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Image retrieval using Nash equilibrium and Kalai–Smorodinsky solution

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In this paper, we propose a new formulation of Nash games for solving a general multi-objectives optimization problems. The objective of this approach is to split the optimization variables, allowing us to determine numerically the strategies between two players. The first player minimizes his function cost using the variables of the first table P and the second player, using the second table Q. The original contribution of this work concerns the construction of the two tables of allocations that lead to a Nash equilibrium on the Pareto front. The second proposition of this paper is to find a Nash Equilibrium solution, which coincides with the Kalai–Smorodinsky solution. Two algorithms that calculate P, Q and their associated Nash equilibrium, by using some extension of the normal boundary intersection approach, are tried out successfully. Then, we propose a search engine to look for similar images of a given image based on multiple image representations using Color, Texture and Shape Features.

Keywords: *Nash Equilibrium, Kalai–Smorodinsky solution, fuzzy K-means, concurrent optimization, color, Gist and SIFT descriptors.*

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1. Introduction

There exists several approaches to solve problems of multi-objective optimization [1–3]. All these methods, until now, deal with the multidisciplinary problem by considering a kind of implicit weighting of all the disciplinary criteria. Another idea consists in assigning to each discipline its own criterion. This multi-criterion problem can be solved by allowing to each criterion a weight [4] (a coefficient of substitution); we get back to a mono criterion problem. This approach has a serious disadvantage. The choice of the criteria weights is arbitrary and influences on the optimum reached. Another alternative which can be used to solve the multicriterion problems consists to identify the Pareto front [5,6] which represent the set of not-dominated strategies. This approach is generally expensive since it needs a great number of evaluations of several criterions. The second difficulty is related to the choice of the best point on the Pareto front. The game theory defines another framework to solve the problems of multicriterion optimization. This theory was studied by J. Périaux [7, 8] and by J. A. Désidéri [9] as a powerful way to solve multidisciplinary optimization problems. B. Abou El Majd considered in [10, 11] an aerodynamic and structural optimization problem of a business-jet wingshape solved by a Nash game and an adapted split of variables. In [12], A. Habbal et al. solved a multidisciplinary optimization problem using a non-cooperative game (Nash game), where the strategy of the players is naturally defined. Several multidisciplinary optimization problems arise in the form:

$$(M)\begin{cases}\min\limits_{y\in R^n} f_1(y),\\ \min\limits_{y\in R^n} f_2(y),\end{cases}\tag{1}$$

f_1 and f_2 two convex function.

To solve the problem (M), there is a lot of methods, weighted method, NBI, ...

In this paper, we propose to solve this problem by using the Nash equalibrium, since it is simple to calculate numerically the solution of problem (M). The split of the variable y amounts to construct two allocation tables P and Q in $\{0,1\}^n$, where $P_i + Q_i = 1$, $1 \leqslant i \leqslant n$. Let $I_{12} = \{1, \ldots, n\}$ be a set of indices of cardinality n, I_1 is a subset of I_{12} of cardinal $n-p$, and I_2 is its complement of cardinal p, that is to say $I_{12} = I_1 \cup I_2$.

Suppose that:

$$\begin{cases} U = (y_i), & \text{for } i \in I_1, \\ V = (y_i), & \text{for } i \in I_2. \end{cases} \tag{2}$$

Define in this case the integer allocation table P of size n:

$$P_i = 1,\ \forall i \in I_1,\ P_i = 0,\ \forall i \in I_2,$$

so that

$$y = P \cdot y + (\mathcal{I} - P) \cdot y = (U, V), \qquad \text{where} \qquad \mathcal{I} = (1, \ldots, 1), \tag{3}$$

where "$\cdot$" denote the Hadamard product (i.e. $(P \cdot y)_i = P_i y_i$, $P \cdot y \in \mathbb{R}^n$), and (U, V) is defined in (2).

$(U^*, V^*) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$ is a Nash equilibrium if and only if:

$$\begin{cases} f_1(U^*, V^*) = \min\limits_U f_1(U, V^*), \\ f_2(U^*, V^*) = \min\limits_V f_2(U^*, V). \end{cases} \tag{4}$$

Let's consider two positive convex functions f_1 and f_2, and the Nash game (5) which is written in the following form:

$$\begin{cases} \text{Find } y_{EN} \text{ solution of:} \\ \min\limits_U f_1(P \cdot y + (\mathcal{I} - P) \cdot y_{EN}), \\ \min\limits_V f_2((\mathcal{I} - P) \cdot y + P \cdot y_{EN}), \end{cases} \tag{5}$$

where $y_{EN} = (U^*, V^*)$.

Consider the following fixed point problem (6):

$$\begin{cases} \text{Find } y_{EN} \text{ solution of:} \\ \min\limits_y f_1(P \cdot y + (\mathcal{I} - P) \cdot y_{EN}) + f_2((\mathcal{I} - P) \cdot y + P \cdot y_{EN}) = f_1(y_{EN}) + f_2(y_{EN}). \end{cases} \tag{6}$$

The allocation table P is fixed, and then strategies of each player are the variables corresponding to P and $\mathcal{I} - P$, i.e R^p and R^{n-p}. If y_{EN} is a solution of (6), then y_{EN} is a Nash Equilibrium of (5), and conversely. For the proof, it suffices, write the optimality condition of problem (5) and (6).

For each choice of P, we find a Nash equilibrium, in this case, we have at most 2^n (where n is the size of y) Nash equilibria. The natural question is, how to choose among all these equilibriums the best Nash equilibrium. That means how to choose the best splitting of territories between the two players that gives an equilibrium belonging to the Pareto front if it exists, which is not always the case. Mixed allocations (the elements of P belonging to $[0,1]$) are obtained by convexification of the set of pure ones. We also drop the mutual exclusivity constraint, to allow both players to share the same variable. In [13], Aboulaich et al. proposes two heuristic algorithms in order to split the territory. These algorithms allow to compute successfully the Nash equilibrium, but the obtained equilibriums are not on the Pareto front.

In this work we will test in the first part a splitting using P and $Q = \mathcal{I} - P$. In the second part we propose two algorithms NS1 and NS2. The first algorithm calculates the two tables P, Q and the Nash equilibrium associated. In such case, this equilibrium belongs to Pareto front, we use the strategy of Nash games coupled with an extension of the approach "Normal Boundary Intersection" NBI (NBI-Nash).

In the second one, we present a new technique to split the optimization variable y, of such kind the Nash equilibrium associated with this splitting is a solution of Kalai–Smorodinsky [14]. To calculate the Kalai–Smorodinsky solution, it is enough to find the intersection between, the line joining the ideal solution and the disagreement point D, and the Pareto front.

In the following we recall briefly the NBI approch [15] and the splitting algorithm proposed in [10, 13, 16, 17].

2. Preliminary result

Let F^* denotes the shadow minimum, i.e., the vector with components $f_i^* = f_i(x_i^*)$, and let Φ denotes the shifted pay-off matrix, where i^th column is $F(x_i^*) - F^*$. The Convex Hull of Individual Minima or CHIM is defined as the set of points that are convex combinations of the columns of Φ, i.e., $\{\Phi\beta\colon \beta_i \geqslant 0, \sum_i \beta_i = 1\}$.

The central idea of the NBI method is that the point of intersection between the Pareto Front and the normal n, emanating from any point in the CHIM and pointing to the origin is a point located on the part of the Front containing the efficient points. Let n be the normal vector to CHIM pointing towards the origin, $\Phi\beta + nt$ represents the set of points belonging to this normal. The point of intersection of the normal with the Pareto Front is the global maximum of the following NBI_β subproblem:

$$\begin{cases} \text{Maximize } t, \\ \text{Subject to: } x \in A, \\ \Phi\beta + tn = F(x) - F^*, \end{cases} \tag{7}$$

where A is the set of feasible solutions.

We solve NBI_β (7) for different β, various points on the efficient frontier can be generated. The advantage of the β parameter is that an even spread of β parameters corresponds to an even sped of points on the CHIM.

Aboulaich et al. [16] demonstrates the equivalence between the research of the Nash equilibrium of the problem (5) and fixed point of the problem (6), for values of P binary. This equivalence is true only if P is binary. In the following we propose an extension of the algorithm introduced in [17] to the non binary case. We search for the Nash equilibrium associated to two given tables of allocation P and $(\mathcal{I} - P)$ which are not necessarily binary, the elements of P belong to the interval $[0, 1]$. In this case we solve the following problem:

$$\begin{cases} \text{Find } y_{EN} \text{ solution of:} \\ \min\limits_y f_1(P \cdot y + (\mathcal{I} - P) \cdot y_{EN}^{k-1}), \Rightarrow y_{opt1}^k, \\ \min\limits_y f_2((\mathcal{I} - P) \cdot y + P \cdot y_{EN}^{k-1}), \Rightarrow y_{opt2}^k, \\ \text{with the update: } \Rightarrow y_{EN}^k = P \cdot y_{opt1}^k + (\mathcal{I} - P) \cdot y_{opt2}^k. \end{cases} \tag{8}$$

Algorithm (NS0):

1. Initialization: y_{opt1}^0, y_{opt2}^0 and $y_{EN}^0 = P \cdot y_{opt1}^0 + (\mathcal{I} - P) \cdot y_{opt2}^0$.
2. for $k \geqslant 1$:
   (a) solve the problem: $\min\limits_y f_1(P \cdot y + (\mathcal{I} - P) \cdot y_{EN}^{k-1}) \Rightarrow y_{opt1}^k$,
   (b) solve the problem: $\min\limits_y f_2((\mathcal{I} - P) \cdot y + P \cdot y_{EN}^{k-1}) \Rightarrow y_{opt2}^k$,
   (c) the update
   $$y_{EN}^k = P \cdot y_{opt1}^k + (\mathcal{I} - P) \cdot y_{opt2}^k, \text{ the update of } y_{EN},$$

$$y^k_{EN} = t y^k_{EN} + (1-t) y^{k-1}_{EN}, \text{ where } t \in]0,1], \text{ the relaxation of } y_{EN},$$

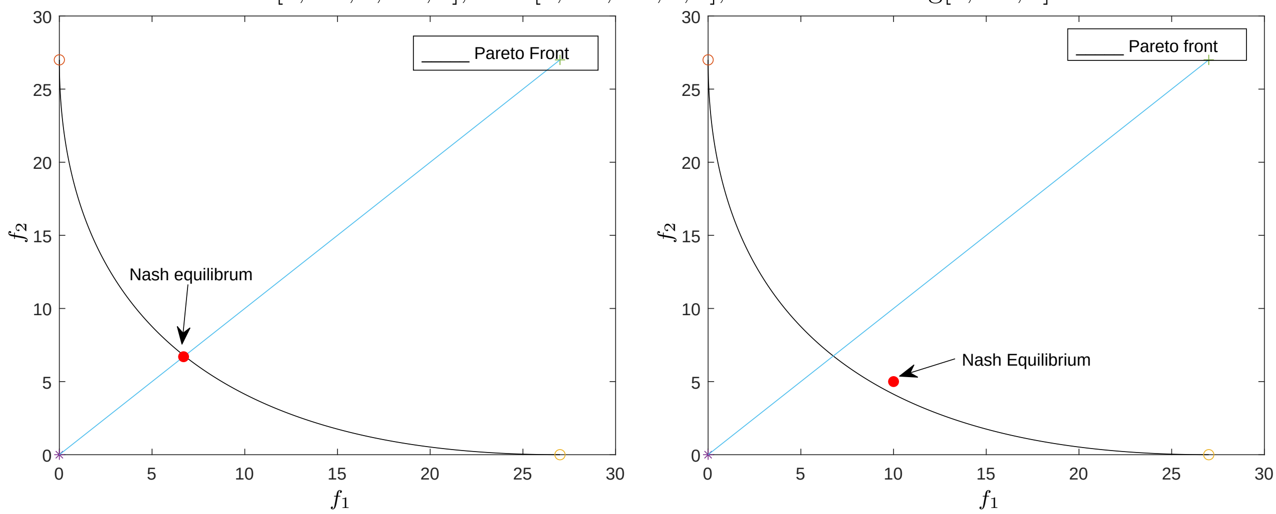
3. while $||y^{(k)}_{EN} - y^{(k-1)}_{EN}|| > test$, set $k = k+1$, and repeat 2.

We present in the following, the results obtained by the algorithm $(NS0)$ for some tests, by considering two functions f_1 and f_2 defined by:

$$f_1(y) = \frac{1}{2}||Ay - b||^2 \text{ et } f_2(y) = \frac{1}{2}||Cy - d||^2, \quad y \in R^{n\times 1}, \tag{9}$$

where A and C are two $n \times n$ matrices, b and d $n \times 1$ matrices, and $\|\cdot\|$ is the Euclidean norm.

$b = [1; -2; 0; -1; 2]$; $d = [1; -3; -1; 3; 5]$; $A = C = \text{tridiag}[1; -2; 1]$.

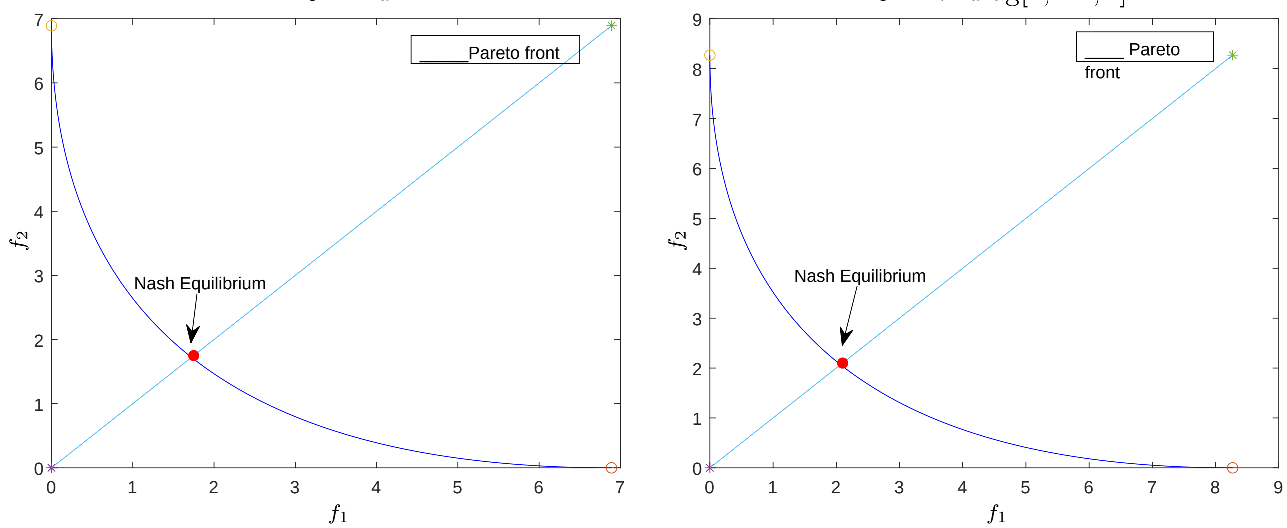


$P = [0.1174; 0.2967; 0.3188; 0.4242; 0.5079]$ $P = [0.0005; 1; 0.9995; 0.0001; 0.9985]$

Fig. 1. Test 1 and 2: The Nash overall loop converged in 21 iterations (left) and in 29 iterations (right).

$b = \text{rand}(50,1)$; $d = \text{rand}(50,1)$.

$A = C = \text{Id}$ $A = C = \text{tridiag}[1; -2; 1]$

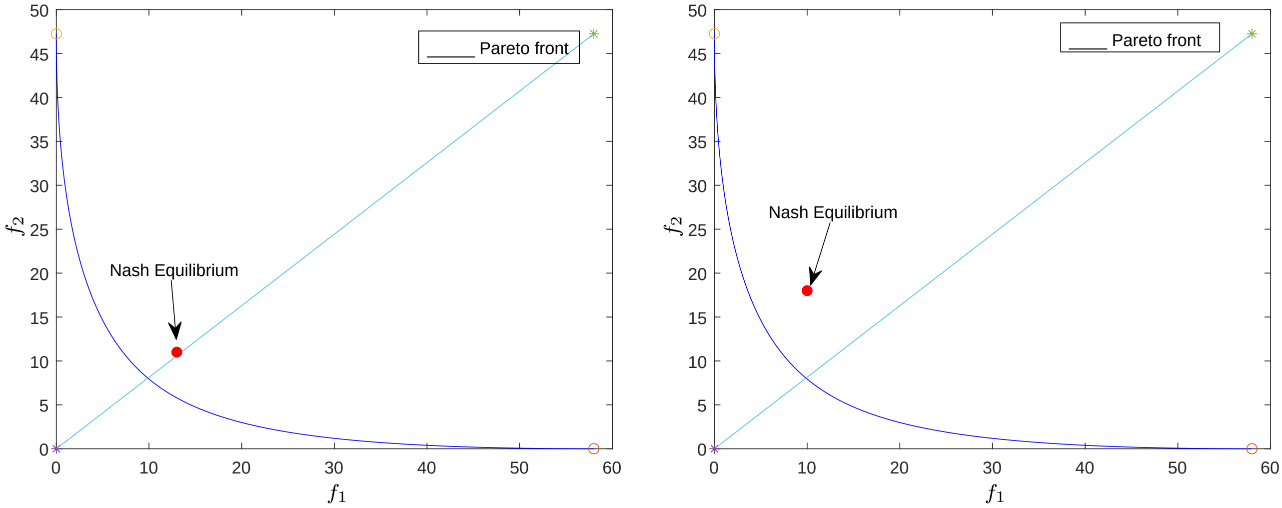


$0.1 < P = \text{rand}(50,1) < 0.99$ $0.1 < P = \text{rand}(50,1) < 0.99$

Fig. 2. Test 3 and 4: The Nash overall loop converged in 51 iterations (left) and in 103 iterations (right).

According to the results obtained in the tests, we note that, in the tests 1, 3 and 4 we have $A = C$ and the elements of P are not all close to 0 and 1 then the Nash equilibrium coincides with the Kalai–

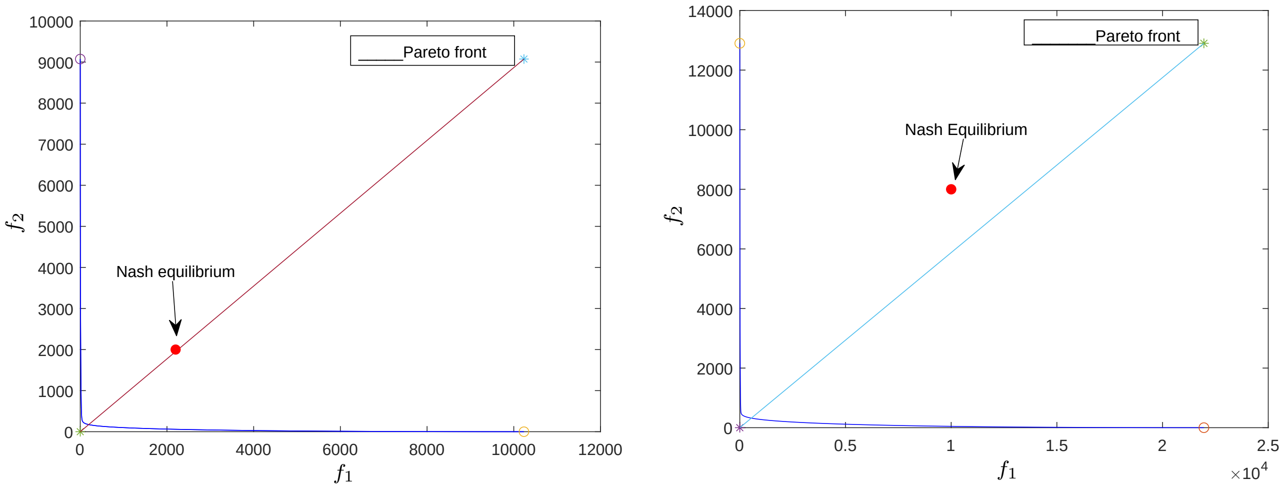
$b = [1; -2; 0; -1; 2]$; $d = [1; -3; -1; 3; 5]$; $A = \text{tridiag}[1; -2; 1]$; $C = \text{diagsup}[1; -1]$.



$P = [0.2760; 0.6797; 0.6551; 0.1626; 0.1190]$ $P = [0.0005; 1; 0.9995; 0.0001; 0.9985]$

Fig. 3. Test 5 and 6: The Nash overall loop converged in 205 iterations (left) and in 252 iterations (right).

$b = 2\,\text{rand}(50, 1)$; $d = 5\,\text{rand}(50, 1)$; $A = \text{tridiag}[1; -2; 1]$; $C = \text{diagsup}[1; -1]$.



$0 < P = \text{rand}(50, 1) < 1$ $0 \cong P = \text{rand}(50, 1) \cong 1$

Fig. 4. Test 7 and 8: The Nash overall loop converged in 520 iterations (left) and in 463 iterations (right).

Smorodinsky solution. In test 2, we have $A = C$ and elements of P are not far from 0 and 1, then the Nash equilibrium is not any more a Kalai–Smorodinsky solution and it is not in the Pareto front but it's close to the minimum of f_1. In tests 5 and 7, we have $A \neq C$ and the elements of P are not all close to 0 and 1, then the Nash equilibrium is on the line passing through the point of disagreement and the utopia point. And in test 6 and 8, we have $A \neq C$ and elements of P are not far from 0 and 1, then the Nash equilibrium is not in the Pareto front.

In the next part we present two new algorithms in order to construct the allocation tables P and Q.

3. Algorithm 1 (NS1): Nash equilibrium and NBI approach

The goal of this algorithm is to search among the Nash equilibria that are on the Pareto front, using an extension of the NBI approach [15]. NBI is a technique that seeks Part space which contains the Pareto optimal points. The idea behind NBI is to pick an even spread of points on the Convex Hull of Individual Minima (CHIM), and to find the intersection point between the efficient front and a set of parallel normals emanating from the chosen set of points on the CHIM. This point belongs to the set of the effective points which are on Pareto front. The pure allocation tables are any elements P and Q from $\{0,1\}^n$ that satisfy $P_i+Q_i=1$ for $1 \leqslant i \leqslant n$. Mixed allocations are obtained by convexification of the set of pure ones. We also drop the mutual exclusivity constraint, to allow both players to share the same variable. To split the optimization variable, we construct two sequences of tables for allocation $P^{(m)}$ and $Q^{(m)}$ in $[0,1]^n$, using the approach proposed in [13] as the initialization step. We build $P^{(0)}$ and $Q^{(0)}$ using the iterations results giving by the iterative minimization of f_1 and f_2, the iteration consists in solving successively two optimization problems (M1) and (M2) by combining NBI and Nash games.

In the first step, we use a heuristic approach to construct the allocations tables. It is based on the observation of preferred directions of descent algorithm to optimize each functional separately. For example, the component P_i is the ratio of the number of times (relative to the total number of optimization iterations) where the direction j was used to reduce the test f_1.

Step1: Let $m=0$, from an initial point $x^{(0)}$ et $y^{(0)} \in R^n$, $P^{(0)}$ and $Q^{(0)}$ are calculated by:

$$\begin{cases} \min\limits_{x \in R^n} f_1(x), \quad x^{(k+1)} = x^{(k)} - \rho_k \nabla f_1(x^{(k)}), \quad k \geqslant 0, \quad P_j^{(0)} = \frac{\sum_k |x_j^{(k+1)} - x_j^{(k)}|}{\sum_k \|x^{(k+1)} - x^{(k)}\|}, \\ \min\limits_{y \in R^n} f_2(y), \quad y^{(k+1)} = y^{(k)} - \rho_k \nabla f_2(y^{(k)}), \quad k \geqslant 0, \quad Q_j^{(0)} = \frac{\sum_k |y_j^{(k+1)} - y_j^{(k)}|}{\sum_k \|y^{(k+1)} - y^{(k)}\|}, \end{cases} \tag{10}$$

set,

$$y_{EN}^{(0)} = P^{(0)} \cdot x^* + Q^{(0)} \cdot y^*, \quad F(x) = (f_1(x), f_2(x))^T \text{ and } F^* = (f_1(x^*), f_2(y^*))^T, \tag{11}$$

where,

$$\begin{cases} x^* = \operatorname{Arg}\min\limits_x f_1(x) \\ y^* = \operatorname{Arg}\min\limits_y f_2(y). \end{cases} \tag{12}$$

Step2: For $m > 0$, solve,

$$(M1) \begin{cases} \max\limits_{x,t,\beta,P} t, \\ s.c.\ F\big(P \cdot x + Q^{(m-1)} \cdot y_{EN}^{(m-1)}\big) = F^* + tn + \Phi\beta, \end{cases} \tag{13}$$

and,

$$(M2) \begin{cases} \max\limits_{y,t,\beta,Q} t, \\ s.c.\ F\big(Q \cdot y + P^{(m-1)} \cdot y_{EN}^{(m-1)}\big) = F^* + tn + \Phi\beta. \end{cases} \tag{14}$$

$$y_{EN}^{(m)} = P^{(m)} \cdot x_{opt}^{(m)} + Q^{(m)} \cdot y_{opt}^{(m)},$$

where $x_{opt}^{(m)}$ (resp $P^{(m)}$) is a solution of the problem (M1) with respect to x (resp P), and $y_{opt}^{(m)}$ (resp $Q^{(m)}$) is a solution of the problem (M2) with respect to y (resp Q).

While $||y_{EN}^{(m)} - y_{EN}^{(m-1)}|| > test$, pose $m = m+1$, and repeat ***Step2***.

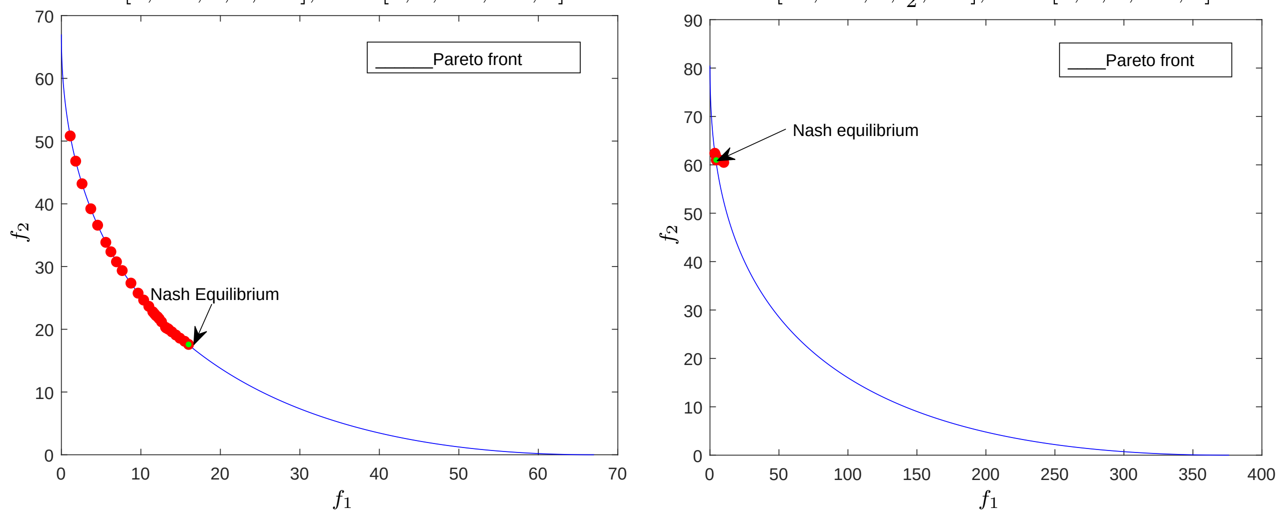
In the following we present the results obtained by the algorithm (NS1) for some tests.

Several other tests have been made ($n = 20$, $n = 50 \ldots$), the results show that the algorithm (NS1) numerically converges to a Nash equilibrium on the Pareto Front for functions defined by (9).

4. Algorithm 2 (NS2)

In this section, we present a new technique to split the optimization variable y using the two tables P and $(\mathcal{I} - P)$ and the algorithm of Kalai–Smorodinsky [14]. This technique is based on the calculation of the utopian point, the disagreement point and the Nash equilibrium associated to P.

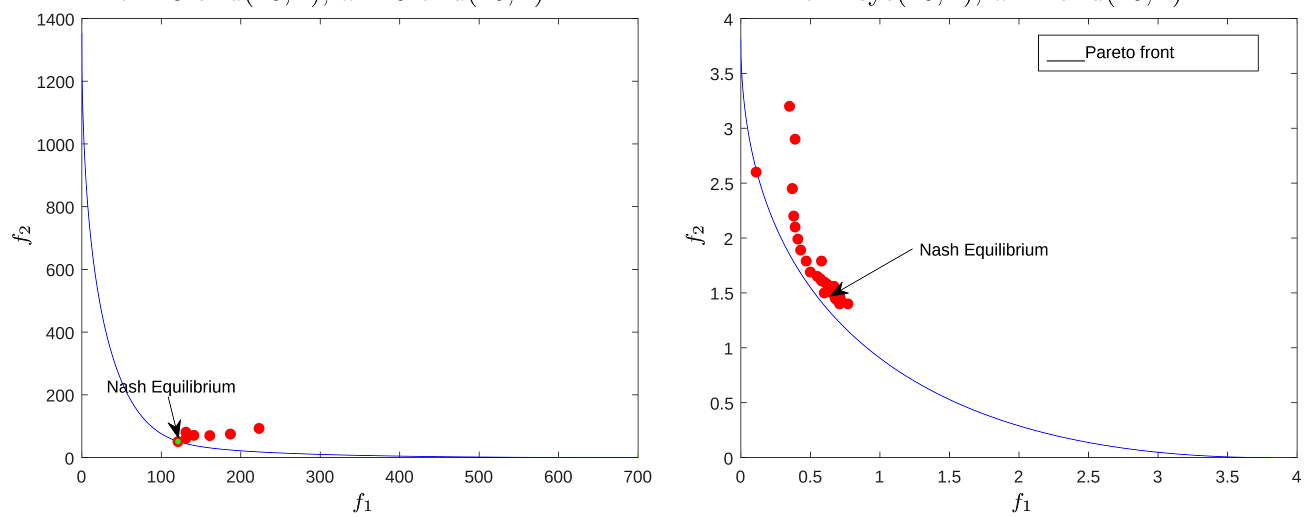
$b = [1; -2; 2; 0; -1]$; $d = [5; 1; -3; -1; 3]$ $\quad$ $b = [10; -2; 2; \frac{1}{2}; -1]$, $d = [6; 1; 9; -1; 3]$



$A = C = \mathrm{Id}$ $\quad$ $A = \mathrm{tridiag}[1; -2; 1]$, $C = \mathrm{triangsup}[-1; 1]$

Fig. 5. Test 1 and 2: The Nash overall loop converged in 25 iterations (left) and in 7 iterations (right).

$b = 8\mathrm{rand}(10, 1)$; $d = 5\mathrm{rand}(10, 1)$ $\quad$ $b = \mathrm{eye}(10, 1)$, $d = \mathrm{rand}(10, 1)$



$A = \mathrm{tridiag}[1; -2; 1]$, $C = \mathrm{triangsup}[-1; 1]$ $\quad$ $A = C = \mathrm{Id}$

Fig. 6. Test 3 and 4: The Nash overall loop converged in 17 iterations (left) and in 32 iterations (right).

We are looking at each iteration for the Nash equilibrium associated to the allocation table calculated, while approaching the intersection between the Pareto front and the line joining the utopian point Ut and the disagreement point D.

Note,

$$Ut = \begin{pmatrix} f_1(x^*) \\ f_2(y^*) \end{pmatrix}, \quad D = \begin{pmatrix} f_1(y^*) \\ f_2(x^*) \end{pmatrix} \quad \text{and} \quad \tau = \frac{Ut - D}{\|Ut - D\|},$$

where

$$\begin{cases} x^* = \operatorname{Arg}\min\limits_{x} f_1(x), \\ y^* = \operatorname{Arg}\min\limits_{y} f_2(y). \end{cases} \tag{15}$$

We look for the splitting of the optimization variable y (we search table P) in order that the Nash equilibrium coincides with the Kalai–Smorodinsky solution, via the following algorithm:

1. Initialization, $m = 0$: ***Step1*** of (NS1).

2. For $m > 0$, solve the problem

$$(KS1)\begin{cases} \max\limits_{y,t,P} t, \\ s.c.\ F(P \cdot y + (\mathcal{I} - P) \cdot y_{EN}^{(m-1)}) = D + t\tau, \end{cases} \tag{16}$$

$$y_{EN}^{(m)} = P^{(m)} \cdot K_{opt}^{(m)} + (\mathcal{I} - P^{(m)}) \cdot y_{EN}^{(m-1)},$$

where $K_{opt}^{(m)}$ (resp $P^{(m)}$) is a solution of the problem (KS1) over y (resp P) and F is defined in 11.

3. While

$$||y_{EN}^{(m)} - y_{EN}^{(m-1)}|| \geqslant test,$$

pose $m = m + 1$, and repeat 2.

In the following we present the results obtained by the algorithm (NS1) for some tests.

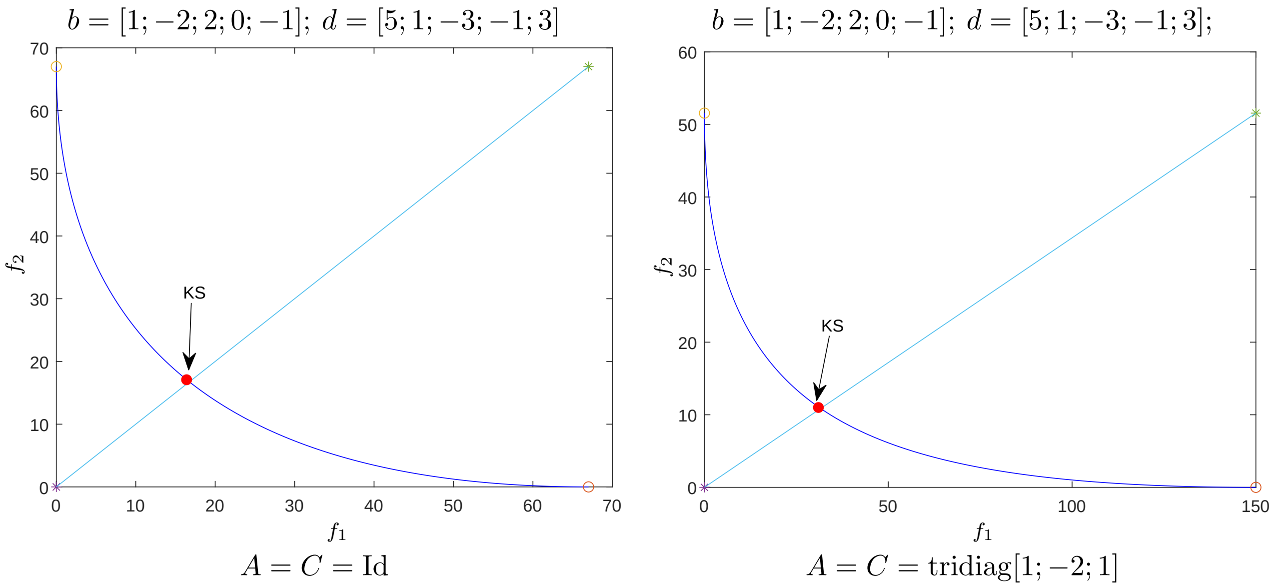


Fig. 7. Test 1 and 2: The Kalai–Smorodinsky solution overall loop converged in 9 iterations (left) and in 4 iterations (right).

According to the obtained results, we note that the Kalai–Smorodinsky solution (KS) is determined as a Nash equilibrium.

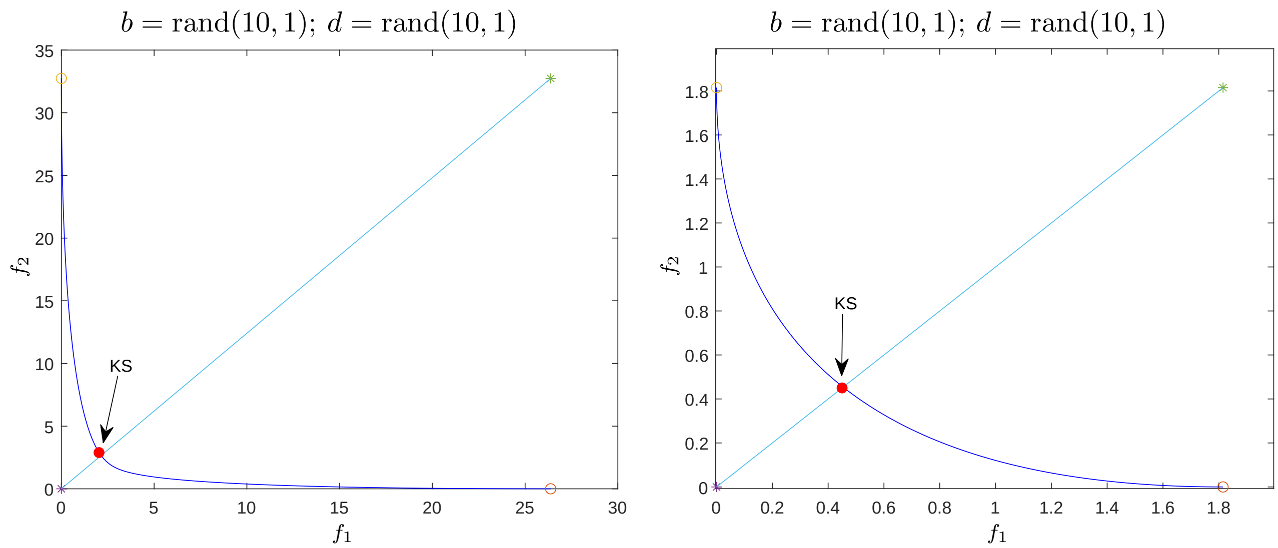
The proposed algorithm allows to construct two allocation tables P and Q, and a Nash equilibrium that is a Kalai–Smorodinsky solution.

5. Split of image in concurrent optimization

This paper aims at highlighting the practical function of a new method/procedure when the similar image solution, coupling color, Segmentation-based Fractal Texture analysis (SFTA) and Shape Features (Zernike) named from now on as strategies within the scope of Content-Based Image Retrieval (CBIR).

Each image $I_0 = (I_{0C}, I_{0S}, I_{0Z})$ of the database will be extracted according to three classes Color (denoted by C), Segmentation-based Fractal Texture analysis (SFTA) (denoted by S) class and Shape Features (Zernike) (denoted by Z) class [18].

We propose to determine the similar images as a Nash Equilibrium, which coincide with the Kalai–Smorodinsky solution. To determine the optimal images we will use in both cases three criteria j_C, j_S and j_Z associated with the three players respectively. The three players play a static game with



$A = \text{tridiag}[1; -2; 1]$, $C = \text{triangsup}[-1; 1]$ $\quad$ $A = C = \text{Id}$

Fig. 8. Test 3 and 4: The Kalai–Smorodinsky solution overall loop converged in 6 iterations (left) and in 11 iterations (right).

complete information, the first player is the Color descriptor that is used to control color classes in an image, C. The second player is the Segmentation-based Fractal Texture analysis (SFTA) classes, denoted by S. Then, the third is the Shape Features (Zernike) descriptors which controls the Zernike classes, denoted by Z.

Let us denote by $I(I_C, I_S, I_Z)$ an similar image, and let I_0 be image of database defined by three vectors (Colors(C), SFTA(S), Zernike(Z)). Retrieve the similar images I from $I0$, by simply minimizing the quadratic misfit. Solving the Nash equilibrium requires solving the following three problems, namely

$$j_C(I_C, I_S, I_Z), \quad j_S(I_C, I_S, I_Z) \quad \text{and} \quad j_Z(I_C, I_S, I_Z)$$

defined by:

$$j_C(I_C, I_S, I_Z) = \frac{1}{2}||I_C - I_{0C}||^2 + \frac{\varepsilon}{2}||\nabla(I_C + I_S + I_Z)||^2, \quad I_C \in H_0^1(\Omega),$$

$$j_S(I_C, I_S, I_Z) = \frac{1}{2}||I_S - I_{0S}||^2 + \frac{\varepsilon}{2}||\nabla(I_C + I_S + I_Z)||^2, \quad I_S \in H_0^1(\Omega),$$

$$j_Z(I_C, I_S, I_Z) = \frac{1}{2}||I_Z - I_{0Z}||^2 + \frac{\varepsilon}{2}||\nabla(I_C + I_S + I_Z)||^2, \quad I_Z \in H_0^1(\Omega),$$

where ε is some parameter to be adjusted [3, 16].

We say that the couple $(I_C^\star, I_S^\star, I_Z^\star)$ is a point of Nash equilibrium [18], if and only if

$$\begin{cases} \text{Find } (I_C^\star, I_S^\star, I_Z^\star) \in H_0^1(\Omega) \text{ such that:} \\ \min\limits_{I_C} j_C(I_C, I_S^\star, I_Z^\star) = J_C(I_C^\star, I_S^\star, I_Z^\star), \\ \min\limits_{I_S} J_S(I_C^\star, I_S, I_Z^\star) = J_S(I_C^\star, I_S^\star, I_Z^\star), \\ \min\limits_{I_Z} j_Z(I_C^\star, I_S^\star, I_Z) = J_Z(I_C^\star, I_S^\star, I_Z^\star). \end{cases} \tag{17}$$

5.1. Simulation results

The collection consists of 20 000 images from a private photographic image collection [18]. Our main objective was to develop a system for big data base so the execution time was our main priority.

Unlike classic system based on the calculation of the distance of the requested Image and all database Image, our system reduces the number of Image to be checked in to small number (less than 30). To minimize the number of images to check we work with a number of clusters $k = 35$ for color, $k = 20$ for Segmentation-based Fractal Texture analysis (SFTA), and $k = 30$ for Zernike. To verify the effectiveness of our approach, we have performed a comparative study between [18] with the presented descriptors and our algorithm.

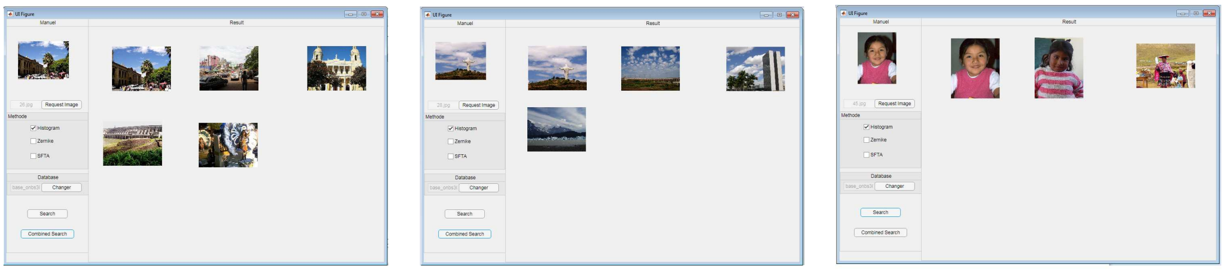
The extraction of three components and the integration of Kalay Smorodinsky need a considerable time. We evaluate automatically the outputs of every system, if the number of Image considered similar is more than five the result is considered correct.

The following figure present examples of request results.

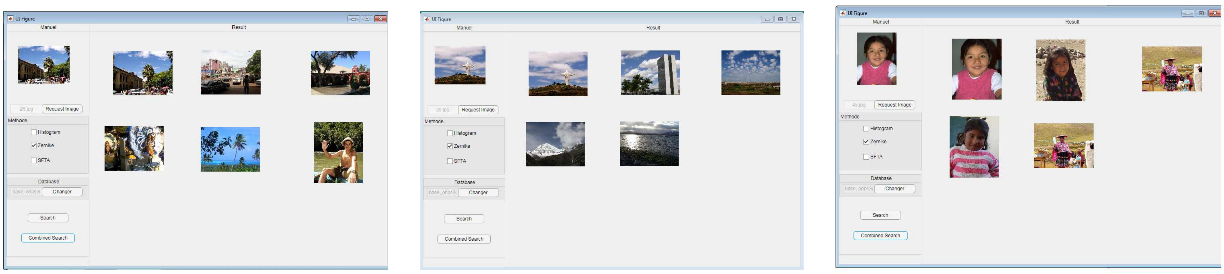
Table 1. Comparative study between the proposed method and classic method.

	Times	20000p
proposed method	$45S$	95%
Moussaid et al [18]	$55S$	92%
KNN(K=30)+Zernike	$100S$	50%
KNN(K=20)+SFTA	$90S$	60%
KNN(K=35)+color	$120S$	78%

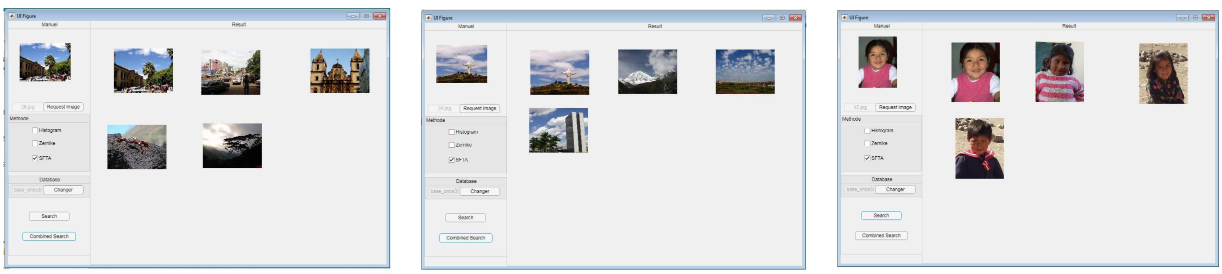
5.1.1. Search Images similar using color



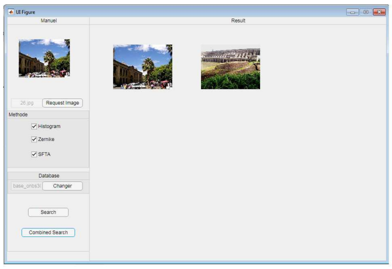
5.1.2. Search Images similar using Zernike

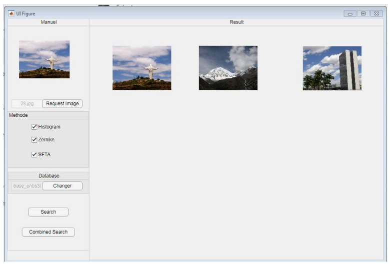


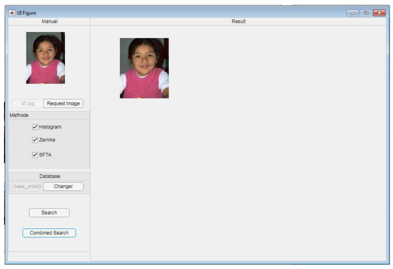
5.1.3. Search Images similar using SFTA



5.1.4. Search Images similar using Method proposed







Experimental results show that our method achieves favorable performance against other methods.

6. Conclusion

In this paper a new approach is proposed to solve a multi-criteria optimization problem using a game theory: a new approach for the splitting the territory in the case of concurrent optimization.

The first algorithm, permits to compute the Nash equilibria which is at the Pareto front. While the second one determines the Nash equilibrium as a Kalai–Smorodinsky solution. The numerical examples confirm that our algorithms have powerful ability to find the Nash equilibrium.

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Пошук зображень за допомогою рівноваги Неша та розв'язку Калаі–Смородинського

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У статті запропоновано нове формулювання ігор Неша для розв'язання загальних багатоцільових задач оптимізації. Мета цього підходу — розділити змінні оптимізації, що дозволить чисельно визначати стратегії між двома гравцями. Перший гравець мінімізує вартість своєї функції, використовуючи змінні першої таблиці P, а другий гравець — з другої таблиці Q. Оригінальність цієї роботи полягає, по-перше, в системі побудови двох таблиць розподілу, які приводять до рівноваги Неша на фронті Парето. По-друге, знайдено розв'язок рівноваги Неша, який співпадає з розв'язком Калаі–Смородинського. Для цього запропоновано та успішно випробувано два алгоритми, які обчислюють P, Q та пов'язану з ними рівновагу Неша, використовуючи деяке розширення підходу нормального перетину границь. Після цього запропоновано, щоб пошукова система шукала подібні зображення до заданого зображення на основі декількох представлень зображень з використанням функцій кольору, текстури та форми.

Ключові слова: *рівновага Неша, розв'язок Калаі–Смородинського, нечітка кластеризація, паралельна оптимізація, дескриптори кольору, Gist та SIFT дескриптори.*