Functional spaces are considered:

$$\begin{cases} L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)) = \left\{\sigma \colon \mathbb{R}^d \mapsto \mathcal{L}^{sym}(\mathbb{R}^d) \Big/ \displaystyle\int_{\mathbb{R}^d} |\sigma|^2\,dx < \infty\right\}, \\ H^{sym}(\operatorname{div}; \mathbb{R}^d) = \left\{\boldsymbol{\sigma} \in L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)),\ \operatorname{div}\boldsymbol{\sigma} \in [L^2(\mathbb{R}^d)]^d\right\}, \\ \mathbb{H}_\alpha^{sym} = \left\{\varphi \in L^2(\mathbb{R}_+, dM_\alpha(\xi)),\ \varphi_{ij} = \varphi_{ji}\ \forall i,j = 1,\ldots,d\right\}, \\ \tilde{V}_\alpha^{sym} = \left\{\varphi \in L^2(\mathbb{R}_+, \xi dM_\alpha(\xi)),\ \varphi_{ij} = \varphi_{ji},\ \forall i,j = 1,\ldots,d\right\}, \\ V_\alpha^{sym} = \{\varphi \in L^2(\mathbb{R}_+, (1+\xi)dM_\alpha(\xi)),\ \varphi_{ij} = \varphi_{ji}\ \forall i,j = 1,\ldots,d\}. \end{cases} \tag{19}$$

For any symmetrical tensor $\boldsymbol{C}$, we denote by:

– $(\cdot : \cdot)_C$ the scalar product in $\mathcal{L}^{sym}(\mathbb{R}^d)$ and $|\cdot|_C$ its associated norm:

$$\begin{aligned} \mathcal{L}^{sym}(\mathbb{R}^d) \times \mathcal{L}^{sym}(\mathbb{R}^d) &\longmapsto \mathbb{R}, \\ (\sigma, \varepsilon) &\longmapsto \boldsymbol{C}\sigma : \varepsilon \equiv (\sigma, \varepsilon)_C; \end{aligned}$$

– $\langle \cdot : \cdot \rangle_C$ the scalar product in $L^2\big(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)\big)$ and $\|\,.\,\|_C$ its associated norm:

$$\begin{aligned} L^2\big(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)\big) \times L^2\big(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)\big) &\longmapsto \mathbb{R} \\ (\sigma, \varepsilon) &\longmapsto \int_{\mathbb{R}^d} C\sigma : \varepsilon\,dx \equiv \langle \sigma, \varepsilon \rangle_C. \end{aligned}$$

Let introduce the Hilbert space:

$$H = \left[H^1(\mathbb{R}^d)\right]^d \times \left[L^2(\mathbb{R}^d)\right]^d \times \left[L^2(\mathbb{R}^d)\right]^d \times L^2(\mathbb{R}^d; \tilde{V}_\alpha). \tag{20}$$

Let $\boldsymbol{U}_1=(u_1,v_1,s_1,\varphi_1)^t$ and $\boldsymbol{U}_2=(u_2,v_2,s_2,\varphi_2)^t$ are two elements of H. Then note $\tilde{Z}=Z\tau_0$. The scalar product in H is defined by:

$$\langle U_1,U_2\rangle_H=(u_1,u_2)_\rho+(\varepsilon(u_1):\varepsilon(u_2))_C+(v_1,v_2)_\rho+(s_1,s_2)_{Z^{-1}}+\langle\varphi_1,\varphi_2\rangle_{\tilde{Z}^{-1},\tilde{V}_\alpha},$$

with $(u_1,u_2)_\eta=\int_{\mathbb{R}^d}\eta(x)\,u_1\cdot u_2\,dx$, $\forall\eta=\rho,C,Z^{-1}$ and $(\varphi_1,\varphi_2)_{\tilde{Z}^{-1},\tilde{V}_\alpha}=\int_{\mathbb{R}}\tilde{Z}^{-1}\langle\varphi_1,\varphi_2\rangle_{\tilde{V}_\alpha}\,dx$.

We define the operator domain A, $D(A)\subset H\longmapsto H$ the space defined by:

$$D(A):=\left\{(u,v,s,\varphi)^t\in H,\;\middle|\;\begin{array}{l}v\in[H^1(\mathbb{R}^d)]^d\\ s-C\varepsilon(u)\in H^{sym}(\mathrm{div};\mathbb{R}^d)\\ (s-\xi\varphi)\in L^2(\mathbb{R}^d;V_\alpha)\end{array}\right\}.$$

The operator $A\colon D(A)\subset H\longrightarrow H$ is well defined, it is a bounded operator, we will show it in the same way when the modimensional case (see [15]).

The proof of the existence and uniqueness of the strong solution of the model problem (10) is based on the use of Hille–Yosida theorem, which requires the following lemma.

Lemma 1. *The operator $A+\lambda I$ is maximal monotone for all $\lambda\geqslant 1/2$.*

Proof.

– **Monotonity:**

Let $\boldsymbol{U}=(u,v,s,\varphi)^t\in D(A)$. Then:

$$\begin{aligned}\langle \boldsymbol{AU},\boldsymbol{U}\rangle_H&=-(u,v)_\rho-(\varepsilon(u),\varepsilon(v))_C-\left(\frac{1}{\rho}\mathrm{div}\,(C\varepsilon(u)+s),v\right)_\rho\\&\quad+\left(\frac{1}{\tau_0}\int_0^{+\infty}(s-\xi\varphi)\,dM_\alpha-Z\varepsilon(v),s\right)_{Z^{-1}}+\langle(s-\xi\varphi),\varphi\rangle_{\tilde{Z}^{-1},\tilde{V}_\alpha}\\&=-\int_{\mathbb{R}^d}\rho u\cdot v\,dx\int_{\mathbb{R}^d}C\varepsilon(u):\varepsilon(v)\,dx-\int_{\mathbb{R}^d}\mathrm{div}(C\varepsilon(u)+s)\cdot v\,dx\\&\quad+\tilde{Z}^{-1}\int_{\mathbb{R}^d}\int_0^{+\infty}(s-\xi\varphi)s\,dM_\alpha\,dx-\int_{\mathbb{R}^d}\varepsilon(v)s\,dx-\tilde{Z}^{-1}\int_{\mathbb{R}^d}\int_0^{+\infty}\xi\varphi(s-\xi\varphi)\,dM_\alpha\,dx.\end{aligned}$$

Using the green formula:

$$\langle AU,U\rangle_H=-\int_{\mathbb{R}^d}\rho\,u\cdot v\,dx+\int_{\mathbb{R}^d}\tilde{Z}^{-1}\|s-\xi\varphi\|^2_{\mathbb{H}_\alpha^{sym}}\,dx,$$

so that,

$$\langle AU,U\rangle_H\geqslant-\frac{1}{2}\left(\int_{\mathbb{R}^d}\rho\,u\,dx+\int_{\mathbb{R}^d}\rho\,v\,dx\right)+\int_{\mathbb{R}^d}\tilde{Z}^{-1}\|s-\xi\varphi\|^2_{\mathbb{H}_\alpha^{sym}}\,dx,$$

on the other hand,

$$\|U\|_H=\int_{\mathbb{R}^d}\rho|u|^2\,dx+\int_{\mathbb{R}^d}C\varepsilon(u):\varepsilon(u)\,dx+\int_{\mathbb{R}^d}\rho|v|^2\,dx+\int_{\mathbb{R}^d}Z^{-1}s.s\,dx+\tilde{Z}^{-1}\int_{\mathbb{R}^d}\|\varphi\|^2_{\tilde{V}_\alpha^{sym}}\,dx,$$

thus,

$$\begin{aligned}\langle AU,U\rangle_H+\lambda\|U\|_H=-\int_{\mathbb{R}^d}\rho\,u\cdot v\,dx+\int_{\mathbb{R}^d}\tilde{Z}^{-1}\|s-\xi\varphi\|_{\mathbb{H}_\alpha^{sym}}\,dx+\lambda\int_{\mathbb{R}^d}\rho|u|^2\,dx,\\ \lambda\int_{\mathbb{R}^d}C\varepsilon(u):\varepsilon(u)\,dx+\lambda\int_{\mathbb{R}^d}\rho|v|^2\,dx+\lambda\int_{\mathbb{R}^d}Z^{-1}s\cdot s\,dx+\lambda\tilde{Z}^{-1}\int_{\mathbb{R}^d}\|\varphi\|^2_{\tilde{V}_\alpha^{sym}}\,dx.\end{aligned}$$

Consequently,

$$\langle AU, U\rangle_H + \lambda\|U\|_H \geqslant \left(\lambda - \frac{1}{2}\right)\int_{\mathbb{R}^d} \rho|u|^2\,dx + \left(\lambda - \frac{1}{2}\right)\int_{\mathbb{R}^d} \rho|v|^2\,dx.$$

Namely, $A+\lambda I$ is monotony for all $\lambda \geqslant \frac{1}{2}$.

– **Surjectivity:**

Let us show that $(A+\eta I)\colon D(A) \longrightarrow H$ is surjetive for all $\eta \geqslant 1$. This is equivalent to show that for all $F=(f_1,f_2,f_3,f_4)^t \in H$, there exist $U=(u,v,s,\varphi)^t \in D(A)$ solution of the system:

$$\begin{cases} -v+\eta u = f_1, & \text{(21a)}\\ -\dfrac{1}{\rho}\operatorname{div}(C\varepsilon(u)) - \dfrac{1}{\rho}\operatorname{div} s + \eta v = f_2, & \text{(21b)}\\ -\dfrac{1}{\tau_0}\displaystyle\int_0^{+\infty}(s-\xi\varphi)\,dM_\alpha(\xi) - Z\varepsilon(v) + \eta s = f_3, & \text{(21c)}\\ \xi\varphi - s + \eta\varphi = f_4. & \text{(21d)} \end{cases}$$

If the above system is solvable, v and φ can be easily eliminated and we find that (u,s) satisfies the following system:

$$\begin{cases} -\operatorname{div} s - \operatorname{div}(C\varepsilon(u)) + \rho\eta^2 u = \rho f_2 + \rho\eta f_1,\\ s = \dfrac{\tilde{Z}}{(c_\eta+\tau_0)}\varepsilon(u) + \dfrac{\tau_0}{(c_\eta+\tau_0)\eta}f_3 - \dfrac{\tilde{Z}}{(c_\eta+\tau_0)\eta}\varepsilon(f_1) + \dfrac{1}{(c_\eta+\tau_0)\eta}\displaystyle\int_0^{+\infty}\frac{\xi}{\xi+\eta}f_4\,dM_\alpha(\xi), \end{cases} \tag{22}$$

with $c_\eta = \int_0^{+\infty}\frac{1}{\xi+\eta}\,dM_\alpha(\xi) < +\infty$, $\forall \eta > 0$. Replacing s in the first equation of (22) gives the next:

$$\begin{aligned} -\operatorname{div}\left(\frac{\tilde{Z}}{(c_\eta+\tau_0)}\varepsilon(u)\right) - \operatorname{div}(C\varepsilon(u)) + \rho\eta^2 u = \operatorname{div}\left(\frac{\tau_0}{(c_\eta+\tau_0)\eta}f_3\right) - \operatorname{div}\left(\frac{\tilde{Z}}{(c_\eta+\tau_0)\eta}\varepsilon(f_1)\right)\\ + \operatorname{div}\left(\frac{1}{(c_\eta-\tau_0)\eta}\int_0^{+\infty}\frac{\xi}{\xi+\eta}f_4\,dM_\alpha(\xi)\right) + \rho f_2 + \rho\eta f_1, \end{aligned} \tag{23}$$

The variational formulation of (23) can be written as follows:

$$\begin{cases} \text{Find } u \in H^1(\mathbb{R}) \text{ such that:}\\ a(u,\tilde{u}) = l(\tilde{u}),\ \forall \tilde{u} \in H^1(\mathbb{R}), \end{cases} \tag{24}$$

with

$$\begin{aligned} a(u,\tilde{u}) &= \left(\left[\frac{\tilde{Z}}{(c_\eta+\tau_0)} + C\right]\varepsilon(u), \varepsilon(\tilde{u})\right) + \left(\rho\,\eta^2 u, \varepsilon(\tilde{u})\right),\\ l(\tilde{u}) &= -\left(\frac{\tau_0}{(c_\eta+\tau_0)\eta}f_3 - \frac{\tilde{Z}}{(c_\eta+\tau_0)\eta}\varepsilon(f_1), \varepsilon(\tilde{u})\right) - \left(\frac{1}{(c_\eta-\tau_0)\eta}\int_0^{+\infty}\frac{\xi}{\xi+\eta}f_4\,dM_\alpha(\xi), \varepsilon(\tilde{u})\right)\\ &\quad + \left(\rho f_2 + \rho\eta f_1, \tilde{u}\right), \end{aligned}$$

where $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ designate the scalar product in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d;\mathcal{L}^{sym}(\mathbb{R}^d))$.

According to (11), (15) and Korn's Inequality [19], the bilinear form $a(\cdot,\cdot)$ is continuous and coercive on $[H^1(\mathbb{R}^d)]^d$ for any $\eta \neq 0$. Lax–Milgram theorem allows then to affirm that the problem (24) admits a unique solution u in $[H^1(\mathbb{R}^d)]^d$.

We can conclude the proof of surjectivity by showing that, for $U=(u,v,s,\varphi)^t \in D(A)$ using same step in our work [15]. ∎

Therefore it is demonstrated that the operator $A+\eta I$ is surjective $\forall \eta \geqslant 1$. Finally, to prove the lemma, it is enough to reason with $\eta = \lambda + 1 \geqslant 1$.

Theorem 1. *For all initial conditions* $(u_0, u_1, \sigma_0) \in [H^1(\mathbb{R}^d)]^d \times [H^1(\mathbb{R}^d)]^d \times H^{sym}(\mathrm{div}; \mathbb{R}^d)$ *and all* $f \in C^1\big(0, T; [L^2(\mathbb{R}^d)]^d\big)$*, there is a unique solution* (u, σ) *the problem (10) which satisfies:*

$$\left\{\begin{array}{l} u \in C^0\big(0, T; [H^2(\mathbb{R}^d)]^d\big) \cap C^1\big(0, T; [H^1(\mathbb{R}^d)]^d\big) \cap C^2\big(0, T; [L^2(\mathbb{R}^d)]^d\big), \\ \sigma \in C^0\big(0, T; H^{sym}(\mathrm{div}, \mathbb{R}^d)\big) \cap C^1\big(0, T; L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d))\big). \end{array}\right.$$

Proof. Under the assumption $U_0 \in D(A)$ and the Hille–Yosida's theorem, we deduce that the problem (10) has a unique solution $U \in C^0(0, T; D(A)) \cap C^1(0, T; H)$, then

- $u \in C^0\big(0, T; [H^2(\mathbb{R}^d)]^d\big) \cap C^1\big(0, T; [H^1(\mathbb{R}^d)]^d\big)$.
- $v = \partial_t u \in C^0\big(0, T; [L^2(\mathbb{R}^d)]^d\big) \cap C^1\big(0, T; [L^2(\mathbb{R}^d)]^d\big) \Longrightarrow u \in C^1\big(0, T; H^1(\mathbb{R})\big) \cap C^2\big(0, T; L^2(\mathbb{R})\big)$.
- $\sigma = s + \mu \partial_x u \in C^0\big(0, T; [H^1(\mathbb{R}^d)]^d\big)$ and $s \in C^1\big(0, T; L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d))\big)$.

This leads to:

$$\left\{\begin{array}{l} u \in C^0\big(0, T; [H^2(\mathbb{R}^d)]^d\big) \cap C^1\big(0, T; [H^1(\mathbb{R}^d)]^d\big) \cap C^2\big(0, T; [L^2(\mathbb{R}^d)]^d\big), \\ \sigma \in C^0\big(0, T; H^{sym}(\mathrm{div}, \mathbb{R}^d)\big) \cap C^1\big(0, T; L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d))\big). \end{array}\right.$$

and completes the proof. ■

3.4. Energy decay result

Definition 1. *Let* (u, σ) *be the strong solution of system (10). We call the energy of* (u, σ) *at time* t *the quantity:*

$$E(t) = \frac{1}{2}\left(\left\|\frac{\partial u}{\partial t}\right\|_\rho^2 + \|\varepsilon(u)\|_C^2 + \|s\|_{Z^{-1}}^2 + \int_{\mathbb{R}^d} \tilde{Z}^{-1} \|\varphi\|_{\tilde{V}_\alpha}^2\right). \tag{25}$$

With

$$s = \sigma - C\varepsilon(u), \quad \tilde{Z} = Z\tau_0.$$

Remark 1. We notice that the quantity of energy is composed of three parts

- the quantity $\frac{1}{2}\left(\|\frac{\partial u}{\partial t}\|_\rho^2 + \|\varepsilon(u)\|_C^2\right)$ corresponds to the classical energy of the purely elastic case (i.e. when $\tau_0 = 0$ is null);
- the quantity $\frac{1}{2}\|s\|_{Z^{-1}}^2$, due to the effects of viscoelasticity, is expressed as the standard of the difference between visco-elastic stress and elastic stress;
- the quantity $\int_{\mathbb{R}^d} \tilde{Z}^{-1} \|\varphi\|_{\tilde{V}_\alpha}^2$, it is added when working with the fractional derivative.

Theorem 2. *The* $E(t)$ *energy associated with the problem* (10) *satisfies the identity:*

$$\frac{dE(t)}{dt} = \left(f, \frac{\partial u}{\partial t}\right) - \int_{\mathbb{R}^d} \tilde{Z}^{-1} \parallel s - \xi\varphi \parallel_{H_\alpha}^2 \, dx. \tag{26}$$

and it decreases in the absence of a source term $(f = 0)$.

Proof. Let's multiply the equation

$$\frac{\partial v}{\partial t} - \frac{1}{\rho}\mathrm{div}\, s - \frac{1}{\rho}\mathrm{div}\,(C\varepsilon(u)) = \frac{f}{\rho}$$

by $\frac{\partial u}{\partial t}$,

$$Z^{-1}\frac{\partial s}{\partial t} + \int_0^{+\infty} \tilde{Z}^{-1}\frac{\partial \varphi}{\partial t}(x, t, \xi)\, dM_\alpha(\xi) - \varepsilon(v) = 0,$$

by s, let's integrate in space.

We put $v = \frac{\partial u}{\partial t}$, $s = \frac{\partial \varphi}{\partial t} + \xi\varphi$:

$$\int_0^{+\infty} s \cdot \frac{\partial \varphi}{\partial t}\, dM_\alpha = \int_0^{+\infty} \left(\frac{\partial \varphi}{\partial t} + \xi\varphi\right) \frac{\partial \varphi}{\partial t}\, dM_\alpha = \int_0^{+\infty} \left(\left(\frac{\partial \varphi}{\partial t}\right)^2 + \xi \frac{\partial \varphi}{\partial t}\varphi\right) dM_\alpha(\xi)$$

$$= \| s - \xi\varphi \|^2_{\mathbb{H}_\alpha} + \frac{1}{2}\frac{d}{dt}\int_0^{+\infty} \xi\varphi^2(x,t,\xi)\, dM_\alpha(\xi).$$

Then

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} \rho \frac{\partial u^2}{\partial t^2}\frac{\partial u}{\partial t}\, dx - \int_{\mathbb{R}^d} \operatorname{div}\left(C\varepsilon(u)\right)\frac{\partial u}{\partial t}\, dx - \int_{\mathbb{R}^d} \operatorname{div} s \frac{\partial u}{\partial t}\, dx = \int_{\mathbb{R}^d} f \frac{\partial u}{\partial t}\, dx, \\ \displaystyle\int_{\mathbb{R}^d} Z^{-1}\frac{\partial s}{\partial t} s\, dx + \int_{\mathbb{R}^d} \tilde{Z}^{-1} \int_0^{+\infty} \frac{\partial \varphi}{\partial t} s\, dM_\alpha(\xi)\, dx - \int_{\mathbb{R}^d} \varepsilon\left(\frac{\partial u}{\partial t}\right) s\, dx = 0, \end{cases}$$

which equals to

$$\begin{cases} \displaystyle\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \rho \left(\frac{\partial u}{\partial t}\right)^2 dx + \int_{\mathbb{R}^d} C\varepsilon(u).\varepsilon\left(\frac{\partial u}{\partial t}\right) dx - \int_{\mathbb{R}^d} \operatorname{div} s \frac{\partial u}{\partial t}\, dx = \int_{\mathbb{R}^d} f \frac{\partial u}{\partial t}\, dx, \\ \displaystyle\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} Z^{-1} s^2\, dx + \int_{\mathbb{R}^d} \operatorname{div} s \frac{\partial u}{\partial t}\, dx + \int_{\mathbb{R}^d} \tilde{Z}^{-1} \| s - \xi\varphi \|^2_{\mathbb{H}_\alpha}\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \tilde{Z}^{-1} \| \varphi \|^2_{\tilde{V}_\alpha}\, dx = 0. \end{cases}$$

By summing them we obtain:

$$\frac{dE}{dt} = \int_{\mathbb{R}^d} f \frac{\partial u}{\partial t}\, dx - \int_{\mathbb{R}^d} \tilde{Z}^{-1} \| s - \xi\varphi \|^2_{\mathbb{H}_\alpha}\, dx.$$

■

3.5. Estimations a priori

In this section we are interested in the a priori estimation of the solution of the model problem (10).

Theorem 3. *The solution to the model problem (10) satisfies the following estimates:*

$$\left\| \frac{\partial u}{\partial t} \right\|_\rho \leqslant \sqrt{2E(0)} + \int_0^t \|f(\tau)\|_{1/\rho}\, d\tau, \quad \forall t > 0, \tag{27}$$

$$\|u(t)\|_\rho \leqslant \|u_0\|_\rho + t\sqrt{2E(0)} + \int_0^t (t-\tau)\|f(\tau)\|_{1/\rho}\, d\tau, \quad \forall t > 0, \tag{28}$$

$$\|\nabla u(t)\| + \|\sigma(t)\|_{Z^{-1}} \leqslant C\left(\sqrt{2E(0)} + \int_0^t \|f(\tau)\|_{1/\rho}\, d\tau\right), \quad \forall t > 0, \tag{29}$$

where $C = C(d, M_-, M_+)$ and $E(0)$ is the initial energy:

$$E(0) = \frac{1}{2}\left(\|u_1\|^2_\rho + \|\varepsilon(u_0)\|^2_C\right),$$

Proof. By integrating between 0 and t equality (26):

$$\int_0^t \frac{dE}{dt}\, d\tau = \int_0^t \left(f, \frac{\partial u}{\partial t}\right) d\tau - \int_0^t \left(\int_{\mathbb{R}^d} \tilde{Z}^{-1} \| s - \xi\varphi \|^2_{H_\alpha}\, dx\right) d\tau,$$

we get:

$$E(t) \leqslant E(0) + \int_0^t \left(f, \frac{\partial u}{\partial t}\right) d\tau \leqslant E(0) + \int_0^t \|f\|_{1/\rho} \left\|\frac{\partial u}{\partial t}\right\|_\rho d\tau,$$

on the other hand, by definition of the energy quantity (25):

$$\left\| \frac{\partial u(\tau)}{\partial t} \right\|^2_\rho \leqslant 2E(\tau),\ \forall\, \tau > 0,$$

thus

$$E(t) \leqslant E(0) + \sqrt{2}\int_0^t \|f\|_{1/\rho}\sqrt{E(\tau)}\, d\tau.$$

Thanks to Gronwall's lemma, the next inequality is true:

$$E(t) \leqslant \left(\sqrt{E(0)} + \frac{1}{\sqrt{2}}\int_0^t \|f\|_{1/\rho}\, d\tau\right)^2. \tag{30}$$

Using the definition of the quantity of energy (25), then:

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_\rho \leqslant \sqrt{2E(t)}.$$

So the first estimate (27)is obtained:

$$\left\|\frac{\partial u}{\partial t}\right\|_\rho \leqslant \sqrt{2E(0)} + \int_0^t \|f(\tau)\|_{1/\rho}\, d\tau.$$

For the second estimate (28), we notice that u is a primitive of $\partial_t u$:

$$u(t) = u_0 + \int_0^t \frac{\partial u(\tau)}{\partial t}\, d\tau,$$

which implies

$$\|u(t)\|_\rho \leqslant \|u_0\|_\rho + \left\|\int_0^t \frac{\partial u(\tau)}{\partial t}\, d\tau\right\|_\rho \leqslant \|u_0\|_\rho + \int_0^t \left\|\frac{\partial u(\tau)}{\partial t}\right\|_\rho d\tau,$$

since the first estimate:

$$\left\|\frac{\partial u(\tau)}{\partial t}\right\|_\rho \leqslant \sqrt{2E(0)} + \int_0^\tau \|f(\tilde\tau)\|_{1/\rho}\, d\tilde\tau,$$

we have

$$\int_0^t \left\|\frac{\partial u(\tau)}{\partial t}\right\|_\rho d\tau \leqslant t\sqrt{2E(0)} + \int_0^t (t-\tau)\|f(\tau)\|_{1/\rho}\, d\tau,$$

we get the second estimate (28):

$$\|u(t)\|_\rho \leqslant \|u_0\|_\rho + t\sqrt{2E(0)} + \int_0^t (t-\tau)\|f(\tau)\|_{1/\rho}\, d\tau.$$

For the third estimate (29),

$$\sqrt{M_-} \leqslant \|\varepsilon(u(t))\|_C \leqslant \sqrt{2E(t)} \leqslant \sqrt{2E(0)} + \int_0^t \|f(\tau)\|_{1/\rho}\, d\tau, \tag{31}$$

on the other hand, we first use the inequality of Korn [19, 20]: there is a constant C_k depending on d the dimension of the space, such as that:

$$\|\nabla u\| \leqslant C_k \|\varepsilon(u)\|. \tag{32}$$

From the equality $s = \sigma - C\varepsilon(u)$, one deducts

$$\|\sigma(t)\|_{Z^{-1}} \leqslant \|s(t)\|_{Z^{-1}} + \|C\varepsilon(u)\|_{Z^{-1}},$$

with

$$\|s(t)\|_{Z^{-1}} \leqslant \sqrt{2E(0)} + \int_0^t \|f(\tau)\|_{1/\rho}\, d\tau.$$

Finally, from (31) and the last inequality, we obtain:

$$\left\|\frac{\partial u(t)}{\partial x}\right\|_\mu + \|\sigma(t)\|_\zeta \leqslant C\left(\sqrt{2E(0)} + \int_0^t \|f(\tau)\|_{1/\rho}\, d\tau\right),$$

where C is a constant depending on d, M_-, M_+. ■

3.6. Existence and uniqueness of weak solutions

Let note:

$$Q_T = \mathbb{R}^d \times [0,T], \quad Q_T^* = \mathbb{R}^d \times]0,T], \tag{33}$$

and the spaces

$$\begin{aligned} \mathcal{H}(Q_T) &= \left\{ v \in C^2(0,T;[L^2(\mathbb{R}^d)]^d) \cap C^0(0,T;[H^1(\mathbb{R}^d)]^d),\ v(T)=0 \text{ and } \partial_t v(T) = 0 \right\}, \\ \mathcal{L}(Q_T) &= \left\{ \tilde{\sigma} \in C^1\left(0,T;L^2\left(\mathbb{R}^d, \mathcal{L}^{sym}\left(\mathbb{R}^d\right)\right)\right),\ \tilde{\sigma}(T) = 0 \right\}. \end{aligned} \tag{34}$$

It is said that (u,σ) is a weak solution of the problem (10) if it verifies:

$$\begin{cases} \displaystyle\int_{Q_T} \left[\rho\, u \cdot \partial_{tt}^2 v + \sigma : \varepsilon(v) - f \cdot v\right] dx\, dt = \int_{\mathbb{R}} \rho \left[u_1 \cdot v(x,0) - u_0 \cdot \partial_t v(x,0)\right] dx, \\ \displaystyle\int_{Q_T} \left[\sigma : \tilde{\sigma} - \tau_0 \sigma : \partial_t^\alpha \tilde{\sigma} - C\varepsilon(u) : \tilde{\sigma} + \tau_0 D \varepsilon(u) \partial_t^\alpha \tilde{\sigma}\right] dx\, dt \\ \displaystyle = \int_{\mathbb{R}} \left[\tau_0 \mathbf{I}^{1-\alpha}(\sigma_0) : \tilde{\sigma}(x,0) - \tau_0 \boldsymbol{D} \varepsilon\left(\boldsymbol{I}^{1-\alpha}(u_0)\right) : \tilde{\sigma}(x,0)\right] dx, \\ \forall (v,\tilde{\sigma}) \in \mathcal{H}\left(Q_T\right) \times \mathcal{L}\left(Q_T\right). \end{cases} \tag{35}$$

The integral per part with fractional derivative is used, see [9]. For any $0 < \alpha < 1$, $\boldsymbol{I}^{1-\alpha}$ is the fractional integral of Caputo

$$\boldsymbol{I}^{1-\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g(\tau)\, d\tau.$$

The existence and uniqueness theorem of the weak solution can be formulated.

Theorem 4. *If the initial data verify*

$$(u_0, u_1, \sigma_0, f) \in \left[H^1(\mathbb{R}^d)\right]^d \times \left[L^2(\mathbb{R}^d)\right]^d \times L^2\left(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)\right) \times L^1\left(0,T;[L^2(\mathbb{R}^d)]^d\right).$$

Then the problem (35) admits a unique solution:

$$(u,\sigma) \in C^0\left(0,T;[H^1(\mathbb{R}^d)]^d\right) \cap C^1\left(0,T;[L^2(\mathbb{R}^d)]^d\right) \times C^0\left(0,T;L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d))\right).$$

Proof.

– **Existence**.

By density, there is a sequence

$$(u_0^n, u_1^n, \sigma_0^n, f^n) \in \left[H^2(\mathbb{R}^d)\right]^d \times \left[H^1(\mathbb{R}^d)\right]^d \times \underline{X}^{sym}(\mathbb{R}^d) \times C^1\left(0,T;[L^2(\mathbb{R}^d)]^d\right),$$

verifying:

$$\begin{cases} u_0^n \to u_0 & \text{in} \quad \left(H^1\left(\mathbb{R}^d\right)\right)^n, \\ u_1^n \to u_1 & \text{in} \quad \left(L^2\left(\mathbb{R}^d\right)\right)^n, \\ \sigma_0^n \to \sigma_0 & \text{in} \quad L^2\left(\mathbb{R}^d, \mathcal{L}^{sym}\left(\mathbb{R}^d\right)\right), \\ f^n \to f & \text{in} \quad L^1\left(0,T,\left(L^2\left(\mathbb{R}^d\right)\right)^n\right). \end{cases}$$

Let (u^n, σ^n) be the strong solution of the problem (10) associated with the initial data $(u_0^n, u_1^n, \sigma_0^n, f^n)$ (cf. theorem 1). Let us apply the estimations of the theorem 3 to the differences $(u^p - u^q, \sigma^p - \sigma^q)$, we notice that the continuation (u^n, σ^n) is a continuation of Cauchy in the space of Banach $W\left(0,T;\mathbb{R}^d\right)$ defined by:

$$C^0\left(0,T;[H^1(\mathbb{R}^d)]^d\right) \times C^1\left(0,T;[L^2(\mathbb{R}^d)]^d\right) \times C^0\left(0,T;L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d))\right),$$

its assocaited norm:

$$\|(u,\sigma)\|_W = \sup_{t\in[0,T]} \Big[\|u(t)\|_\rho + \|\partial_t u\|_\rho + \|\nabla u(t)\| + \|\sigma(t)\|_{Z^{-1}}\Big]. \tag{36}$$

We deduce the existence of $(u,\sigma) \in W\left(0,T;\mathbb{R}^d\right)$ verifying:

$$\begin{cases} u^n \to u & \text{in} \quad C^0\big(0,T;[H^1(\mathbb{R}^d)]^d\big) \times C^1\big(0,T;[L^2(\mathbb{R}^d)]^d\big), \\ \sigma^n \to \sigma & \text{in} \quad C^0\big(0,T;L^2(\mathbb{R}^d,\mathcal{L}^{sym}(\mathbb{R}^d))\big). \end{cases}$$

Finally, just note that if (u^n,σ^n) is a strong solution then in particular it is a weak solution to the problem:

$$\begin{cases} \displaystyle\int_{Q_T} \big[\rho u^n \cdot \partial^2_{tt} v + \sigma^n : \varepsilon(v)\big]\,dx\,dt = \int_{Q_T} f\cdot v\,dx\,dt + \int_{\mathbb{R}^d} \rho\big[u^n_1\cdot v(x,0) - u^n_0 \cdot \partial_t v(x,0)\big]\,dx, \\ \displaystyle\int_{Q_T} \left[\sigma^n : \tilde\sigma - \tau_0 \sigma^n : \partial_t\tilde\sigma - C\varepsilon\left(u^n\right) : \tilde\sigma + \tau_0 D\varepsilon\left(u^n\right) : \partial_t \tilde\sigma\right] dx\,dt \\ \displaystyle= \int_{\mathbb{R}^d} \big[\tau_0 \boldsymbol{I}^{1-\alpha}(\sigma^n_0) : \tilde\sigma(x,0) - \tau_0 D\,\boldsymbol{I}^{1-\alpha}\varepsilon\left(u^n_0\right) : \tilde\sigma(x,0)\big]\,dx, \\ \forall (v,\tilde\sigma) \in \mathcal{H}\left(Q_T\right) \times \mathcal{L}\left(Q_T\right) \end{cases}$$

by passing the limit when $n \to +\infty$

– **Uniqueness**.

By linearity, it is enough to show the uniqueness of the solution of the problem (35) in no source and with zero initial data. Let (u,σ) be a solution to the problem (35) with $u_0 = 0, u_1 = 0, \sigma_0 = 0$ and $f = 0$:

$$\begin{cases} \displaystyle\int_{Q_T} \big[\rho u \cdot \partial^2_{tt} v + \sigma : \varepsilon(v)\big]\,dx\,dt = 0, \\ \displaystyle\int_{Q_T} \left[\sigma : \tilde\sigma - \tau_0 \sigma : \partial^\alpha_t \tilde\sigma - C\varepsilon(u) : \tilde\sigma + \tau_0 D\varepsilon(u) : \partial^\alpha_t \tilde\sigma\right] dx\,dt = 0, \\ \forall (v,\tilde\sigma) \in \mathcal{H}(Q_T) \times \mathcal{L}(Q_T). \end{cases} \tag{37}$$

The next problem is considered:

$$\begin{cases} \rho\,\partial^2_{tt}\bar u - \operatorname{div}\bar\sigma = u, & (x,t) \in \mathbb{R}^d \times [0,T[, \\ \bar\sigma - \tau_0\,\partial^\alpha_t \bar\sigma = C\varepsilon(\bar u) - \tau_0\,D\,\varepsilon\left(\partial^\alpha_t \bar u\right), & (x,t) \in \mathbb{R}^d \times [0,T[, \\ \bar u(x,T) = \partial_t \bar u(x,T) = 0, \;\; \bar\sigma(x,T) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{38}$$

This problem admits a unique strong solution $(\bar u,\bar\sigma)$ in space:

$$C^1\big(0,T;[H^1(\mathbb{R}^d)]^d\big)^n \cap C^2\big(0,T;[L^2(\mathbb{R}^d)]^d\big) \times C^0\big(0,T;\underline{X}^{sym}(\mathbb{R}^d)\big) \cap C^1\big(0,T;L^2(\mathbb{R}^d),\mathcal{L}^{sym}(\mathbb{R}^d)\big),$$

because as $u \in C^1\big(0,T;[L^2(\mathbb{R}^d)]^d\big)$ simply apply the theorem 1 by rewriting the last problem in the form of the system (10), by making the change of variable $s = T - t$ and taking as data the functions

$$(u_0,u_1,\sigma_0,f) = (0,0,0,u(T-s)).$$

Noting $\pi = \sigma - D\varepsilon(u)$, the system (37) is rewritten in the following form:

$$\begin{cases} \displaystyle\int_{Q_T} \big[\rho\,u\cdot\partial^2_{tt} v + \pi : \varepsilon(v) + \boldsymbol{D}\varepsilon(u) : \varepsilon(v)\big]\,dx\,dt = 0, \\ \displaystyle\int_{Q_T} \big[\pi : \tilde\sigma - \tau_0 \pi : \partial^\alpha_t \tilde\sigma + Z\varepsilon : \tilde\sigma\big]\,dxdt = 0, \\ \forall (v,\tilde\sigma) \in \mathcal{H}(Q_T) \times \mathcal{L}(Q_T). \end{cases} \tag{39}$$

Due to the construction of $(\bar{u}, \bar{\sigma})$ we can choose as test functions in (37):

$$(v, \tilde{\sigma}) = \left(\bar{u}, Z^{-1}\bar{\pi}\right) \in \mathcal{H}(Q_T) \times \mathcal{L}(Q_T). \quad \bar{\pi} = \bar{\sigma} - \tau_0 D\varepsilon(\bar{u}). \tag{40}$$

The system (39) then gives:

$$\int_{Q_T} \left[\rho\, u \cdot \partial_{tt}^2 \tilde{u} + \pi : \varepsilon(\tilde{u}) + \boldsymbol{D}\varepsilon(u) : \varepsilon(\tilde{u})\right] dx\, dt = 0, \tag{41}$$

$$\int_{Q_T} \left[Z^{-1}\pi : \bar{\pi} - Z^{-1}\tau_0 \pi : \partial_t^\alpha \bar{\pi} + \varepsilon : \bar{\pi}\right] dx\, dt = 0, \tag{42}$$

where $\varepsilon = \varepsilon(u)$ and $\bar{\varepsilon} = \varepsilon(\bar{u})$. We then make the scalar product in the sense of the tensors of the second system equation (38) with $Z^{-1}\pi$. After integration on Q_T:

$$\int_{Q_T} \left[Z^{-1}\bar{\pi} : \pi - \tau_0 Z^{-1} \partial_t^\alpha \bar{\pi} : \pi + \bar{\varepsilon} : \pi\right] dx\, dt = 0. \tag{43}$$

This last equation and the equation (42) imply:

$$\int_{Q_T} \pi : \bar{\varepsilon}\, dx\, dt = \int_{Q_T} \bar{\pi} : \varepsilon\, dx\, dt.$$

Hence (41) becomes:

$$\int_{Q_T} \left[\rho\, u \cdot \partial_{tt}^2 \bar{u} - \bar{\pi} : \varepsilon + D\, \varepsilon : \bar{\varepsilon}\right] dx\, dt = 0. \tag{44}$$

By replacing $\bar{\pi}$ by its value given by (40) and after an integration by part:

$$\int_{Q_T} u \left[\rho\, \partial_{tt}^2 \bar{u} - \operatorname{div} \bar{\sigma}\right] \, dx\, dt = 0. \tag{45}$$

Now, since $(\bar{u}, \bar{\sigma})$ is the solution to the problem (38):

$$\int_{Q_T} |u(x,t)|^2\, dx\, dt = 0. \tag{46}$$

which results in $u = 0$. Finally, we show that $\sigma = 0$ using the second system equation (35):

$$\int_{Q_T} \left[\sigma : \tilde{\sigma} - \tau_0 \sigma : \partial_t^\alpha \tilde{\sigma}\right] dx\, dt = 0, \qquad \forall \tilde{\sigma} \in \mathcal{L}(Q_T).$$

Hence the uniqueness of the solution is proved. ∎

4. Conclusion

Mathematical modeling of waves propagation in viscoelastic media associated with a generalized fractional model are studied. Firstly, we have studied homogeneous media using plane waves, this method allows us to distinguish the different modes and their properties. Secondly, we have introduced some auxiliary variables which permit us to transform the hyperbolic second order system to a first order evolution one, then via Hille–Yosida theory, it is showed the existence and uniqueness of the strong solution. Afterward, it was obtained an energy decay result which guarantees the dissipation of the model problem. Finally, a prior estimation of the model solution is given to show the weak solution of the model problem.

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Математичне моделювання поширення хвиль у в'язкопружних середовищах за допомогою дробової моделі Зенера

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У цій роботі розглянута задача математичого моделювання поширення хвилі в дисипативних середовищах. Розглянуто узагальнену дробову модель Зенера вимірності d ($d = 1, 2, 3$). Ця робота присвячена математичному аналізу такої моделі, а саме: існування та єдиність сильного та слабкого розв'язку та загасання енергії, що забезпечує розсіювання хвиль. Також подаються апріорні оцінки розв'язків, що допомагають показати існування слабкого розв'язку.

Ключові слова: *дробова похідна, сильний розв'язок, слабкий розв'язок, загасання енергії, плоскі хвилі, в'язкопружні хвилі, модель Зенера.*