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Hemivariational inverse problem for contact problem with locking materials

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The aim of this work is to study an inverse problem for a frictional contact model for locking material. The deformable body consists of electro-elastic-locking materials. Here, the locking character makes the solution belong to a convex set, the contact is presented in the form of multivalued normal compliance, and frictions are described with a sub-gradient of a locally Lipschitz mapping. We develop the variational formulation of the model by combining two hemivariational inequalities in a linked system. The existence and uniqueness of the solution are demonstrated utilizing recent conclusions from hemivariational inequalities theory and a fixed point argument. Finally, we provided a continuous dependence result and then we established the existence of a solution to an inverse problem for piezoelectric-locking material frictional contact problem.

Keywords: *locking piezoelectric material, frictional contact problem, inverse problem, hemivariational inequality.*

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1. Introduction

Recently, the theory of variational and hemivariational inequalities has become more attractive in the mathematical and physics domain. This type of inequality was first introduced by Panagiotopolous in 1980 to generalize variational inequalities for non-convex and non-monotone operators see [1]. Based on the generalized gradient of Clarke [2], it is used to study engineering problems involving non-smooth, non-monotone, and multivalued functionals, e.g. in the variational formulation of mechanical problems whenever nonconvex energy functionals (non-smooth constitutive laws) are involved [3, 4]. However, more of their mathematical and applied developments can be found in [5–7] and the reference therein. In the last years, inverse problems have grown in popularity as a subject of applied mathematics with numerous practical applications [8–12]. The goal of this research is to look into the inverse problem of identifying parameters in a hemivariational inequality. On another side, the theory of locking materials was firstly discussed by Prager [13,14]. We consider elastic ideally locking materials, as defined in [15]. Then, we shall deal with piezoelectric materials for which the constitutive laws are given as follows

$$\sigma \in \mathcal{E}(l, \varepsilon(u)) - \mathcal{B}^T(l, E(\varphi)) + \partial I_B(l, \varepsilon(u)) \quad \text{in} \quad \Omega, \tag{1}$$

$$D \in \mathcal{B}(l, \varepsilon(u)) + \beta(l, E(\varphi)) + \partial I_C(l, E(\varphi)) \quad \text{in} \quad \Omega, \tag{2}$$

where $\partial I_B \colon \mathcal{L} \times \mathbb{S}^d \longrightarrow 2^{\mathbb{S}^d}$ and $\partial I_C \colon \mathcal{L} \times L^2(\Omega) \longrightarrow 2^{L^2(\Omega)}$ stands for the subdifferential, respectively, of the indicators functions of sets B and C, given by

$$I_{\mathcal{B}}(l, \varepsilon) = \begin{cases} 0 & \text{if} \quad \varepsilon \in B, \\ +\infty & \text{if} \quad \varepsilon \notin B, \end{cases} \qquad I_C(l, \psi) = \begin{cases} 0 & \text{if} \quad \psi \in C, \\ +\infty & \text{if} \quad \psi \notin C. \end{cases}$$

The physics point of view of locking materials can be found in [16]. The sets $B \subset \mathbb{S}^d$ and $C \subset L^2(\Omega)$ design the locking constraints and define the properties of the materials. Moreover, these sets have many forms, see [16]. In this paper, we discuss the perfectly locking materials forms, for which the sets B and C are given by

$$B = \{\varepsilon \in \mathbb{S}^d \colon Q_1(\varepsilon) \leqslant 0\}, \quad C = \{\psi \in L^2(\Omega) \colon Q_2(\psi) \leqslant 0\}, \tag{3}$$

where the locking functions $Q_1 \colon \mathbb{S}^d \longrightarrow \mathbb{R}$ and $Q_2 \colon L^2(\Omega) \longrightarrow \mathbb{R}$ are convexes continuous functions verifying the condition $Q_i(0) \leqslant 0$ for $i = 1, 2$. To study these problems, we consider the following abstract variational-hemivariational inequality, which has been discussed in [17].

Problem (P). *Given $l \in L$, find $u = u(l) \in K$ such that*

$$\langle A(l,u) - f(l), v - u\rangle_X + j^0(l,u,u;v-u) \geqslant 0, \quad \forall v \in K. \tag{4}$$

where $A \colon L \times X \to X^*$ is an operator from a Banach space X to its dual X^*, $f \colon L \to X^*$ and $J \colon L \times X \times X \to \mathbb{R}$ are two real valued functions, K is a subset of X and $\langle \cdot, \cdot \rangle$ denotes the duality pairing of X and X^*. For $z \in X$ fixed, the notation $J^0(l,z,u;v)$ represents the generalized directional derivative of the function $J(l,z,\cdot)$ at $u \in X$ in the direction $v \in X$. For existence and uniqueness of a solution to inverse problems (P) have been studied by [18]. To apply the obtained result on Problem (P), the contact problem with piezoelectric locking materials is considered. The novelty of this paper is study of the existence and uniqueness solution of a static frictional contact problem electro-elastic-locking materials and also proof of Lipschitz continuous dependence of this solution. Furthermore, we study an inverse problem for the frictional electro-elastic contact problem and show that it possesses a solution.

The paper is structured as follows. In Section 2, we introduce the mathematical model of frictional contact for locking materials, for example, we consider a static electroelastic-locking materials contact problem in which the frictional contact with a conductive foundation. There are described the equations and boundary conditions, list the data assumption on the data and derive formulation variational is in a form of a coupled system of two hemi-variational inequalities. Section 3 is devoted to study of the existence and unique solution of this problem. Moreover, we proved Lipschitz continuous dependence of solution of this model and used this dependence result to study the solvability of the inverse problem for piezoelectric-locking material frictional contact problem.

2. Contact problem for piezoelectric-locking material

In this section, there is discussed a static contact problem for a nonlinear electro-elastic and locking material body which is described by unilateral constraints with multi-valued normal compliance function, and non-monotone multi-valued friction condition with slip dependent coefficient. We describe the physical setting of the problem and we provide its classical variational-hemivariational formulation, which is a system of coupled hemi-variational inequalities.

There is considered a piezoelectric-locking material body that occupies the domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ with Lipschitz boundary $\Gamma = \partial\Omega$ and a unit outward normal ν at Γ. Body forces f_0 and volume free electric charges q_0 act on the body. It is also constrained mechanically and electrically on Γ: to describe these constraints, let consider three open and measurable parts Γ_1, Γ_2 and Γ_3 such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 = \Gamma$ and $\operatorname{meas}(\Gamma_1) > 0$, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two measurable parts Γ_a and Γ_b such that $\operatorname{meas}(\Gamma_a) > 0$, on the other hand.

The space of second order symmetric tensors on $\mathbb{R}^d$ is denoted by $\mathbb{S}^d$, while $\cdot$ and $\|\cdot\|$ represent the inner product and the associated Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ given for all $u, v \in \mathbb{R}^d$ and $\sigma, \tau \in \mathbb{S}^d$ by

$$u \cdot v = u_i\, v_i, \quad \|v\| = (v \cdot v)^{1/2} \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij}\, \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{1/2}.$$

The normal and tangential components of the displacement vector $v \in \mathbb{R}^d$ and the stress tensor $\sigma \in \mathbb{S}^d$ on the boundary Γ are given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu \quad \text{and} \quad \sigma_\nu = (\sigma\nu) \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Then, the classical formulation of the frictional electro-elastic-locking material contact problem is as follows.

Problem (P). *Given $l \in \mathcal{L}$, find a displacement $u = u(l) \colon \Omega \longrightarrow \mathbb{R}^d$, an electric potential $\varphi = \varphi(l) \colon \Omega \to \mathbb{R}$ such that*

$$\sigma \in \mathcal{E}(l, \varepsilon(u)) - \mathcal{B}^T(l, E(\varphi)) + \partial I_B(l, \varepsilon(u)) \quad \text{in} \quad \Omega, \tag{5}$$

$$D \in \mathcal{B}(l, \varepsilon(u)) + \beta(l, E(\varphi)) + \partial I_C(l, E(\varphi)) \quad \text{in} \quad \Omega, \tag{6}$$

$$\operatorname{Div} \sigma + f_0(l) = 0 \quad \text{in} \quad \Omega, \tag{7}$$

$$\operatorname{div} D - q_0(l) = 0 \quad \text{in} \quad \Omega, \tag{8}$$

$$u = 0 \quad \text{on} \quad \Gamma_1, \tag{9}$$

$$\sigma\nu = f_2(l) \quad \text{on} \quad \Gamma_2, \tag{10}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{11}$$

$$D \cdot \nu = q_b(l) \quad \text{on} \quad \Gamma_b, \tag{12}$$

$$\left\{ \begin{array}{l} u_\nu \leqslant g_0\,, \ \sigma_\nu + \gamma \leqslant 0\,, \ (\sigma_\nu + \gamma)(u_\nu - g_0) = 0, \\ \gamma \in w_\nu(l, \varphi - \varphi_0)\, \partial j_\nu(l, u_\nu - g_0), \end{array} \right. \quad \text{on} \quad \Gamma_3, \tag{13}$$

$$-\,\sigma_\tau \in w_\tau(l, \varphi - \varphi_0, u_\nu - g_0)\, \mu(\|u_\tau\|)\, \partial j_\tau(l, u_\tau) \quad \text{on} \quad \Gamma_3, \tag{14}$$

$$D \cdot \nu \in w_e(l, u_\nu - g_0)\, \partial j_e(l, \varphi - \varphi_0) \quad \text{on} \quad \Gamma_3. \tag{15}$$

Here, (5), (6) represent the electro-elastic-locking materials constitutive law of the material see [19,20] for more details, where $\mathcal{E} = (\mathcal{E}_{ijkl})$ is the elastic tensor, $\mathcal{B} = (\mathcal{B}_{ijk})$ and $\beta = (\beta_{ij})$ are the piezoelectric and the electric permittivity tensors. In addition, $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ is the linearized strain tensor, $E(\varphi) = -\nabla\varphi$ is the electric field and $\mathcal{B}^T = (\mathcal{B}_{kij})$ is the transpose tensor of $\mathcal{B}$. Equations (7), (8) represent the equilibrium equations for the stress and the electric displacement fields. Moreover, (9)–(12) are the mechanical and electrical boundary conditions, the relation (13) represents the multivalued normal compliance contact condition with unilateral constraints of Signorini type coupled with the electric potential through the stiffness coefficient w_ν which depends on the difference between the electric potential on the body and the electrically conductive foundation and g_0 represents the gap function between the body and the foundation on the contact surface. Condition (14) represents the friction law, the function w_τ models the influence of the electric potential and normal displacement on the frictional contact, and μ denotes a positive function called the coefficient of friction. Finally, relation (15) represents a regularized condition for the electrical contact on Γ_3 in which φ_0 represents the electric potential of the foundation and w_e, j_e are given functions.

We explore the following spaces in order to obtain the variational formulation of Problem (P)

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathcal{H} = \left\{\tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$

which are real Hilbert spaces for the following inner products and their associated norms

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \quad (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx.$$

Let introduce the following variational subspaces

$$\begin{gathered} V = \{v \in H_1, v = 0 \ \text{ on } \Gamma_1\}, \\ W = \{\psi \in H^1(\Omega), \psi = 0 \ \text{ on } \ \Gamma_a\}, \\ K_1 = \{v \in V, v_\nu \leqslant g_0 \ \text{ on } \ \Gamma_3\}. \end{gathered}$$

Over V and W, we look at the inner products and Euclidean norms that go along with them

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad \|u\|_V = (u,u)_V^{1/2}, \tag{16}$$

$$(\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_H, \quad \|\varphi\|_W = (\varphi,\varphi)_W^{1/2}. \tag{17}$$

and sets with locking constraints

$$V_1 = \{v \in V, \varepsilon(v(x)) \in B \ \text{ a.e. } \ x \in \Omega\}, \tag{18}$$

$$V_2 = \{\xi \in W, E(\xi(x)) \in C \ \text{ a.e. } \ x \in \Omega\}. \tag{19}$$

Since V is a closed subspace of H_1 and $\text{meas}(\Gamma_1) > 0$, the Korn's inequality holds and there exists a constant $c_k > 0$ depending on Ω and Γ_1 such that

$$\|v\|_{H_1} \leqslant c_k \|\varepsilon(v)\|_{\mathcal{H}}, \quad \forall v \in V. \tag{20}$$

Hence, the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V and then $(V,\|\cdot\|_V)$ is a real Hilbert space. Furthermore, by Sobolev trace theorem, there exists a constant $c_0 > 0$ depending on Ω, Γ_3 and Γ_1 such that

$$\|v\|_{L^2(\Gamma)^d} \leqslant c_0 \|v\|_V, \quad \forall v \in V. \tag{21}$$

Since $\text{meas}(\Gamma_a) > 0$, the Friedrichs–Poincaré inequality holds and thus

$$\|\psi\|_{H^1(\Omega)} \leqslant c_F \|\nabla\psi\|_H, \quad \forall \psi \in W, \tag{22}$$

where a constant $c_F > 0$ depends only on Ω and Γ_a. It follows from (17) and (22) that the norms $\|\cdot\|_W$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on W, and so $(W,\|\cdot\|_W)$ is a real Hilbert space. In addition, the Sobolev trace theorem implies that there exists $c_1 > 0$ depending on Ω, Γ_a and Γ_3 such that

$$\|\xi\|_{L^2(\Gamma_3)} \leqslant c_1 \|\xi\|_W, \quad \forall \xi \in W. \tag{23}$$

From the first constitutive law (5) of locking piezoelectric materials, one can obtain

$$\sigma = \mathcal{E}(l,\varepsilon(u)) - \mathcal{B}^T(l,E(\varphi)) + \zeta(l,u) \quad \text{where} \quad \zeta(l,u) \in \partial I_B(l,\varepsilon(u)) \ \text{ in } \ \Omega.$$

Hence, for all $u, v \in V_1$, we get $\langle \zeta(l,u), (\varepsilon(v) - \varepsilon(u))\rangle \leqslant I_B(l,\varepsilon(v)) - I_B(l,\varepsilon(u)) \leqslant 0$, and then

$$(\sigma, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} \leqslant (\mathcal{E}(l,\varepsilon(u)) - \mathcal{B}^T(l,E(\phi)), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}}. \tag{24}$$

Also, from the second constitutive law (6) of locking piezoelectric materials, it follows

$$D = \mathcal{B}(l,\varepsilon(u)) + \beta(l,E(\varphi)) + p(l,\varphi) \quad \text{where} \quad p(l,\varphi) \in \partial I_C(l,E(\varphi)) \ \text{ in } \ \Omega.$$

Then, for all φ, $\phi \in V_2$, we have $\langle p(l,\varphi), E(\phi) - E(\varphi)\rangle \leqslant I_C(l,E(\phi)) - I_C(l,E(\varphi)) \leqslant 0$, and thus

$$(D, \nabla\varphi - \nabla\phi)_{L^2(\Omega)} \leqslant (\mathcal{B}(l,\varepsilon(u)) + \beta(l,E(\phi)), \nabla\varphi - \nabla\phi)_{L^2(\Omega)}. \tag{25}$$

The study of Problem (P) requires the following hypotheses.

($\mathcal{A}_1$) The tensor $\mathcal{E}\colon \Omega \times \mathcal{L} \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ is such that
  (i) $\mathcal{E}(\cdot,l,\xi)$ is measurable on Ω for all $l \in \mathcal{L}$ and all $\xi \in \mathbb{S}^d$,
  (ii) $\mathcal{E}(y,l,\cdot)$ is continuous on $\mathbb{S}^d$ for all $l \in \mathcal{L}$ and all $y \in \Omega$,
  ($3i$) there exist $L_{\mathcal{E}} > 0$ such that for all $l_1, l_2 \in \mathcal{L}$, $\xi_1, \xi_2 \in \mathbb{S}^d$ and $y \in \Omega$,

$$\|\mathcal{E}(y,l_1,\xi_1) - \mathcal{E}(y,l_2,\xi_2)\| \leqslant L_{\mathcal{E}}(\|l_1 - l_2\|_{\mathcal{L}} + \|\xi_1 - \xi_2\|), \tag{26}$$

$(4i)$ there exist $\alpha_{\mathcal{E}} > 0$ such that for all $l \in \mathcal{L}$, $\xi_1, \xi_2 \in \mathbb{S}^d$ and $y \in \Omega$,

$$(\mathcal{E}(y,l,\varepsilon_1) - \mathcal{E}(y,l,\varepsilon_2)) \cdot (\xi_1 - \xi_2) \geqslant \alpha_{\mathcal{E}} \|\xi_1 - \xi_2\|^2, \tag{27}$$

$(5i)$ $\mathcal{E}(y,l,0) = 0$ for all $l \in \mathcal{L}$ and $y \in \Omega$.

$(\mathcal{A}_2)$ The tensor of piezoelectric $\mathcal{B} = (\mathcal{B}_{ijk})\colon \Omega \times \mathcal{L} \times \mathbb{S}^d \longrightarrow \mathbb{R}^d$ is such that

(i) $\mathcal{B}_{ijk} \in L^\infty(\Omega)$,

(ii) there exist $L_{\mathcal{B}} > 0$ such that for all $l_1, l_2 \in \mathcal{L}$, $\xi_1, \xi_2 \in \mathbb{S}^d$ and $y \in \Omega$,

$$\|\mathcal{B}(y,l_1,\xi_1) - \mathcal{B}(y,l_2,\xi_2)\| \leqslant L_{\mathcal{B}}(\|l_1 - l_2\|_{\mathcal{L}} + \|\xi_1 - \xi_2\|). \tag{28}$$

$(\mathcal{A}_3)$ The permittivity tensor $\beta = (\beta_{ijk})\colon \Omega \times \mathcal{L} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is such that

(i) $\beta(\cdot,l,\xi)$ is measurable on Ω for all $l \in \mathcal{L}, \xi \in \mathbb{R}^d$,

(ii) $\beta(y,l,\cdot)$ is continuous on $\mathbb{R}^d$ for all $l \in \mathcal{L}, y \in \Omega$,

$(3i)$ there exist $L_\beta > 0$ such that for all $l_1, l_2 \in \mathcal{L}$, $\xi_1, \xi_2 \in \mathbb{R}^d$ and $y \in \Omega$,

$$\|\beta(y,l_1,\xi_1) - \beta(y,l_2,\xi_2)\| \leqslant L_\beta(\|l_1 - l_2\|_{\mathcal{L}} + \|\xi_1 - \xi_2\|), \tag{29}$$

$(4i)$ there exist $\alpha_\beta > 0$ such that for all $l \in \mathcal{L}$, $\xi_1, \xi_2 \in \mathbb{R}^d$ and $y \in \Omega$,

$$(\beta(y,l,\xi_1) - \beta(y,l,\xi_2)) \cdot (\xi_1 - \xi_2) \geqslant \alpha_\beta \|\xi_1 - \xi_2\|^2, \tag{30}$$

$(5i)$ $\beta(y,l,0) = 0$ for all $l \in \mathcal{L}$ and $y \in \Omega$.

$(\mathcal{A}_4)$ The functions $j_\nu\colon \Gamma_3 \times \mathcal{L} \times \mathbb{R} \to \mathbb{R}$, $j_\tau\colon \Gamma_3 \times \mathcal{L} \times \mathbb{R}^d \to \mathbb{R}$ and $j_e\colon \Gamma_3 \times \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ satisfy

$(i)\,(a)$ $j_\nu(\cdot,l,s)$ is measurable on Γ_3 for all $l \in \mathcal{L}$ and $s \in \mathbb{R}$,

(b) $j_\nu(y,l,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for all $l \in \mathcal{L}$ and $y \in \Gamma_3$,

(c) there exist $c_{0\nu}, c_{1\nu}, c_{2\nu} \geqslant 0$ such that for all $l \in \mathcal{L}$, $s \in \mathbb{R}$ and $y \in \Gamma_3$,

$$\|\partial j_\nu(y,l,s)\| \leqslant c_{0\nu} + c_{1\nu}|s| + c_{2\nu}\|l\|_{\mathcal{L}}, \tag{31}$$

(d) there exist positive constants $\alpha_{j\nu}$ and $\beta_{j\nu}$ such that

$$j_\nu^0(y,l_1,s_1;s_2 - s_1) + j_\nu^0(y,l_2,s_2;s_1 - s_2) \leqslant \alpha_{j\nu}|s_1 - s_2|^2 + \beta_{j\nu}\|l_1 - l_2\|_{\mathcal{L}}|s_1 - s_2| \tag{32}$$

for all l_1, $l_2 \in \mathcal{L}$, $s_1, s_2 \in \mathbb{R}$ and $y \in \Gamma_3$.

$(ii)(a)$ $j_\tau(\cdot,l,\xi)$ is measurable on Γ_3 for all $l \in \mathcal{L}$ and $\xi \in \mathbb{R}^d$,

(b) $j_\tau(y,l,\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for all $l \in \mathcal{L}$ and $y \in \Gamma_3$,

(c) there exist $c_{0\tau}, c_{1\tau}, c_{2\tau} \geqslant 0$ such that for all $l \in \mathcal{L}$, $\xi \in \mathbb{R}^d$ and $y \in \Gamma_3$,

$$\|\partial j_\tau(y,l,\xi)\| \leqslant c_{0\tau} + c_{1\tau}\|\xi\|_{\mathbb{R}^d} + c_{2\tau}\|l\|_{\mathcal{L}}, \tag{33}$$

(d) there exist positive constants $\alpha_{j\tau}$ and $\beta_{j\tau}$ such that

$$j_\tau^0(y,l_1,\xi_1;\xi_2 - \xi_1) + j_\tau^0(y,l_2,\xi_2;\xi_1 - \xi_2) \leqslant \alpha_{j\tau}\|\xi_1 - \xi_2\|^2_{\mathbb{R}^d} + \beta_{j\tau}\|l_1 - l_2\|_{\mathcal{L}}\|\xi_1 - \xi_2\|_{\mathbb{R}^d} \tag{34}$$

for all $l_1, l_2 \in \mathcal{L}$, $\xi_1, \xi_2 \in \mathbb{R}^d$ and $y \in \Gamma_3$.

$(3i)(a)$ $j_e(\cdot,l,s)$ is measurable on Γ_3 for all $l \in \mathcal{L}$ and $s \in \mathbb{R}$,

(b) $j_e(y,l,.)$ is locally Lipschitz on $\mathbb{R}$ for all $l \in \mathcal{L}$ and $y \in \Gamma_3$,

(c) there exist $c_{0e}, c_{1e}, c_{2e} \geqslant 0$ such that for all $l \in \mathcal{L}$, $s \in \mathbb{R}$ and $y \in \Gamma_3$, we have

$$\|\partial j_e(y,l,s)\| \leqslant c_{0e} + c_{1e}|s| + c_{2e}\|l\|_{\mathcal{L}}, \tag{35}$$

(d) there exist positive constants α_{je} and β_{je} such that

$$j_e^0(y, l_1, s_1; s_2 - s_1) + j_e^0(y, l_2, s_2; s_1 - s_2) \leqslant \alpha_{je}|s_1 - s_2|^2 + \beta_{je}\|l_1 - l_2\|_{\mathcal{L}}|s_1 - s_2| \tag{36}$$

for all $l_1, l_2 \in \mathcal{L}$, $s_1, s_2 \in \mathbb{R}$ and $y \in \Gamma_3$.

$(\mathcal{A}_5)$ The function $w_\nu : \Gamma_3 \times \mathcal{L} \times \mathbb{R} \to \mathbb{R}$, $w_\tau \colon \Gamma_3 \times \mathcal{L} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $w_e \colon \Gamma_3 \times \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ and $\mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy

$(i)\,(a)$ $w_\nu(\cdot, l, s)$ is measurable on Γ_3 for all $l \in \mathcal{L}$ and $s \in \mathbb{R}$,

(b) $w_\nu(y, l, \cdot)$ is continuous on $\mathbb{R}$ for all $l \in \mathcal{L}$ and $y \in \Gamma_3$,

(c) there exists $\overline{w}_\nu > 0$ such that for all $l \in \mathcal{L}$, $s \in \mathbb{R}$ and $y \in \Gamma_3$, we have

$$0 \leqslant w_\nu(y, l, s) \leqslant \overline{w}_\nu, \tag{37}$$

$(ii)(a)$ $w_\tau(\cdot, l, s_1, s_2)$ is measurable on Γ_3 for all $l \in \mathcal{L}$ and $s_1, s_2 \in \mathbb{R}$,

(b) $w_\tau(y, l, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}$ for all $l \in \mathcal{L}$, $y \in \Gamma_3$,

(c) there exists $\overline{w}_\tau > 0$ such that for all $l \in \mathcal{L}$, $s_1, s_2 \in \mathbb{R}$ and $y \in \Gamma_3$, we have

$$0 \leqslant w_\tau(y, l, s_1, s_2) \leqslant \overline{w}_\tau, \tag{38}$$

$(3i)(a)$ $w_e(., l, s)$ is measurable on Γ_3 for all $l \in \mathcal{L}$ and $s \in \mathbb{R}$,

(b) $w_e(y, l, .)$ is continuous on $\mathbb{R}$ for all $l \in \mathcal{L}$, $y \in \Gamma_3$,

(c) there exists $\overline{w}_e > 0$ for all $l \in \mathcal{L}$, $s \in \mathbb{R}$ and $y \in \Gamma_3$,

$$0 \leqslant w_e(y, l, s) \leqslant \overline{w}_e, \tag{39}$$

$(4i)(a)$ $\mu(\cdot, s)$ is measurable on Γ_3 for all $s \in \mathbb{R}_+$,

(b) there exists $L_\mu > 0$ such that for all $s_1, s_2 \in \mathbb{R}_+$ and $y \in \Gamma_3$, we have

$$\|\mu(y, s_1) - \mu(y, s_2)\| \leqslant L_\mu |s_1 - s_2|, \tag{40}$$

(c) there exists $\mu_0 > 0$ such that for all $s \in \mathbb{R}_+$ and $y \in \Gamma_3$,

$$\mu(y, s) \leqslant \mu_0. \tag{41}$$

$(\mathcal{A}_6)$ The forces, tractions, volume and surface charge densities, gap and foundation's potential satisfy

(i) for all $l \in \mathcal{L}$, the following regularity conditions are true

$$f_0(l) \in L^2(\Omega)^d, \quad f_2(l) \in L^2(\Gamma_2)^d, \quad q_0(l) \in L^2(\Omega), \quad q_b(l) \in L^2(\Gamma_b),$$

(ii) there exists $L_{f_0}, L_{f_2}, L_{q_0}, L_{q_b} > 0$ such that for all l_1, $l_2 \in \mathcal{L}$, we have

$$\begin{aligned}
&\|f_0(l_1) - f_0(l_2)\|_{L^2(\Omega)^d} \leqslant L_{f_0}\|l_1 - l_2\|_{\mathcal{L}}, \\
&\|f_2(l_1) - f_2(l_2)\|_{L^2(\Gamma_2)^d} \leqslant L_{f_2}\|l_1 - l_2\|_{\mathcal{L}}, \\
&\|q_0(l_1) - q_0(l_2)\|_{L^2(\Omega)} \leqslant L_{q_0}\|l_1 - l_2\|_{\mathcal{L}}, \\
&\|q_b(l_1) - q_b(l_2)\|_{L^2(\Gamma_b)} \leqslant L_{q_b}\|l_1 - l_2\|_{\mathcal{L}},
\end{aligned} \tag{42}$$

$(3i)$ the functions g_0 and φ_0 are such that $g_0 \geqslant 0 \in L^2(\Gamma_3)$ and $\varphi_0 \in L^2(\Gamma_3)$.

($\mathcal{A}_7$) B and C are nonempty closed convex subset, resp. of $\mathbb{S}^d$ and $L^2(\Omega)$ with

$$0_{\mathbb{S}^d} \in B, \quad 0_{L^2(\Omega)} \in C.$$

Next, let $l \in \mathcal{L}$, consider two elements $f(l) \in V$ and $\mathbf{q}(l) \in W$ defined by

$$\big(f(l), v\big) = \big(f_0(l), v\big) + \big(f_2(l), v\big) \quad \text{for all} \quad v \in V, \tag{43}$$

$$\big(\mathbf{q}(l), \psi\big) = \big(q_0(l), \psi\big) - \big(q_b(l), \psi\big) \quad \text{for all} \quad \psi \in W. \tag{44}$$

Using standard techniques, one can get the following variational formulation of Problem (P).

Problem (PV). *Given $l \in \mathcal{L}$, find a displacement field $u \in K_1 \cap V_1$ and an electric potential field $\varphi \in V_2$ such that*

$$\big(\mathcal{E}(l, \varepsilon(u)) + \mathcal{B}^T(l, \nabla\varphi), \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + \int_{\Gamma_3} w_\nu(l, \varphi - \varphi_0)\, j_\nu^0(l, u_\nu - g_0; v_\nu - u_\nu)\, da$$
$$+ \int_{\Gamma_3} w_\tau(l, \varphi - \varphi_0, u_\nu - g_0)\, \mu(\|u_\tau\|)\, j_\tau^0(l, u_\tau; v_\tau - u_\tau)\, da \geqslant \big(f(l), v - u\big)_V, \quad \forall\, v \in K_1 \cap V_1, \tag{45}$$

$$\big(\beta(l, \nabla\varphi) - \mathcal{B}(l, \varepsilon(u)), \nabla(\psi - \varphi)\big)_H + \int_{\Gamma_3} w_e(l, u_\nu - g_0)\, j_e^0(l, \varphi - \varphi_0; \psi - \varphi)\, da$$
$$\geqslant \big(\mathbf{q}(l), \psi - \varphi\big)_W, \quad \forall\, \psi \in V_2. \tag{46}$$

The previous Problem can be reformulated as follows.

Problem (PV). *Given $l \in \mathcal{L}$, find $(u, \varphi) \in (K_1 \cap V_1) \times V_2$*

$$\begin{aligned}
&\big(\mathcal{E}(l, \varepsilon(u)) + \mathcal{B}^T(l, \nabla\varphi), \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + \big(\beta(l, \nabla\varphi) - \mathcal{B}(l, \varepsilon(u)), \nabla(\psi - \varphi)\big)_H \\
&+ \int_{\Gamma_3} \big[w_\nu(l, \varphi - \varphi_0)\, j_\nu^0(l, u_\nu - g_0; v_\nu - u_\nu) + w_e(l, u_\nu - g_0)\, j_e^0(l, \varphi - \varphi_0; \psi - \varphi)\big]\, da \\
&+ \int_{\Gamma_3} w_\tau(l, \varphi - \varphi_0, u_\nu - g_0)\, \mu(\|u_\tau\|)\, j_\tau^0(l, u_\tau; v_\tau - u_\tau)\, da \\
&\geqslant \big(f(l), v - u\big)_V + \big(\mathbf{q}(l), \psi - \varphi\big)_W, \quad \forall\, (v, \psi) \in (K_1 \cap V_1) \times V_2.
\end{aligned} \tag{47}$$

Now, consider the real Hilbert product space $Y = V \times W$ endowed by the usual inner product

$$\big(y, k\big)_Y = \big(u, v\big)_V + \big(\varphi, \psi\big)_W \ \text{ for all } \ y = (u, \varphi),\ k = (v, \psi) \in Y, \tag{48}$$

Consider a nonempty closed convex $U = (K_1 \cap V_1) \times V_2$ of Y and the operator $\mathrm{A}\colon \mathcal{L} \times Y \longrightarrow Y^*$ defined by

$$\langle \mathrm{A}(l, y), k\rangle_Y = \big(\mathcal{E}(l, \varepsilon(u)) + \mathcal{B}^T(l, \nabla\varphi), \varepsilon(v)\big)_{\mathcal{H}} + \big(\beta(l, \nabla\varphi) - \mathcal{B}(l, \varepsilon(u)), \nabla\psi\big)_H, \tag{49}$$

for all $y = (u, \varphi)$, $k = (v, \psi) \in Y$ and $l \in \mathcal{L}$, the functional $\mathsf{J} : \mathcal{L} \times U \times Y \longrightarrow \mathbb{R}$ given by

$$\begin{aligned}
\mathsf{J}(l, k, y) = &\int_{\Gamma_3} w_\nu(l, \varphi - \varphi_0)\, j_\nu(l, u_\nu - g_0)\, da + \int_{\Gamma_3} w_e(l, u_\nu - g_0)\, j_e(l, \varphi - \varphi_0)\, da \\
&+ \int_{\Gamma_3} w_\tau(l, \varphi - \varphi_0, v_\nu - g_0)\, \mu(\|v_\tau\|)\, j_\tau(l, u_\tau)\, da
\end{aligned} \tag{50}$$

for all $y = (u, \varphi)$, $k = (v, \psi) \in Y$ and $l \in \mathcal{L}$, and the element $f_q(l) \in Y^*$ given by

$$\langle f_q(l), k\rangle_Y = \big(f(l), v\big)_V + \big(\mathbf{q}(l), \psi\big)_W, \quad \forall\, k = (v, \psi) \in Y,\ l \in \mathcal{L}. \tag{51}$$

Let state the following problem using the preceding notations.

Problem $(\mathcal{Q}V)$. Given $l \in \mathcal{L}$, find $y \in U$ such that

$$\langle \mathrm{A}(l, y) - f_q(l), k - y\rangle_Y + \mathsf{J}^0(l, y, y; k - y) \geqslant 0, \quad \forall\, k \in U. \tag{52}$$

As result, the solution of Problem $(\mathcal{Q}V)$ is a solution of Problem (PV).

The analysis of Problem $(\mathcal{Q}V)$, including its unique solvability is based on the abstract result on hemi-variational inequality which has been discussed in [17]. Then we study the inverse problem for the contact problem and deliver a result and its solvability.

3. Analysis of Problem (PV)

Moreover, for the problem (PV), we obtain the existence and uniqueness result.

Theorem 1. *Assume hypotheses $(\mathcal{A}_1)$-$(\mathcal{A}_7)$ and the following smallness condition are satisfied*

$$\max\left\{\overline{w}_\nu \alpha_{j\nu} c_0^2 + \overline{w}_\tau \mu_0 \alpha_{j\tau} c_0^2, \overline{w}_e \alpha_{je} c_1^2\right\} \leqslant \min(\alpha_{\mathcal{E}}, \alpha_\beta) \tag{53}$$

Then, for all $l \in \mathcal{L}$, the problem (PV) has a unique solution $y(l) = (u(l), \varphi(l)) \in U$. Moreover, for all $l_1, l_2 \in \mathcal{L}$, there exists a constant $c > 0$ such that

$$\|u(l_1) - u(l_2)\|_V + \|\varphi(l_1) - \varphi(l_2)\|_W \leqslant c\,\|l_1 - l_2\|_{\mathcal{L}}. \tag{54}$$

where $(u(l_i), \varphi(l_i))$ is the unique solution of Problem (PV) corresponding to $l_i \in \mathcal{L}$ with $i = 1, 2$.

Proof. The proof is based on the Banach fixed point arguments and some results for hemi-variational inequality. By the definition of U it is clear that U is a nonempty, closed and convex subset of Y. Moreover, from the definitions (43), (44) and (51) of f, $\mathbf{q}$ we get $f_q(l) \in Y^*$ for all $l \in \mathcal{L}$.

Lemma 1. *Under the assumptions $(\mathcal{A}_1) - (\mathcal{A}_3)$. The operator* A *defined by* (49) *satisfies the properties*

(i) *for all $l \in \mathcal{L}$, the mapping $\mathrm{A}(l, \cdot)$ is a pseudo-monotonous one,*

(ii) *there exist $\alpha_{\mathrm{A}} > 0$ such that for all $l \in \mathcal{L}$ and u_1, $u_2 \in Y$, it yields*

$$\langle A(l, u_1) - A(l, u_2), u_1 - u_2\rangle_X \geqslant \alpha_A \|u_1 - u_2\|_Y^2. \tag{55}$$

Proof. First, it follows from $(\mathcal{A}_1)(3i)$, $(\mathcal{A}_1)(5i)$, $(\mathcal{A}_2)(ii)$, $(\mathcal{A}_3)(3i)$ and $(\mathcal{A}_3)(5i)$ that for all $l \in \mathcal{L}$, the operator $\mathrm{A}(l, \cdot)$ is bounded one. Hence, for all $k_1 = (v_1, \varphi_1)$, $k_2 = (v_2, \varphi_2) \in Y$,

$$\begin{aligned}
\langle \mathrm{A}(l, k_1) - \mathrm{A}(l, k_2), k_1 - k_2\rangle_Y &= \langle \mathrm{A}(l, k_1), k_1 - k_2\rangle_Y - \langle \mathrm{A}(l, k_2), k_1 - k_2\rangle_Y \\
&= \big(\mathcal{E}(l, \varepsilon(v_1)) + \mathcal{B}^T(l, \nabla\varphi_1), \varepsilon(v_1) - (v_2)\big)_{\mathcal{H}} + \big(\beta(l, \nabla\varphi_1) - \mathcal{B}(l, \varepsilon(v_1)), \nabla\varphi_1 - \nabla\varphi_2\big)_H \\
&\quad - \big(\mathcal{E}(l, \varepsilon(v_2)) - \mathcal{B}^T(l, \nabla\varphi_2), \varepsilon(v_1) - (v_2)\big)_{\mathcal{H}} - \big(\beta(l, \nabla\varphi_2) + \mathcal{B}(l, \varepsilon(v_2)), \nabla\varphi_1 - \nabla\varphi_2\big)_H \\
&= \big(\mathcal{E}(l, \varepsilon(v_1)) - \mathcal{E}(l, \varepsilon(v_2)), \varepsilon(v_1) - (v_2)\big)_{\mathcal{H}} + \big(\beta(l, \nabla\varphi_1) - \beta(l, \nabla\varphi_2), \nabla\varphi_1 - \nabla\varphi_2\big)_H,
\end{aligned}$$

Thus by assumptions $(\mathcal{A}_1)(4i)$ and $(\mathcal{A}_3)(4i)$,

$$\langle \mathrm{A}(l, k_1) - \mathrm{A}(l, k_2), k_1 - k_2\rangle_Y \geqslant \alpha_{\mathcal{E}} \|k_1 - k_2\|_V^2 + \alpha_\beta \|\varphi_1 - \varphi_2\|_W^2, \tag{56}$$

which implies the inequality (5) with $\alpha_{\mathrm{A}} = \min(\alpha_{\mathcal{E}}, \alpha_\beta)$. In addition, since $\mathrm{A}(l, \cdot)$ is bounded, monotonous and hemi-continuous operator, for all $l \in \mathcal{L}$, it is also pseudo-monotonous one. ■

Lemma 2. *Under the assumptions $\mathcal{A}_4$ and $\mathcal{A}_5$, the function* J *defined by* (50) *satisfies the properties*

(i) *for all $l \in \mathcal{L}$, $z \in Y$, the function $\mathsf{J}(l, z, \cdot)$ is a locally Lipschitz on Y,*

(ii) *there exist positive constants a_0, a_1, a_2 and a_3 such that for all $l \in \mathcal{L}$, one has*

$$\|\partial \mathsf{J}(l, z, u)\|_{Y^*} \leqslant a_0 + a_1\|z\|_Y + a_2\|u\|_Y + a_3\|l\|_{\mathcal{L}}, \quad \textit{for all} \quad u, z \in Y, \tag{57}$$

(*iii*) *there exist* $\alpha_{\mathsf{J}} > 0$, $\beta_{\mathsf{J}} \geqslant 0$ *and* $\gamma_{\mathsf{J}} \geqslant 0$ *such that for all* $l_1, l_2 \in \mathcal{L}$, *one has*

$$\begin{aligned} &\mathsf{J}^0(l_1, z_1, u_1; u_2 - u_1) + \mathsf{J}^0(l_2, z_2, u_2; u_1 - u_2) \leqslant \alpha_{\mathsf{J}} \|u_1 - u_2\|_Y^2 + \beta_{\mathsf{J}} \|l_1 - l_2\|_{\mathcal{L}} \|u_1 - u_2\|_Y \\ &\quad + \gamma_{\mathsf{J}} \|z_1 - z_2\|_Y \|u_1 - u_2\|_Y, \quad \text{for all} \quad u_1, u_2, z_1, z_2 \in Y. \end{aligned} \tag{58}$$

Proof. Consider the following functions

$$j_1 \colon \mathcal{L} \times \mathbb{R}^2 \to \mathbb{R}, \qquad j_1(l, s_1, s_2) = w_\nu(l, s_1)\, j_\nu(l, s_2), \tag{59}$$

$$j_2 \colon \mathcal{L} \times \mathbb{R}^3 \times \mathbb{R}^d \to \mathbb{R}, \qquad j_2(l, s_1, s_2, s_3, \xi) = w_\tau(l, s_1, s_2)\, \mu(\|s_3\|)\, j_\tau(l, \xi), \tag{60}$$

$$j_3 \colon \mathcal{L} \times \mathbb{R}^2 \to \mathbb{R}, \qquad j_3(l, s_1, s_2) = w_e(l, s_1)\, j_e(l, s_2). \tag{61}$$

Then, let represent $\mathsf{J} = \mathsf{J}_1 + \mathsf{J}_2 + \mathsf{J}_3$ such that

$$\mathsf{J}_1(l, z, y) = \int_{\Gamma_3} j_1(l, \psi - \varphi_0, u_\nu - g_0)\, da, \tag{62}$$

$$\mathsf{J}_2(l, z, y) = \int_{\Gamma_3} j_2(l, \psi - \varphi_0, v_\nu - g_0, v_\tau, u_\tau)\, da, \tag{63}$$

$$\mathsf{J}_3(l, z, y) = \int_{\Gamma_3} j_3(l, v_\nu - g_0, \varphi - \varphi_0)\, da, \tag{64}$$

for all $l \in \mathcal{L}$ and $z = (v, \psi)$, $y = (u, \varphi) \in Y$. First, it is clear that J is well defined and $\mathsf{J}(l, z, \cdot)$ is locally Lipschitz on Y for all $l \in \mathcal{L}$ and $z \in U$. Next, we use $(\mathcal{B}_4)(i)(c)$ and $(\mathcal{B}_5)(i)(c)$ to obtain

$$\begin{aligned} \|\partial \mathsf{J}_1(l, z, y)\|_Y &\leqslant \int_{\Gamma_3} \overline{w}_\nu \big(c_{0\nu} + c_{1\nu} \|u_\nu - g_0\| + c_{2\nu} \|l\|_{\mathcal{L}}\big)\, da \\ &\leqslant \overline{w}_\nu \big\{c_{0\nu} \mathrm{meas}(\Gamma_3) + c_{1\nu} c_0 \|u_\nu\|_V \sqrt{\mathrm{meas}(\Gamma_3)} + c_{1\nu} \|g_0\|_{L^2(\Gamma_3)} \sqrt{\mathrm{meas}(\Gamma_3)} + c_{2\nu} \|l\|_{\mathcal{L}} \mathrm{meas}(\Gamma_3)\big\}. \end{aligned} \tag{65}$$

In a similar way, the assumptions $(\mathcal{A}_4)(ii)(c)$, $(\mathcal{A}_5)(ii)(c)$ and $(\mathcal{A}_5)(4i)(c)$ imply

$$\|\partial \mathsf{J}_2(l, z, y)\|_Y \leqslant \overline{w}_\tau \mu_0 \big\{c_{0\tau} \mathrm{meas}(\Gamma_3) + c_{1\tau} c_0 \|u\|_V \sqrt{\mathrm{meas}(\Gamma_3)} + c_{2\tau} \mathrm{meas}(\Gamma_3) \|l\|_{\mathcal{L}}\big\}, \tag{66}$$

and the assumptions $(\mathcal{A}_4)(3i)(c)$ and $(\mathcal{A}_5)(3i)(c)$ imply

$$\begin{aligned} \|\partial \mathsf{J}_3(l, z, y)\|_Y \leqslant \overline{w}_e \big\{&c_{0e} \mathrm{meas}(\Gamma_3) + c_{1e} c_2 \|\varphi\|_W \sqrt{\mathrm{meas}(\Gamma_3)} \\ &+ \|\varphi_0\|_{L^2(\Gamma_3)} \sqrt{\mathrm{meas}(\Gamma_3)} + c_{2e} \|l\|_{\mathcal{L}} \mathrm{meas}(\Gamma_3)\big\}. \end{aligned} \tag{67}$$

From the previous estimations (65)–(67), One can deduce

$$\|\partial \mathsf{J}(l, z, y)\| \leqslant C_0 + C_1 \|z\|_Y + C_2 \|y\|_Y + C_3 \|l\|_{\mathcal{L}} \quad \text{for all} \quad l \in \mathcal{L} \quad \text{and} \quad (z, y) \in U \times Y, \tag{68}$$

where the constants C_0, C_1, C_2 and C_3 are given by

$$C_0 = \big(\overline{w}_\nu c_{0\nu} + \overline{w}_\tau \mu_0 c_{0\tau} + \overline{w}_e c_{0e}\big) \mathrm{meas}(\Gamma_3) + \big(\overline{w}_\nu c_{1\nu} \|g_0\|_{L^2(\Gamma_3)} + \overline{w}_e \|\varphi_0\|_{L^2(\Gamma_3)}\big) \sqrt{\mathrm{meas}(\Gamma_3)}, \tag{69}$$

$$C_1 = 0, \tag{70}$$

$$C_2 = \big(\overline{w}_\nu c_{1\nu} c_0 + \overline{w}_\tau \mu_0 c_{1\tau} c_0 + \overline{w}_e c_{1e} c_2\big) \sqrt{\mathrm{meas}(\Gamma_3)}, \tag{71}$$

$$C_3 = \big(\overline{w}_\nu c_{2\nu} + \overline{w}_\tau \mu_0 c_{2\tau} + \overline{w}_e c_{2e}\big) \mathrm{meas}(\Gamma_3). \tag{72}$$

Next, using Corollary 4.15 in [21], we get for $l \in \mathcal{L}$ and $z = (v, \psi)$, $y = (u, \varphi)$, $\overline{y} = (\overline{u}, \overline{\varphi}) \in Y$ that

$$\mathsf{J}_1^0(l, z, y, \overline{y}) \leqslant \int_{\Gamma_3} w_\nu(l, \psi - \varphi_0) j_\nu^0(l, u_\nu - g_0; \overline{u}_\nu)\, da, \tag{73}$$

$$\mathsf{J}_2^0(l,z,y,\overline{y}) \leqslant \int_{\Gamma_3} w_\tau(l,\psi-\varphi_0,v_\nu-g_0)\mu(\|v_\tau\|)j_\tau^0(l,u_\tau;\overline{u}_\tau)\,da, \tag{74}$$

$$\mathsf{J}_3^0(l,z,y,\overline{y}) \leqslant \int_{\Gamma_3} w_e(l,v_\nu-g_0)j_e^0(l,\varphi-\varphi_0;\overline{\varphi})\,da. \tag{75}$$

For the functional J_1^0, we use $(\mathcal{A}_4)(i)(d)$ and $(\mathcal{A}_5)(i)(c)$ to find

$$\begin{aligned}\mathsf{J}_1^0(l_1,z_1,y_1;y_2-y_1)&+\mathsf{J}_1^0(l_2,z_2,y_2;y_1-y_2)\\ &\leqslant \int_{\Gamma_3}\overline{w}_\nu\big|j_\nu^0(l_1,u_{1\nu}-g_0;u_{2\nu}-u_{1\nu})+j_\nu^0(l_2,u_{2\nu}-g_0;u_{1\nu}-u_{2\nu})\big|\,da\\ &\leqslant \overline{w}_\nu\alpha_{j\nu}c_0^2\,\|u_1-u_2\|_V^2+\overline{w}_\nu\beta_{j\nu}c_0\,\mathrm{meas}(\Gamma_3)\,\|l_1-l_2\|_{\mathcal{L}}\|u_1-u_2\|_V.\end{aligned} \tag{76}$$

Similarly, for functionals J_2^0 and J_3^0, we conclude

$$\begin{aligned}\mathsf{J}_2^0(l_1,z_1,y_1;y_2-y_1)&+\mathsf{J}_2^0(l_2,z_2,y_2;y_1-y_2)\\ &\leqslant \overline{w}_\tau\mu_0\alpha_{j\tau}c_0^2\,\|u_1-u_2\|_V^2+\overline{w}_\tau\beta_{j\tau}c_0\,\mathrm{meas}(\Gamma_3)\,\|l_1-l_2\|_{\mathcal{L}}\|u_1-u_2\|_V,\end{aligned} \tag{77}$$

$$\begin{aligned}\mathsf{J}_3^0(l_1,z_1,y_1;y_2-y_1)&+\mathsf{J}_3^0(l_2,z_2,y_2;y_1-y_2)\\ &\leqslant \overline{w}_e\alpha_{je}c_1^2\,\|\varphi_1-\varphi_2\|_W^2+\overline{w}_e\beta_{je}c_1\,\mathrm{meas}(\Gamma_3)\,\|l_1-l_2\|_{\mathcal{L}}\|\varphi_1-\varphi_2\|_W.\end{aligned} \tag{78}$$

Consequently from the inequalities (76)–(78) one can obtain

$$\mathsf{J}^0(l_1,z_1,y_1;y_2-y_1)+\mathsf{J}^0(l_2,z_2,y_2;y_1-y_2)\leqslant \alpha_J\|y_1-y_2\|_Y^2+\beta_J\|l_1-l_2\|_{\mathcal{L}}\|y_1-y_2\|_Y, \tag{79}$$

where the constants α_J and β_J are given by

$$\begin{aligned}\alpha_J&=\max\left\{\overline{w}_\nu\alpha_{j\nu}c_0^2+\overline{w}_\tau\mu_0\alpha_{j\tau}c_0^2,\ \overline{w}_e\alpha_{je}c_1^2\right\}\\ \beta_J&=\max\left\{\overline{w}_\tau\beta_{j\tau}c_0 meas(\Gamma_3)+\overline{w}_\tau\beta_{j\tau}c_0\mathrm{meas}(\Gamma_3),\ \overline{w}_e\beta_{je}c_1\mathrm{meas}(\Gamma_3)\right\}\end{aligned}$$

Then, assumption (3.6) holds with the previous constants α_J, β_J and $\gamma_\mathsf{J}=0$. ■

Then, from Theorem 10 in [17] and the smallness conditions (53), one can conclude that for all $l\in\mathcal{L}$, the Problem (PV) has a unique solution $y(l)=(u(l),\varphi(l))\in U$. ■

Now, we derive a second continuous dependence result of the weak solution of problem (P) with respect to the constraints.

Theorem 2. *Assume that the assumptions of theorem 1 then we have*

$$\|y(l_1)-y(l_2)\|\leqslant\frac{L_\mathcal{E}+2\,L_\mathcal{B}+L_\beta+\beta_J+L_{f_0}c_k+L_{f_2}c_1+L_{q_0}c_F+L_{q_b}c_2}{\alpha_A-\alpha_J}\|l_1-l_2\|_\mathcal{L} \tag{80}$$

where $y(l_1)=(u(l_1),\varphi(l_1))$ and $y(l_2)=(u(l_2),\varphi(l_2))$ are the unique solution of Problem (P) corresponding to l_1, l_2, respectively.

Proof. Let $y(l_1),y(l_2)\in K$ be the solution of Problem (QV) corresponding to $l_1,l_2\in\mathcal{L}$, then

$$\langle \mathrm{A}(l_1,u(l_1))-f_q(l_1),z-u(l_1)\rangle_Y+\mathsf{J}^0(l_1,u(l_1),u(l_1);z-u(l_1))\geqslant 0,\quad \text{for all } z\in U, \tag{81}$$

$$\langle \mathrm{A}(l_2,u(l_2))-f_q(l_2),z-u(l_2)\rangle_Y+\mathsf{J}^0(l_1,u(l_2),u(l_2);z-u(l_2))\geqslant 0,\quad \text{for all } z\in U, \tag{82}$$

Taking $z=y(l_2)$ in (81) and $z=y(l_1)$ in (82), then we add the obtained inequalities to find

$$\begin{aligned}&\langle \mathrm{A}(l_1,u(l_1))-\mathrm{A}(l_2,y(l_2)),y(l_1)-y(l_2)\rangle_Y\\ &\leqslant\langle f_q(l_2)-f_q(l_1),y(l_2)-y(l_1)\rangle_Y+\mathsf{J}^0(l_1,u(l_1),u(l_1);u(l_2)-u(l_1))\\ &\quad+\mathsf{J}^0(l_2,y(l_2),y(l_2);y(l_1)-y(l_2)).\end{aligned} \tag{83}$$

As a result, the previous inequality can be stated like this

$$\begin{aligned}
&\langle \mathrm{A}(l_2, y(l_1)) - \mathrm{A}(l_2, y(l_2)), y(l_1) - y(l_2)\rangle_Y \\
&\qquad \leqslant \langle f_q(l_2) - f_q(l_1), y(l_2) - y(l_1)\rangle_Y + \langle \mathrm{A}(l_2, y(l_1)) - \mathrm{A}(l_1, y(l_1)), y(l_1) - y(l_2)\rangle_Y \\
&\qquad\quad + \mathrm{J}^0(l_1, y(l_1), y(l_1); y(l_2) - y(l_1)) + \mathrm{J}^0(l_2, y(l_2), y(l_2); y(l_1) - y(l_2)).
\end{aligned} \tag{84}$$

By $(\mathcal{A}_1)(3i)$, $(\mathcal{A}_2)(ii)$ and $(\mathcal{A}_3)(3i)$, we find, for all $l_1, l_2 \in \mathcal{L}$ and $y = (u, \varphi)$, $z = (v, \psi) \in Y$, that

$$\begin{aligned}
\langle \mathrm{A}(l_1, y) - \mathrm{A}(l_2, y), z\rangle_Y &= \langle \mathrm{A}(l_1, y), z\rangle_Y - \langle \mathrm{A}(l_2, y), z\rangle_Y \\
&= \big(\mathcal{E}(l_1, \varepsilon(u)) + \mathcal{B}^T(l_1, \nabla\varphi), \varepsilon(v)\big)_{\mathcal{H}} + \big(\beta\nabla(l_1, \varphi) - \mathcal{B}(l_1, \varepsilon(u)), \nabla\psi\big)_H \\
&\quad - \big(\mathcal{E}(l_2, \varepsilon(u)) + \mathcal{B}^T(l_2, \nabla\varphi), \varepsilon(v)\big)_{\mathcal{H}} - \big(\beta(l_2, \nabla\varphi) - \mathcal{B}(l_2, \varepsilon(u)), \nabla\psi\big)_H \\
&= \big(\mathcal{E}(l_1, \varepsilon(u)) - \mathcal{E}(l_2, \varepsilon(u)), \varepsilon(v)\big)_{\mathcal{H}} + \big(\beta(l_1, \nabla\varphi) - \beta(l_2, \nabla\varphi), \nabla\psi\big)_H \\
&\quad + \big(\mathcal{B}^T(l_1, \nabla\varphi) - \mathcal{B}^T(l_2, \nabla\varphi), \varepsilon(v)\big)_{\mathcal{H}} - \big(\mathcal{B}(l_1, \varepsilon(u)) - \mathcal{B}(l_2, \varepsilon(u)), \nabla\psi\big)_H \\
&\leqslant L_{\mathcal{A}}\|l_1 - l_2\|_{\mathcal{L}}\|v\|_V + L_\beta\|l_1 - l_2\|_{\mathcal{L}}\|\psi\|_W + L_{\mathcal{B}}\|l_1 - l_2\|_{\mathcal{L}}\big[\|v\|_V + \|\psi\|_W\big] \\
&\leqslant \big(L_{\mathcal{E}} + 2\,L_{\mathcal{B}} + L_\beta\big)\|l_1 - l_2\|_{\mathcal{L}}\|z\|_Y,
\end{aligned} \tag{85}$$

Next, by definitions (43), (44) and (51) of f, q and f_q, and assumption $(\mathcal{A}_6)(ii)$ to have

$$\begin{aligned}
\langle f_q(l_1) - f_q(l_2), z\rangle_Y &= \big(f(l_1), v\big)_V + \big(q(l_1), \psi\big)_W - \big(f(l_2), v\big)_V - \big(q(l_2), \psi\big)_W \\
&= \big(f_0(l_1) - f_0(l_2), v\big)_{L^2(\Omega)^d} + \big(f_2(l_1) - f_2(l_2), v\big)_{L^2(\Gamma_2)^d} \\
&\quad + \big(q_0(l_1) - q_0(l_2), \psi\big)_{L^2(\Omega)} - \big(q_b(l_1) - q_b(l_2), \psi\big)_{L^2(\Gamma_2)}
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
\langle f_q(l_1) - f_q(l_2), z\rangle_Y &\leqslant \|f_0(l_1) - f_0(l_2)\|_{L^2(\Omega)^d}\|v\|_{L^2(\Omega)^d} + \|f_2(l_1) - f_2(l_2)\|_{L^2(\Gamma_2)^d}\|v\|_{L^2(\Gamma_2)^d} \\
&\quad + \|q_0(l_1) - q_0(l_2)\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} - \|q_b(l_1) - q_b(l_1)\|_{L^2(\Gamma_b)}\|\psi\|_{L^2(\Gamma_b)} \\
&\leqslant \big(L_{f_0}c_k\,\|v\|_V + L_{f_2}c_1\,\|v\|_V + L_{q_0}c_F\,\|\psi\|_W + L_{q_b}c_2\,\|\psi\|_W\big)\,\|l_1 - l_2\|_{\mathcal{L}}.
\end{aligned} \tag{86}$$

Remembering $\|v\|_V \leqslant \|z\|_Y$ and $\|\psi\|_W \leqslant \|z\|_Y$,

$$\|f_q(l_1) - f_q(l_2)\|_{Y^*} \leqslant \big(L_{f_0}c_k + L_{f_2}c_1 + L_{q_0}c_F + L_{q_b}c_2\big)\|l_1 - l_2\|_{\mathcal{L}}. \tag{87}$$

Therefore, it follows from (55), (58) and (84)–(86) with the fact that $\alpha_A - \alpha_J - \gamma_J > 0$ then Theorem 2 holds. ■

It also demonstrates that the contact Problem (P) has a weak solution depending continuously on data. Theorem 2 can be applied to several optimization situations involving inequality (52). Now, we consider an inverse problem for the frictional electro-elastic-locking materials contact Problem (P). Let $\mathcal{L}_{ad} \subset \mathcal{L}$ be an admissible subset of parameters and $F\colon \mathcal{L} \times K_1 \cap V_1 \times V_2 \longrightarrow \mathbb{R}$ be a cost function. Consider the following minimization problem

$$\text{Find } \ l^* \in \mathcal{L}_{ad} \ \text{ such that } \ F(l^*, u(l^*), \varphi(l^*)) = \min_{l \in \mathcal{L}_{ad}} F(l, u(l), \varphi(l)), \tag{88}$$

where $y(l) = (u(l), \varphi(l)) \in K \times W$ is the unique solution of Problem (PV) corresponding to a parameter l, we have the following corollary. In the study of this problem we assume that

$$\mathcal{L}_{ad} \ \text{ is a compact of } \ \mathcal{L}. \tag{89}$$

$$F\colon \mathcal{L} \times K_1 \cap V_1 \times V_2 \longrightarrow \mathbb{R} \quad \text{is a lower semi-continuous function.} \tag{90}$$

Corollary 1. *Assume the hypothesis of Theorem 1,* (89) *and* (90) *hold. Then, Problem* (88) *has at least one solution.*

Various examples and interpretations of cost functionals F that satisfy the previous corollary's hypothesis can be found in [18, 22].

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Гемiварiацiйна обернена задача для контактної задачi зi запiрними матерiалами

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Метою цiєї роботи є дослiдження оберненої задачi для моделi фрикцiйного контакту запiрного матерiалу. Деформiвне тiло складається з електроеластичних запiрних матерiалiв. Характер запирання робить розв'язок належним до опуклої множини, контакт подається у виглядi багатозначної нормальної вiдповiдностi, а тертя описуються субградiєнтом локального вiдображення Лiпшица. Розроблено варiацiйне формулювання моделi, поєднуючи двi гемiварiацiйнi нерiвностi у пов'язану систему. Iснування та єдинiсть розв'язку демонструються на основi нещодавнiх висновкiв теорiї гемiварiацiйних нерiвностей та аргументу з фiксованою точкою. Далi подано результат неперервної залежностi, а потiм встановлено iснування розв'язку оберненої задачi для задачi тертя контакту з п'єзоелектричним запiрним матерiалом.

Ключовi слова: *запiрний п'єзоелектричний матерiал, задача про фрикцiйний контакт, обернена задача, гемiварiацiйнi нерiвностi.*