MATHEMATICAL MODELING AND COMPUTING, Vol. 8, No. 4, pp. 627–637 (2021)

Weak and strong stabilization for time-delay semi-linear systems governed by constrained feedback control

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(Received 23 May 2021; Accepted 7 June 2021)

This paper is concerned with the issue of weak and strong stabilization for distributed semi-linear systems with time delay using a constrained feedback control. The results of the semi-linear systems without delay are generalized for strong and weak stabilization cases. Illustrating applications to hyperbolic and parabolic equations are considered.

Keywords: *semi-linear system, feedback stabilization, polynomial decay estimate, time delay.*

2010 MSC: 93C10, 93D15, 93E99 **DOI:** 10.23939/mmc2021.04.627

1. Introduction

This paper considers the problem of feedback stabilization of distributed semi-linear systems with time delay $r > 0$ described as follows:

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + v(t)Ny(t-r), \quad t \geqslant 0, \\ y(t) = \Phi(t), \quad t \in [-r, 0]\,. \end{cases} \tag{1}$$

Here $y(t)$ is the state on a Hilbert space H endowed with the inner product $\langle \cdot, \cdot \rangle$ and its corresponding norm $\|\cdot\|$. In addition, the linear operator $A\colon D(A) \subset H \to H$ (generally unbounded) generates a strongly continuous semi-group of contractions $S(t)$ on H. If $y \in C\left([-r, +\infty[, H\right)$ and $t \geqslant 0$, then $y_t \in C_r$ is defined by $y_t(\theta) = y(t+\theta)$ for all $\theta \in [-r, 0]$, where $C_r = C([-r, 0], H)$ denotes the Banach space of continuous functions defined from $[-r, 0]$ into H, endowed with the supremum norm $\|\psi\|_{C_r} = \sup_{\theta \in [-r,0]} \|\psi(\theta)\|$ and $\Phi \in C_r$ is a given initial function, while N is a nonlinear operator from H into H such that $N(0) = 0$ (so that 0 is an equilibrium point), whereas $t \to v(t)$ is a scalar function which represents the control. The stabilization problem for distributed semi-linear systems without delay (i.e., for $r = 0$) has been studied in many works, (see e.g. [1, 4, 6]). In [1], it has been shown that if N is weakly sequentially continuous, then the weak stabilization result has been established using the following quadratic feedback control: $v_0(t) = -\langle Ny(t), y(t) \rangle$ provided that

$$\langle NS(t)\phi, S(t)\phi \rangle = 0, \quad \forall t \geqslant 0 \implies \phi = 0, \quad \forall \phi \in H \tag{2}$$

holds. In [2], it has been proved that under the following condition

$$\int_0^T |\langle NS(t)\phi, S(t)\phi \rangle|\, dt \geqslant \delta \|\phi\|^2, \quad \forall \phi \in H, \quad \text{(for some } T > r \text{ and } \delta > 0), \tag{3}$$

the strong stabilization result was obtained using the same feedback control with the following decay estimate:

$$\|y(t)\| = \mathrm{O}\left(\frac{1}{\sqrt{t}}\right), \quad \text{as} \quad t \to +\infty.$$

In [6], the authors had used the following feedback control to show that it guaranteed the weak and strong stabilization to the system (1) without delay:

$$v_{\log}(t) = \rho \log\left(1 - \frac{\langle Ny(t), y(t)\rangle}{1 + |\langle Ny(t), y(t)\rangle|}\right), \quad \forall t \geqslant 0, \quad \rho > 0. \tag{4}$$

The main objective of this paper is to show those results for the system (1) by using the following feedback control:

$$v^r_{\log}(t) = \rho \log\left(1 - \frac{\langle Ny(t-r), y(t)\rangle}{1 + |\langle Ny(t-r), y(t)\rangle|}\right), \quad \forall t \geqslant 0, \quad \rho > 0. \tag{5}$$

Section 2 will focus on demonstrating the existence and uniqueness of the global mild solution of the system (1). In addition, an estimate will be used to prove strong and weak stabilization of the system (1). Sections 3 and 4 are dedicated to discussing strong and weak stabilization respectively, under the conditions (3) and (2). In Sections 5 and 6 we will give some specific applications and simulations to some functional differential equations.

2. Existence and uniqueness of the global mild solution and decay estimate

Next, we will analyze the existence and uniqueness of the global mild solution of the system (1). Additionally, we will establish a useful estimate to show both strong and weak stabilization of the studied system (1).

Theorem 1. *Assume that A generates a semi-group of contractions $S(t)$, and let N be a non linear and locally Lipschitz operator from H into H such that $N(0) = 0$. Then, the system (1) controlled by (5) possesses a unique global mild solution $y \in C([-r, +\infty[, H)$. Moreover, for each $T > r$, we have*

$$\begin{aligned}
&\int_r^T |\langle NS(\sigma - r)y(t), S(\sigma)y(t)\rangle| d\sigma \\
&= \mathrm{O}\left(\int_t^{t+T} \left|\log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right)\langle Ny(\sigma - r), y(\sigma)\rangle\right| d\sigma\right)^{\frac{1}{4}} \ \text{as } t \to +\infty, \quad \forall t \geqslant 0.
\end{aligned} \tag{6}$$

Proof. Using the feedback control (5), the system (1) becomes

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + \rho \log\left(1 - \dfrac{\langle Ny(t-r), y(t)\rangle}{1 + |\langle Ny(t-r), y(t)\rangle|}\right) Ny(t-r), \quad t \geqslant 0, \quad \rho > 0, \\ y(t) = \Phi(t), \quad t \in [-r, 0]. \end{cases} \tag{7}$$

First, let's show the existence and the uniqueness of a mild solution of the system (1) and we will first prove that the function $G\colon C_r \to H$ defined by

$$G(\phi) = \rho \log\left(1 - \frac{\langle N\phi(-r), \phi(0)\rangle}{1 + |\langle N\phi(-r), \phi(0)\rangle|}\right) N\phi(-r), \quad \forall \phi \in C_r$$

is locally Lipschitz. To do this, for any $R > 0$ and $\psi, \phi \in B_{C_r}(0, R) := \{\phi \in C_r; \|\phi\|_{C_r} \leqslant R\}$, we have

$$\begin{aligned}
\|G(\psi) - G(\phi)\| &= \rho \left\|\log\left(1 - \frac{\langle N\psi(-r), \psi(0)\rangle}{1 + |\langle N\psi(-r), \psi(0)\rangle|}\right) N\psi(-r) - \log\left(1 - \frac{\langle N\phi(-r), \phi(0)\rangle}{1 + |\langle N\phi(-r), \phi(0)\rangle|}\right) N\phi(-r)\right\|, \\
\|G(\psi) - G(\phi)\| &= \rho \left\|\log\left(1 - \frac{\langle N\psi(-r), \psi(0)\rangle}{1 + |\langle N\psi(-r), \psi(0)\rangle|}\right) N\psi(-r) - \log\left(1 - \frac{\langle N\phi(-r), \phi(0)\rangle}{1 + |\langle N\phi(-r), \phi(0)\rangle|}\right) N\phi(-r)\right\|,
\end{aligned} \tag{8}$$

$$
\begin{aligned}
&\|G(\psi)-G(\phi)\| \\
&\quad= \rho\left\|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)N\psi(-r)-\log\left(1-\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}\right)N\phi(-r)\right\| \\
&\quad\leqslant \rho\left\|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)N\psi(-r)-\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)N\phi(-r)\right\| \\
&\qquad+\rho\left\|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)N\phi(-r)-\log\left(1-\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}\right)N\phi(-r)\right\| \\
&\quad\leqslant \rho L_R\left|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)\right|\|\psi(-r)-\phi(-r)\| \\
&\qquad+\rho L_R\left|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)-\log\left(1-\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}\right)\right|\|\phi(-r)\|.
\end{aligned} \tag{9}
$$

Let's study each case separately, using the fact that $\log(1+x)\leqslant x$, $\forall x>0$.

Case1: $\langle N\psi(-r),\psi(0)\rangle>0$

$$\left|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)\right|=\log\left(1+\langle N\psi(-r),\psi(0)\rangle\right)\leqslant\langle N\psi(-r),\psi(0)\rangle\leqslant R^2L_R.$$

Case2: $\langle N\psi(-r),\psi(0)\rangle<0$

$$\left|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)\right|\leqslant\frac{|\langle N\psi(-r),\psi(0)\rangle|}{1+|\langle N\psi(-r),\psi(0)\rangle|}\leqslant|\langle N\psi(-r),\psi(0)\rangle|\leqslant R^2L_R.$$

It follows that

$$\left|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)\right|\leqslant|\langle N\psi(-r),\psi(0)\rangle| \tag{10}$$

and from (9), we deduce

$$
\begin{aligned}
\|G(\psi)-G(\phi)\|&\leqslant\rho L_R^2R^2\|\psi-\phi\| \\
&\quad+\rho L_RR\left|\log\left(1-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right)-\log\left(1-\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}\right)\right|.
\end{aligned}
$$

It remains to show that the map g defined by:

$$g(\phi)=\log\left(1-\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}\right)=(\log\circ h)(\phi),\quad\forall\phi\in H,$$

is locally Lipschitz, where $h(\phi)=1-\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}$. Since the function log is of C^1 on the interval $\mathrm{Im}(h):=\left[\frac{1}{1+R^2L_R},1+2R^2L_R\right]$ it suffice to show that the function h is locally Lipschitz. Indeed, $\forall R>0$ and $\forall\phi,\psi\in B_R(0)$ with the fact that $\forall a,b\in\mathbb{R}$, $||a|-|b||\leqslant|a-b|$, we deduce that

$$|h(\psi)-h(\phi)|=\left|\frac{\langle N\phi(-r),\phi(0)\rangle}{1+|\langle N\phi(-r),\phi(0)\rangle|}-\frac{\langle N\psi(-r),\psi(0)\rangle}{1+|\langle N\psi(-r),\psi(0)\rangle|}\right|$$

thus

$$
\begin{aligned}
|h(\psi)-h(\phi)|&\leqslant|\langle N\phi(-r),\phi(0)\rangle-\langle N\psi(-r),\psi(0)\rangle| \\
&\quad+|\langle N\phi(-r),\phi(0)\rangle\,|\langle N\psi(-r),\psi(0)\rangle|-\langle N\psi(-r),\psi(0)\rangle\,|\langle N\phi(-r),\phi(0)\rangle||\\
&\leqslant|\langle N\phi(-r),\phi(0)-\psi(0)\rangle+\langle N\phi(-r)-N\psi(-r),\psi(0)\rangle| \\
&\quad+|\langle N\phi(-r),\phi(0)\rangle\,|\langle N\psi(-r),\psi(0)\rangle|-\langle N\phi(-r),\phi(0)\rangle\,|\langle N\phi(-r),\phi(0)\rangle|| \\
&\quad+|\langle N\phi(-r),\phi(0)\rangle\,|\langle N\phi(-r),\phi(0)\rangle|-\langle N\psi(-r),\psi(0)\rangle\,|\langle N\phi(-r),\phi(0)\rangle|| \\
&\leqslant 2RL_R\|\psi-\phi\|+2R^2L_R|\langle N\psi(-r),\psi(0)\rangle-\langle N\phi(-r),\phi(0)\rangle|
\end{aligned}
$$

$$\begin{aligned}
&\leqslant 2RL_R\|\psi-\phi\| + 2R^2L_R|\langle N\psi(-r),\psi(0)-\phi(0)\rangle + \langle N\psi(-r)-N\phi(-r),\phi(0)\rangle| \\
&\leqslant 2RL_R\|\psi-\phi\| + 2R^3L_R^2\|\psi-\phi\| \\
&\leqslant 2RL_R\left(1+2R^2L_R\right)\|\psi-\phi\|,
\end{aligned}$$

which means that the function h is locally Lipschitz, and then g is. Consequently, G is locally Lipschitz. Then, the system (7) admits a unique mild solution defined on a maximal interval $y \in C\left([-r, t_{\max}[, H\right)$ given by the variation of constants formula:

$$y(t) = \begin{cases} \Lambda(t)\Phi(0) = S(t)y(0) + \displaystyle\int_0^t S(t-\sigma)G(y_\sigma)\,d\sigma, & t \in [0, t_{max}[, \\ \Phi(t), \quad t \in [-r, 0], \end{cases} \tag{11}$$

where $S(t)$ and $\Lambda(t)$ are the semi-groups generated by the operator A and the system (1) respectively (see [8], p. 51, Theorem 2.6).

Next we will show that this solution is globally defined. Indeed, if $y(0) \in D(A)$, the solution of the system (7) becomes a classical one (see [5]). It follows after multiplying (7) by $y(t)$ and using the fact that $S(t)$ is a semi-group of contractions that

$$\frac{d\|y(t)\|^2}{dt} \leqslant 2\rho\log\left(1 - \frac{\langle Ny(t-r), y(t)\rangle}{1+|\langle Ny(t-r), y(t)\rangle|}\right)\langle Ny(t-r), y(t)\rangle \leqslant 0, \quad \forall t > 0, \tag{12}$$

which implies

$$\|y(t)\| \leqslant \|y(0)\|. \tag{13}$$

If $\Phi(0) \notin D(A)$, let $\tau \in [0, t_{\max}[$. Since $g\colon t \to v^r_{\log}(t)Ny(t-r)$ is continuous in $[0,\tau]$, we deduce that there exists a sequence $(g_n) \subset C^1([0,\tau], H)$ such that $g_n \to g$ in $(C([0,\tau],H), \|\cdot\|_\infty)$ as $n \to +\infty$ (see [1]). Moreover, since A generates a semi-group of contractions (i.e, $\overline{D(A)} = H$), so, there exists a sequence $(\nu_n) \subset D(A)$ such that $\nu_n \to \Phi(0)$ in H as $n \to +\infty$. Let $(y_n) \subset C\left([0,\tau]\right)$ such that

$$y_n(t) = \begin{cases} S(t)\nu_n + \displaystyle\int_0^t S(t-\sigma)g_n(\sigma)\,d\sigma, & t \in [0,\tau], \\ y_n(0) = \nu_n, \end{cases} \tag{14}$$

is the unique classical solution of the system:

$$\begin{cases} \dfrac{dy_n(t)}{dt} = Ay_n(t) + g_n(t), & t \in [0,\tau], \\ y_n(0) = \nu_n. \end{cases} \tag{15}$$

That is $(y_n(t)) \subset D(A)$ and the function $t \mapsto y_n(t)$ is continuously differentiable in $[0,\tau]$ (see, Pazy (1983)). Now we will show that $y_n \to y$ as $n \to +\infty$ in $(C([0,\tau],H); \|\cdot\|_\infty)$. By using the fact that $S(t)$ is a semi-group of contractions, it yields from (11) and (14) that for each $t \in [0,\tau]$,

$$\|y_n(t) - y(t)\| \leqslant \|\nu_n - y(0)\| + \tau \sup_{s\in[0,\tau]} \|g_n(s) - g(s)\| \to 0, \quad \text{as} \quad n \to +\infty. \tag{16}$$

Thus, $y_n \to y$ as $n \to \infty$ in $(C([0,\tau],H); \|.\|_\infty)$. By the dissipativity of A, we infer from (15) that

$$\frac{d\|y_n(t)\|^2}{dt} \leqslant 2\left\langle g_n(t), y_n(t)\right\rangle, \quad \forall t \geqslant 0. \tag{17}$$

Integrating the last inequality from s to τ, where $s \in [0,\tau]$, we derive

$$\|y_n(\tau)\|^2 - \|y_n(s)\|^2 \leqslant 2\int_s^\tau \langle g_n(\sigma), y_n(\sigma)\rangle\,d\sigma, \quad \forall \tau \in [0, t_{\max}[. \tag{18}$$

Using the dominated convergence theorem, one can deduce from (18) that

$$\|y(\tau)\|^2 - \|y(s)\|^2 \leqslant 2\rho \int_s^\tau \log\left(1 - \frac{\langle Ny(t-r), y(t)\rangle}{1 + |\langle Ny(t-r), y(t)\rangle|}\right) \langle Ny(t-r), y(t)\rangle \leqslant 0, \quad \forall \tau \in [0, t_{\max}[. \tag{19}$$

It means that $t \mapsto \|y(t)\|$ is a nonincreasing function on $[0, t_{\max}[$. In particular, from (19),

$$\|y(t)\| \leqslant \|y(0)\|, \quad \forall t \in [0, t_{max}[. \tag{20}$$

Since $t \mapsto y(t)$ is continuous in $[-r, 0]$, then, there exists $C_1 > 0$, such that

$$\|y(t)\| \leqslant C_1, \quad \forall t \in [-r, 0]. \tag{21}$$

Combining (20) and (21), it comes

$$\|y(t)\| \leqslant C_* := \max\left\{C_1, \|y(0)\|\right\}, \quad \forall t \in [-r, t_{\max}[. \tag{22}$$

Finally, $y(t)$ is a global solution i.e., $t_{\max} = +\infty$ (see Wu (1996)).

Next, we will establish the estimate (6). By using the variation of constants formula (11) and taking

$$z(t) = y(t) - S(t)y(0), \quad \forall t \geqslant 0,$$

one can get that

$$z(t) = \rho \int_0^t S(t-\sigma) \log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right) Ny(\sigma - r)\, d\sigma.$$

Since (10), (22) and the fact that $S(t)$ is a semi-group of contractions, it follows by Schwartz's inequality for any $T > r$, that

$$\begin{aligned}
\|z(t)\| &\leqslant \rho C_* L_{C_*} \int_0^t \left|\log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right)\right| d\sigma \\
&\leqslant \rho C_* L_{C_*} \int_0^t \left|\log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right) \langle Ny(\sigma-r), y(\sigma)\rangle\right|^{\frac{1}{2}} d\sigma \\
&\leqslant \rho T^{\frac{1}{2}} C_* L_{C_*} \left(\int_0^t \left|\log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right) \langle Ny(\sigma-r), y(\sigma)\rangle\right| d\sigma\right)^{\frac{1}{2}}, \quad \forall t \in [0, T].
\end{aligned} \tag{23}$$

In addition, we have

$$\begin{aligned}
\langle NS(\sigma-r)y(0), S(\sigma)y(0)\rangle &= \langle NS(\sigma-r)y(0) - Ny(\sigma-r), y(\sigma)\rangle - \langle NS(\sigma-r)y(0), z(\sigma)\rangle \\
&\quad + \langle Ny(\sigma-r), y(\sigma)\rangle \\
&= \langle Nz(\sigma-r), y(\sigma)\rangle - \langle NS(\sigma-r)y(0), z(\sigma)\rangle + \langle Ny(\sigma-r), y(\sigma)\rangle, \ \forall \sigma \geqslant r.
\end{aligned}$$

Using (22) and since $S(t)$ is a semi-group of contractions, and that N is locally Lipschitz then we get for all $\sigma \in [r, T]$, that

$$|\langle NS(\sigma-r)y(0), S(\sigma)y(0)\rangle| \leqslant L_{C_*} C_* \|z(\sigma-r)\| + C_* L_{C_*} \|z(\sigma)\| + |\langle Ny(\sigma-r), y(\sigma)\rangle|.$$

Employing (23) one easily gets

$$\begin{aligned}
|\langle NS(\sigma-r)y(0), S(\sigma)y(0)\rangle| &\leqslant |\langle Ny(\sigma-r), y(\sigma)\rangle| \\
&\quad + 2\rho L_{C_*}^2 C_*^2 T^{\frac{1}{2}} \left(\int_0^T \left|\log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right) \langle Ny(\sigma-r), y(\sigma)\rangle\right| d\sigma\right)^{\frac{1}{2}}.
\end{aligned} \tag{24}$$

Replacing $y(0)$ by $y(t) = \Lambda(t)\Phi(0)$, $\forall t \geqslant 0$ in (24) and using the semi-group property of the solution $y(t)$ it yields

$$
\begin{aligned}
&|\langle NS(\sigma - r)y(t), S(\sigma)y(t)\rangle| \leqslant |\langle Ny(\sigma + t - r), y(\sigma + t)\rangle| \\
&\qquad + 2\rho L_{C_*}^2 C_*^2 T^{\frac{1}{2}} \left(\int_t^{t+T} \left| \log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right) \langle Ny(\sigma - r), y(\sigma)\rangle \right| d\sigma \right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $\log(1+x) \geqslant \frac{x}{2}$ and $|\log(1-x)| \geqslant \log(1+x)$, $\forall 0 < x < 1$, it yields that

$$
\left| \log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right) \right| \geqslant \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{2\left(1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|\right)}.
$$

Then

$$
\langle Ny(\sigma - r), y(\sigma)\rangle \leqslant 2\left(1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|\right) \left| \log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right) \right|,
$$

$$
\langle Ny(\sigma - r), y(\sigma)\rangle \leqslant 2\left(1 + L_{C_*} C_*^2\right) \left| \log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right) \right|. \tag{25}
$$

From (24), (25), and using Schwartz's inequalities, it follows that

$$
\begin{aligned}
&|\langle NS(\sigma - r)y(0), S(\sigma)y(0)\rangle| \\
&\quad \leqslant 2^{\frac{1}{4}} L_{C_*}^{\frac{1}{2}} C_* \left(1 + L_{C_*} C_*^2\right)^{\frac{1}{4}} \left| \log\left(1 - \frac{\langle Ny(\sigma + t - r), y(\sigma + t)\rangle}{1 + |\langle Ny(\sigma + t - r), y(\sigma + t)\rangle|}\right) \langle Ny(\sigma + t - r), y(\sigma + t)\rangle \right|^{\frac{1}{4}} \\
&\quad + 2\rho L_{C_*}^{\frac{5}{2}} C_*^{\frac{5}{2}} T^{\frac{3}{4}} \left(\int_t^{t+T} \left| \log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right) \langle Ny(\sigma - r), y(\sigma)\rangle \right| d\sigma \right)^{\frac{1}{4}}.
\end{aligned} \tag{26}
$$

Integrating (26) over the interval $[r, T]$, and using the Schwartz's inequality, we deduce

$$
\begin{aligned}
&\int_r^T |\langle NS(\sigma - r)y(t), S(\sigma)y(t)\rangle| \, d\sigma \\
&\qquad \leqslant C_{**} \left(\int_t^{t+T} \left| \log\left(1 - \frac{\langle Ny(\sigma - r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma - r), y(\sigma)\rangle|}\right) \langle Ny(\sigma - r), y(\sigma)\rangle \right| d\sigma \right)^{\frac{1}{4}},
\end{aligned} \tag{27}
$$

where $C_{**} := L_{C_*}^{\frac{1}{2}} C_* (T - r)^{\frac{3}{4}} \left(2^{\frac{1}{4}} \left(1 + L_{C_*} C_*^2\right)^{\frac{1}{4}} + 2\rho L_{C_*}^2 C_*^{\frac{3}{2}} T^{\frac{3}{4}} (T - r)^{\frac{1}{4}}\right)$. This achieves the proof. ■

3. Strong stabilization

Based on the previous results, we are able to establish the polynomial stability of the system (1), which leads us to the following theorem.

Theorem 2. *Let A generate a semi-group of contractions $S(t)$ on H, and let N be a locally Lipschitz operator from H into itself. Then, under the condition*

$$
\int_r^T |\langle NS(\sigma - r)\phi, S(\sigma)\phi\rangle| \, d\sigma \geqslant \delta \|\phi\|^2, \quad \forall \phi \in H, \quad (\text{for some } T > r \text{ and } \delta > 0), \tag{28}
$$

the feedback control (5) *strongly stabilizes the system* (1) *with the following decay estimate:*

$$
\|y(t)\| = \mathrm{O}\left(t^{-\frac{1}{2}}\right), \quad \text{as} \quad t \to +\infty \tag{29}
$$

Proof. According to Theorem 1, the system (1) controlled by (5) possesses a unique global mild solution $y(t)$ defined on the interval $[-r,+\infty)$, and given by the variation of constants formula (11). From (19),

$$\|y(t+T)\|^2-\|y(t)\|^2\leqslant 2\rho\int_s^{\tau}\log\left(1-\frac{\langle Ny(t-r),y(t)\rangle}{1+|\langle Ny(t-r),y(t)\rangle|}\right)\langle Ny(t-r),y(t)\rangle\leqslant 0,\quad T>r.$$

It follows from (27) and (28) that

$$\|y(t+T)\|^2-\|y(t)\|^2\leqslant -2\rho\frac{\delta^2}{C_{**}^2}\|y(t)\|^4,\quad \forall t\geqslant 0,$$

which implies that

$$2\rho\frac{\delta^2}{C_{**}^2}\|y(t)\|^4\leqslant\|y(t)\|^2-\|y(t+T)\|^2,\quad \forall t\geqslant 0. \tag{30}$$

Let we note that

$$\begin{aligned}\frac{1}{\|y(t+T)\|^2}-\frac{1}{\|y(t)\|^2}&=\int_0^T\frac{d}{d\theta}\left(\frac{\theta}{T}\|y(t+T)\|^2+\left(1-\frac{\theta}{T}\right)\|y(t)\|^2\right)^{-1}d\theta\\&=\frac{1}{T}\left(\|y(t)\|^2-\|y(t+T)\|^2\right)\int_0^T\left(\frac{\theta}{T}\|y(t+T)\|^2+\left(1-\frac{\theta}{T}\right)\|y(t)\|^2\right)^{-2}d\theta.\end{aligned}$$

It yields from (30) that

$$\frac{1}{\|y(t+T)\|^2}-\frac{1}{\|y(t)\|^2}\geqslant 2\rho\frac{\delta^2}{C_{**}^2}.$$

Then, for any $n\in\mathbb{N}$, one can deduce

$$\frac{1}{\|y((n+1)T)\|^2}-\frac{1}{\|y(0)\|^2}\geqslant Cn$$

with $C=2\rho\frac{\delta^2}{C_{**}^2}$, which implies that

$$\|y((n+1)T)\|^2\leqslant\left(\frac{1}{\|y(0)\|^2}+Cn\right)^{-1}$$

by taking $t=(n+1)T$, one can deduce that

$$\|y(t)\|^2\leqslant\left(\frac{1}{\|y(0)\|^2}-C+\frac{t}{T}\right)^{-1},$$

which proves the climate estimate. ■

Remark 1. 1. Since $t\mapsto\|y(t)\|^2$ decreases on $\mathbb{R}^+$, then $\exists t_*\geqslant 0$ such that:

$$y(t_*)=0\iff y(t)=0,\quad \forall t\geqslant t_*.$$

2. If $r=0$, we obtain the same results retrieved as in [6] for infinite dimensional semi-linear systems.
3. Note that the control used is more performed than the control used in [7] and guarantee the same results with gain of energy.

4. Weak stabilization

In the following next result, we will show that if N is sequentially continuous, then the weak stabilization of the system (1) by using the same feedback control (5) under a particular condition.

Theorem 3. *Let A generate a semi-group of contractions $S(t)$. Moreover, we assume that N is a locally Lipschitz and weakly sequentially continuous operator provided that*

$$\langle NS(t-r)y, S(t)y\rangle = 0, \quad \forall t \geqslant r \implies y = 0 \tag{31}$$

holds. Then, the system (1) *is weakly stabilizable using the feedback control* (5).

Proof. According to Theorem 1, the system (1) controlled by (5) possesses a unique global mild solution $y(t)$ defined on the interval $[-r, +\infty)$ and given by the variation of constants formula (11). From (19), we have

$$\rho \int_0^t \log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right)\langle Ny(\sigma-r), y(\sigma)\rangle\, d\sigma \leqslant \|y(0)\|^2, \quad \forall t \geqslant 0. \tag{32}$$

It yields from (32) that the integral

$$\int_0^t \log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right)\langle Ny(\sigma-r), y(\sigma)\rangle\, d\sigma$$

converges for all $t \geqslant 0$. Thus, we deduce from the Cauchy criterion that

$$\int_t^{t+T} \log\left(1 - \frac{\langle Ny(\sigma-r), y(\sigma)\rangle}{1 + |\langle Ny(\sigma-r), y(\sigma)\rangle|}\right)\langle Ny(\sigma-r), y(\sigma)\rangle\, d\sigma \to 0, \ \text{ as } \ t \to +\infty, \quad (\text{for any } T > r). \tag{33}$$

To prove that $y(t) \rightharpoonup 0$, as $t \to +\infty$, let (t_n) be a sequence of real numbers such that $t_n \to +\infty$, as $n \to +\infty$. From (22) and since the space H is reflexive, one can deduce that there exists a subsequence $(t_{\phi(n)})$ of (t_n) and $\psi \in H$ such that

$$y(t_{\phi(n)}) \rightharpoonup \psi, \quad \text{as} \quad n \to +\infty. \tag{34}$$

Since N is weakly sequentially continuous and $S(t)$ is continuous for all $t \geqslant 0$, we deduce that $S(t)y(t_{\phi(n)}) \rightharpoonup S(t)\psi$ and $NS(t)y(t_{\phi(n)}) \to NS(t)\psi$ as $n \to +\infty$. Thus, for all $t \geqslant r$,

$$\lim_{n\to+\infty} \left\langle NS(t-r)y(t_{\phi(n)}), S(t)y(t_{\phi(n)})\right\rangle = \langle NS(t-r)\psi, S(t)\psi\rangle\,.$$

It follows by the dominated convergence theorem that

$$\lim_{n\to+\infty} \int_r^T \left\langle NS(\sigma-r)y(t_{\phi(n)}), S(\sigma)y(t_{\phi(n)})\right\rangle d\sigma = \int_r^T |\left\langle NS(\sigma-r)\psi, S(\sigma)\psi\right\rangle| \, d\sigma. \tag{35}$$

Using (6) and (33), we deduce from (35) that $\int_r^T |\langle NS(\sigma-r)\psi, S(\sigma)\psi\rangle|\, d\sigma = 0$. Since the map $\tau \to S(\tau)\psi$ is continuous on $[0, +\infty)$, we deduce that $\langle NS(t-r)\psi, S(t)\psi\rangle = 0$, $\forall t \geqslant r$. From (31), we get $\psi = 0$. Moreover, from (34), one can prove that

$$y(t_{\phi(n)}) \rightharpoonup 0, \quad \text{as} \quad n \to +\infty. \tag{36}$$

Additionally, noticing that (36) holds for each subsequence $(t_{\phi(n)})$ of (t_n) such that $y(t_{\phi(n)})$ is weakly convergent in H. It yields that $\forall \zeta \in H$,

$$\langle y(t_n), \zeta\rangle \to 0, \quad \text{as} \quad n \to +\infty,$$

and hence,

$$y(t) \rightharpoonup 0, \quad \text{as} \quad t \to +\infty.$$

This achieves the proof of Theorem 3. ∎

Remark 2. 1. Note that the sequential continuity notion coincides with the compactness condition, when the operator is linear.

2. If we replace the sequential continuous condition of N by the compactness condition of $S(t)$, we retrieve the same result of the Theorem 3.

5. Applications

The main goal of this section is to present some applications to illustrate the previous results.

5.1. Strong stabilization

Example 1. Applications to Liénard's equations.
Let's consider the following system:

$$\begin{cases} \ddot{y}(t) = -y(t) + p(t) f\Big(y\left(t - \frac{\pi}{2}\right)\Big)\, \dot{y}\left(t - \frac{\pi}{2}\right), \quad t \geqslant 0, \\ y(t) = \sin(2\pi t), \quad t \in \left[0; \frac{\pi}{2}\right], \end{cases} \tag{37}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz function such that $f(0) = 0$. Here the space $H = \mathbb{R}^2$. The inner product is defined by:

$$\langle y, z\rangle = y_1 z_1 + y_2 z_2, \quad \forall y = (y_1, y_2), \quad z = (z_1, z_2) \in \mathbb{R}^2.$$

If we set $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{N}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_2 f(y_1) \end{pmatrix}$, $\forall (y_1, y_2) \in H$, one can easy deduce that the system (37) has the same form as (1). The operator A is skew adjoint and $e^{tA} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ (see [3]). Moreover,

$$\begin{aligned} &\left\langle N e^{\left(t - \frac{\pi}{2} A\right)} \begin{pmatrix} a \\ b \end{pmatrix}, e^{tA}\begin{pmatrix} a \\ b \end{pmatrix} \right\rangle \\ &\quad = \left(b\cos(t) - a\sin(t)\right)\left(b\cos\left(t - \frac{\pi}{2}\right) - a\sin\left(t - \frac{\pi}{2}\right)\right) f\left(a\cos\left(t - \frac{\pi}{2}\right) + b\sin\left(t - \frac{\pi}{2}\right)\right). \end{aligned} \tag{38}$$

Then (31) holds, as well as (28) since $\dim(H) < +\infty$. We deduce by Theorem 2 that the solution of the system (37) satisfies

$$y^2(t) + \dot{y}^2(t) = \mathrm{O}\left(\frac{1}{t}\right), \quad \text{as} \quad t \to +\infty$$

if $(y(t), \dot{y}(t)) \neq (0,0)$ using the feedback control defined by:

$$p(t) = \begin{cases} \rho \log\left(1 - \dfrac{\dot{y}(t - \frac{\pi}{2})\dot{y}(t) f\left(y\left(t - \frac{\pi}{2}\right)\right)}{1 + |\dot{y}(t - \frac{\pi}{2})\dot{y}(t) f\left(y\left(t - \frac{\pi}{2}\right)\right)|}\right), \quad (y(t), \dot{y}(t)) \neq (0,0), \quad \rho > 0, \\ 0, \quad (y(t), \dot{y}(t)) = (0,0). \end{cases} \tag{39}$$

5.2. Weak stabilization

Example 2. Heat equation.
Consider the following semi-linear system:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \dfrac{\partial^2 y(x,t)}{\partial x^2} + p(t) N y(x,t), \quad (x,t) \in (0,1) \times (0, +\infty), \\ \dfrac{\partial y(0,t)}{\partial x} = \dfrac{\partial y(1,t)}{\partial x} = 0, \quad t \in [-r, +\infty), \\ y(x,t) = t\sin t, \quad t \in [-r, 0], \quad x \in (0,1), \end{cases} \tag{40}$$

where $y(t)$ is the temperature profile at time t. $v(t)$ is the flow rate of a liquid that controlled the system. The state space $H = L^2(0,1)$ and the operator A is defined by $Ay = \frac{\partial^2 y}{\partial x^2}$, with $D(A) = \left\{ y \in H^2(0,1); \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0 \right\}$.

The spectrum of A is given by the simple eigenvalues $\lambda_j = -\pi^2(j-1)^2$, $j \in \mathbb{N}^*$ with its corresponding eigenfunctions $\phi_1(x) = 1$ and $\phi_j(x) = \sqrt{2}\cos((j-1)\pi x)$, $j \geqslant 2$. Moreover, the operator N defined by $Ny = \sum_{j=1}^{+\infty} \frac{1}{j^2} \langle y, \phi_j \rangle \phi_j$ is compact and satisfies

$$\langle NS(t-r)y, S(t)y \rangle = \sum_{j=1}^{+\infty} \frac{e^{\lambda_j(2t-r)}}{j^2} |\langle y, \phi_j \rangle|^2 \geqslant 0.$$

In addition, it is easy to check that (30) holds. According to the Theorem 3, we deduce that the system (40) is weakly stabilizable using the following feedback control

$$p(t) = \begin{cases} -\log\left(1 + \sum_{j=1}^{+\infty} \frac{e^{\lambda_j(2t-r)}}{j^2} |\langle y, \phi_j \rangle|^2\right), & \text{if} \quad y(\cdot,t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

6. Numerical simulation

Consider the system (37). Take $\rho = 1$ and $f(y) = y$. Then, we get the results shown in the Figs. 1–5. Figs. 1 and 2 show the evolution and norm of the free state ($v(t) = 0$). Use feedback control (39), we obtain Fig. 3 and Fig. 4 which show the evolution and the norm of the stabilized state. Fig. 5 shows the evolution of the stabilizing control.

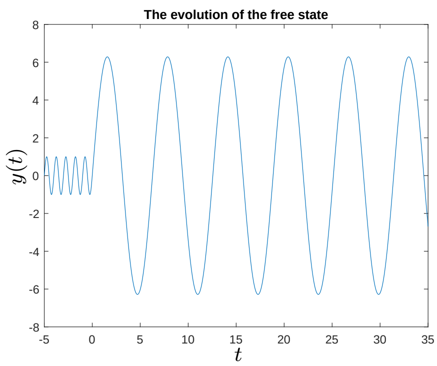

Fig. 1. Evolution of the free state.

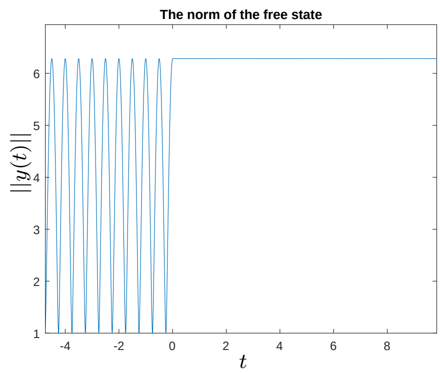

Fig. 2. Norm of the free state.

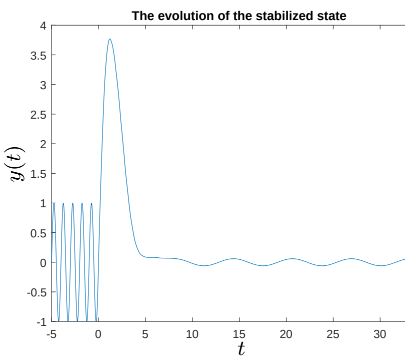

Fig. 3. Evolution of the stabilized state.

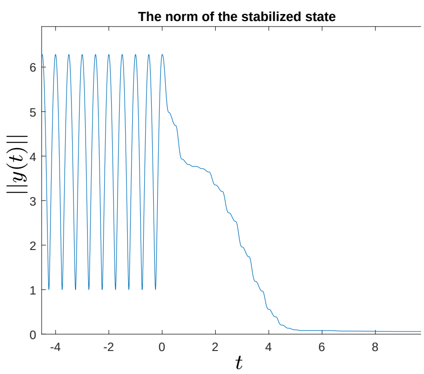

Fig. 4. Norm of the stabilized state.

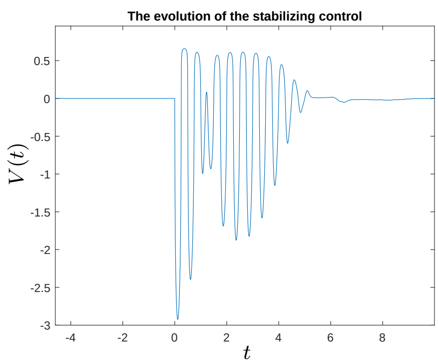

Fig. 5. Evolution of the stabilizing control.

7. Conclusion

Under the exact observability inequality (30) we have established the polynomial stabilization for infinite dimensional semi-linear systems with time delay with a new constrained multiplicative feedback control. The rate of polynomial convergence is explicitly expressed. We also have considered the question of weak stabilization by the same feedback control. Furthermore, some applications are given to illustrates our main results.

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Слабка та сильна стабілізація напівлінійних систем із запізнюванням, керованих обмеженим зворотним зв'язком

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У цій роботі розглядається питання слабкої та сильної стабілізації розподілених напівлінійних систем із часовою затримкою з використанням керування з обмеженим зворотним зв'язком. Результати для напівлінійних систем без запізнювання узагальнені для випадків сильної та слабкої стабілізації. Розглянуто ілюстративні приклади застосування методу до гіперболічних та параболічних рівнянь.

Ключові слова: *напівлінійна система, стабілізація зі зворотним зв'язком, оцінка розпаду полінома, запізнювання.*