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NEW REGULARITIES OF SEGMENT DIVISION ACCORDING TO THE GOLDEN RATIO

The paper investigates four problems on the dividing a unit segment by the "golden" proportion. Namely, the general model of the unit segment "golden" division, the decomposition of a square trinomial, the "golden" division of a unit segment by a point with coordinate $x \langle \frac{1}{2}$, the "golden" division of a unit segment with loss of "memory".

In this article, the concept of decomposition is used as elevation to the degree of a quadratic trinomial. The binary division of a unit segment into two unequal parts with the properties of the "golden" proportion is realized at an arbitrary point in the phase plane $0pq$, and the decomposition of a square trinomial leads to the formation of recurrent sequences with Fibonacci properties. It can be noted that the well-known "golden" ratio between the parts of the binary division is most likely a partial imitation of the theorems of Viet and Poincaré. The rules of the "golden" division for the case $x \rangle \frac{1}{2}$ are well studied. Therefore, the regularities for the case $x \langle 1/2$ were researched. Despite the fact that the numbers ψ, φ are expressed through each other, from the point of view of the "golden" division, both realizations with quantitative characteristics $Y_\varphi\big|_{L=1} = \varphi$ and $X_\psi\big|_{L=1} = \psi$ are independent and equal, although their quantitative characteristics can be related to each other with the appropriate formulas. Geometric progressions were constructed for numbers φ and ψ for whole positive values $n \geq 0$ of the exponent to confirm the independence and equality of both models. Quantitative characteristics of the "golden" division of a unit segment by two points with coordinates in intervals $x \rangle 1/2$ and $x \langle 1/2$ interconnected by a nonlinear relation of parabolic type $\psi = \varphi^2$. In the classical "golden" section theory, it is assumed that after distribution, the parts of the segment do not change their spatial directions, and they coincide with the direction of the original segment, i.e. $\alpha = 0$. In this article the case $\alpha \neq 0$ was studied when, after the distribution, the spatial orientation of the distribution elements changes. The angular dependence of the "golden" division of a unit segment with the loss of "memory" of its parts on the spatial orientation after division, shows a known angle $\alpha\big|_{p \to 0} \to \frac{\pi}{3}$ of inclination on the lateral surface of the Hyops.

Key words: "golden" division, Fibonacci numbers, Vieta's theorem, recurrent sequences.

Introduction

The idea of the "golden" division, which is the oldest scientific paradigm, dates back to the teachings of Pythagoras. [1-9]. Studies of ratios and reflections of the "golden" section are performed in many fields of science. Separately the study of the "golden" proportion between unequal parts x and $1-x$

as a result of unit segment division [1, 3]. Discrete Fibonacci models are also relevant for use in a wide range of problems, including those related to binary structuring of systems [10, 11]. Binary system structuring in the context of segment division can be implemented in two ways: a point with a coordinate within $x \rangle 1/2$ or in $x \langle 1/2$. The regularities of the "golden" proportion by the first method are well studied [1-3, 10]. The regularities of binary structuring for the case $x \langle 1/2$ are insufficiently studied.

A classic example of the "golden" division theory application is the case of determining and studying the roots of a quadratic trinomial. At the phase point with coordinates $p = \pm 1, q = 1$, the solutions of the quadratic trinomial equation are well-known [1, 10, 11]. These are the quantitative characteristics of the "golden" proportion of a unit segment by a point with coordinate $x \rangle \frac{1}{2}$. On the coordinate plane Opq, there are directions along which the discriminant D of the quadratic trinomial solution is an exact quadrant, and the roots are equal to integers a_n. The sequence $\{a_K\}$ of such numbers is determined by the recurrence relation of the first order $a_{K+1} = a_K + 1, \quad K \geq 1$.

On the phase plane Opq, the regularities of the "golden" proportions have been studied sufficiently completely for the point $p = \pm 1, q = \pm 1$ and the phase direction $|p| = |q| = k$. Along this phase direction, the "golden" ratio has a well-known form [11, 12]. There is a scientific interest in decomposing a quadratic trinomial in the context of raising it to a power and studying the ratios of quadratic trinomial solutions. This trinomial is obtained from the general model of the "golden" ratio of division of a unit segment.

A special case of the theory of the "golden" section is the case of dividing a unit segment with thr memory "loss" of. It is known [10-13], that in the classical theory of the "golden" section, it is assumed that after division, the parts of the segment do not change their spatial directions and they coincide with the direction of the original segment, i.e $\alpha = 0$. The partial case, when after separation, the spatial orientation of the parts of the separation changes, is insufficiently studied.

The object of research is the methodology of the segment “golden” division. The subject of investigation is new regularities, rejected as a result of the division of a unit segment by the method of "golden" division.

The purpose of this article is to determine new properties of the "golden" section. The tasks of this article are to study the regularities of binary structuring for the case of $x \langle 1/2$, to build a general model of the "golden" division of a unit segment and to decompose a square trinomial.

Analysis of literary sources

The "golden" division theory and the corresponding mathematical apparatus is well-known and has found its application in various fields of science, technology, arts and reflection in nature. For example, the source [7, 11, 13] presents the main ratios of the methodology of the "golden" section, its proportions. The application of the "golden" ratio is shown not only in exact sciences, but also in living nature. A large number of objects in the living world, from viruses to humans, have basic body or its parts proportions very close to the golden ratio. The dependence of 0.618 or 1.618 is characteristic only for biological beings and some types of crystals, inanimate objects have the golden ratio geometry extremely rarely. The "Golden" proportions in the body structure are the most optimal for the survival of real biological objects. Today, the "golden" proportions are found in the structure of animal bodies, shells and shells of molluscs, the proportions of leaves, branches, trunks and root systems in a fairly large number of shrubs and herbs. Usually [13] the device of the Astraea Heliotropium shell, one of the marine mollusks, is given as an example. The carapace is a coiled spiral calcite shell with a geometry that is almost identical to the proportions of the golden ratio. Similarly to the dimensions of the classic logarithmic spiral.

The source [14] considered the possibility of applying the "golden" division methodology to construct a set of numbers with ratios of the golden proportion. The source [15] showed the possibility of using the "golden" ratio for modeling electrodynamic systems by the method of binary separation of the additive parameter. Transient processes in the oscillator were investigated in the source [16-21]. Special attention of research was focused on oscillations of damping character. As a result of research, it was determined that the damping oscillations of the oscillator have patterns of the golden ratio. As a result of research, it was determined that the damping oscillations of the oscillator have regularity of the golden proportion.

All the examples given, as well as many other works, prove great scientific interest in the theory of the "golden" division. Further studies of the problems of division of a unit segment by the "golden" ratio in the context of the phase plane and studies of the roots of the quadratic equation $x^2 - px - q = 0$ will be relevant and topical.

Research results and their discussion

Before beginning the presentation of the main results, let us make some remarks of a conceptual nature [22, 23]. Consider the equality of a nonlinear monomial $y = x^2$ (parabola equation) and a linear binomial $y = px + q$ (line equation) in the form of a square trinomial:

$$x^2 = px + q \,, \tag{1}$$

where: p, q – quadratic trinomial coefficients, x – coordinate.

On the plane $X0Y$ of cartesian coordinates, equation (1) means that the line $y = px + q$ with the slope p intersects the parabola $y = x^2$ at one or two points where the roots $x_\pm$ are real. In another case, if the roots are absent – complex (2):

$$x_+\big|_{p=\pm 1, q=1} \cong \mp 0.62 = \varphi, \qquad x_-\big|_{p=\pm 1, q=1} \cong 1 \pm .62 = \frac{1}{\varphi} \,, \tag{2}$$

where: φ – "golden" number [1 ,3, 22].

Roots (2) -"golden" numbers - known [1, 3, 22] as quantitative characteristics of the "golden" division of a unit segment by a point with coordinate $x \rangle \frac{1}{2}$.

Single segment "golden" division, general model and square trinomial decomposition. On the coordinate plane Opq, there are directions along which the discriminant D is an exact quadrant, and the roots are equal to integers a_n, the sequence $\{a_K\}$ of which is determined by the first-order recurrence relation $a_{K+1} = a_K + 1, \quad K \geq 1$. These directions are known [23, 24] as root lines, which are described by the equations:

$$-Kp + q \pm K^2 = 0 \,. \tag{3}$$

Root straight lines start at points on the x-axis with coordinates

$$K_\pm = -\frac{p}{2} \pm \sqrt{D} \Rightarrow p = 0 : \begin{cases} K = \pm 1, & \left| q\big|_{p=0} \right| = 1, \\ K = \pm 2, & \left| q\big|_{p=0} \right| = 4, \\ K = \pm 3, & \left| q\big|_{p=0} \right| = 9, \\ \ldots \end{cases} \Rightarrow \begin{cases} +1 \cdot p + q + 1^2 = 0, \\ +2 \cdot p + q + 2^2 = 0 \\ +3 \cdot p + q + 3^2 = 0 \\ \ldots \end{cases} \Rightarrow \begin{cases} q\big|_{p=0, K=+1} = -1, \\ q\big|_{p=0, K=+2} = -4, \\ q\big|_{p=0, K=+3} = -9. \\ \ldots \end{cases} \tag{4}$$

Figure 1 shows the root straight lines of the first order $\pm p + q - 1 = 0$, the slope to the x-axis of which increases with increasing order. In the second and third quadrants, the set of root lines coincides with the set of all tangents to a discriminant parabola, in whose interior the solutions are complex [22].

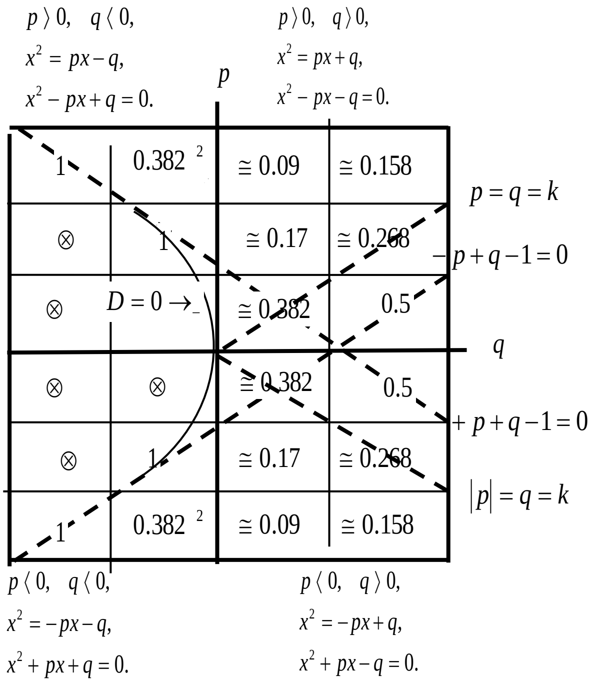


Fig.1. *First order root lines*

In the regions between the root lines, the solutions are irrational.

On the phase plane Opq, the regularities of the "golden" division have been studied quite fully for a point $p = \pm 1, q = \pm 1$ and a phase direction:

$$|p| = |q| = k, \tag{5}$$

along which the "golden" proportion has the form [12, 22]

$$\frac{x(k)}{1} = k\frac{1 - x(k)}{x(k)}. \tag{6}$$

Therefore, we will build a general model of the "golden" proportion of the division of a single segment. For a segment of length L, it is known [23-27] in the form:

$$\frac{x}{L} = \frac{L - x}{x}, \quad x \rangle \frac{L}{2}, \tag{7}$$

where: L – the length of segment.

Let us rewrite (6) for a unit segment in terms of geometric averages

$$\frac{x}{1} = \frac{(L \cdot L) - (L \cdot x)}{x}, \tag{8}$$

and by analogy (8), for the quadratic equation:

$$\xi^2 = q + p\xi, \tag{9}$$

where: $\xi = x$, $q = L^2$, $p = L$,

construct the proportion in the first quadrant:

$$\frac{\xi}{1} = \frac{\left(\sqrt{q} \cdot \sqrt{q}\right) - (p \cdot \xi)}{\xi} \quad \Rightarrow \quad \frac{\xi / \sqrt{q}}{1} = \frac{1 - \left(\frac{p}{q} \cdot \xi\right)}{\xi / \sqrt{q}}. \tag{10}$$

Taking into account condition (5), formula (10) coincides with (6), and the roots of equation (10) satisfy the well-known Vieta theorem:

$$\phi = \frac{\xi}{\sqrt{q}} : \quad \begin{cases} \phi^2 + \frac{p}{\sqrt{q}}\phi - 1 = 0, \\ \phi_+ + \phi_- = \frac{p}{\sqrt{q}}, \quad q \rangle 0. \\ \phi_+ \cdot \phi_- = -1. \end{cases} \tag{11}$$

Let us show that_ solutions (11) with large modulus, inaccordance with the well-known Poincaré theorem, are the limit of the ratios of th eterms of there currents equence. This sequence is formed by the coefficients of the square trinomial decomposition:

$$\phi^n = \alpha_n \phi^1 + \beta_n \cdot \phi^0, n \geq 0, \tag{12}$$

where: α, β - scaling factor.

Concept decomposition is known in mathematical modelingin a broadersense [28]. In thi sarticle, it is used as the exponentiation_of a square trinomial.

The initial conditions of problem (12) are:

$$\begin{cases} \phi_\pm^0 = \alpha_0 \cdot \phi_\pm^1 + 1 \cdot \phi_\pm^0 \Rightarrow \alpha_0 = 0, \beta_0 = 1, \\ \phi_\pm^1 = \alpha_1 \cdot \phi_\pm^1 + 0 \cdot \phi_\pm^0 \Rightarrow \alpha_1 = 1, \beta_1 = 0. \end{cases} \tag{13}$$

In the decomposition problem, conditions (13) remain unchanged, and replacing them with other arbitrary fixed numbers $\alpha_0 = r, \quad \alpha_1 = s$ illegal:

$$\begin{cases} \phi^0 = \alpha_0 \phi^0 + \beta_0 = \beta_0 \neq r \cdot \phi^0 + 1 \quad if \quad r = 0, \\ \phi^1 = \alpha_1 \phi^1 + \beta_1 = \phi^1 \neq s \cdot \phi^1 + 0 \cdot \phi^0 \quad if \quad s = 1 \end{cases}, \tag{14}$$

Let us write the analytical expressions for the first six coefficients:

$$\begin{cases} \begin{cases} \quad \rangle 0, \\ p \rangle 0, \quad q \rangle 0, \end{cases}, \begin{cases} \Lambda = \frac{p^2}{q} + 1, \\ \Omega = \frac{p}{\sqrt{q}}(\Lambda + 1), \end{cases} \\ \phi^0 = 0 \cdot \phi + 1, \\ \phi^1 = 1 \cdot \phi + 0, \\ \phi^2 = -\frac{p}{\sqrt{q}} \cdot \phi + 1, \\ \phi^3 = \Lambda \cdot \phi - \frac{p}{\sqrt{q}}, \\ \phi^4 = -\Omega \cdot \phi + \Lambda, \\ \phi^5 = (\Omega - \Lambda) \cdot \phi - \Omega, \\ \phi^6 = \left[(\Omega - \Lambda)\frac{p}{\sqrt{q}} - \Omega \right] \cdot \phi - (\Omega - \Lambda), \\ \ldots \end{cases} \tag{15}$$

Numerical values of the coefficients α_n, β_n for anarbitrary point with coordinates $p = \sqrt{e}, q = \pi$ presented in the table 1.

Table 1. Numerical values of the coefficients

$\alpha_0, \ \beta_0$	$\alpha_1, \ \beta_1$	$\alpha_2, \ \beta_2$	$\alpha_3, \ \beta_3$	$\alpha_4, \ \beta_4$	$\alpha_5, \ \beta_5$	$\alpha_6, \ \beta_6$
0, 1	1, 0	−0.93, −1	1.865, −0.93	−2.665, 1.865	4.344, −2.665	−6.71, 4.344

We can conclude that the terms of the sequence $\{\beta_n\}$ are equal $\beta_n = \alpha_{n-1}$, and the sequences $\{\alpha_n\}$ are defined by there currence relation (16) and oscillate around zero:

$$\alpha_n = \alpha_2\alpha_{n-1} + \beta_{n-2} = \frac{p}{\sqrt{q}} \cdot \alpha_{n-1} - \cdot\alpha_{n-2}, \quad n \geq 2 . \tag{16}$$

As the term number n increases, the ratio $\frac{\alpha_n}{\alpha_{n-1}}$ oscillates and approaches the solution with a greater absolute value.

$$\lim_{n\to\infty} \frac{\phi_n}{\phi_{n-1}} = \lfloor \phi_\pm \rfloor \quad and \lim_{n\to\infty} \frac{z_n}{z_{n-1}} = \lfloor \phi_\pm \rfloor^2 . \tag{17}$$

Limits (17) for recurrent sequences with constant coefficients form the content of the Poincaré theorem.

Golden divisionof a unit segment by a point with coordinate $x \langle \frac{1}{2}$ **.** Binary structuring of the system in terms of segment division can be implemented in two ways. First, a point with a coordinate in the interval $x \rangle 1/2$ or second, in the interval $x \langle 1/2$. The regularities of the “golden” division are well studied by the first method [1-11, 15, 22]. У [29] запропоновано новий метод визначення золотого січення у контексті застосування у криптосистемах, що покращило їхню швидкодію. Проте недостатньо дослідженим є питання бінарного структурування систем в інтервалі $x \langle 1/2$. Тому дослідимо відповідні закономірності «золотого» січення для випадку $x \langle 1/2$.

In order to distinguish between both cases, we denote the division coordinateas X_ϕ intheinterval $x \langle L/2$, and intheinterval $x \rangle 1/2$ - Y_φ .

Then $\frac{X_\psi}{1} = \psi$, and $\frac{Y_\varphi}{1} = \varphi$. Then the mathematical model of dividing by a point $x \langle 1/2$ can be written as:

$$\frac{1 - X_\psi}{1} = \frac{X_\psi}{1 - X_\psi} . \tag{18}$$

From the (18) there is

$$\psi^2 - 3\psi + 1 = 0 \Rightarrow \begin{cases} \psi_+ = \Psi = +2.618 = 1 + \Phi, \\ \psi_- = \psi = +0.382 = \varphi^2 . \end{cases}, \tag{19}$$

where: $\frac{X_\psi}{1} = \psi$., Φ - Phidias number.

Ingeneral, the numbers $\psi, \Psi, \varphi, \Phi$ are interconnected by theratios:

$$\psi = \varphi^2 \quad and \quad \Psi = p - \varphi^2 = 1 + \Phi . \tag{20}$$

For each pair ψ, Ψ and φ, Φ the connection relationsare satisfied:

$$\varphi = \frac{q}{|\Phi|} = \frac{1}{1.618} \quad or \quad \Phi = \frac{1}{0.618} \text{ And } \psi = \frac{q}{\Psi} = \frac{1}{2.618} \quad or \quad \Psi = \frac{q}{\psi}, \tag{21}$$

Vieta's theorem for (19) :

$$\begin{cases} \varphi + \Phi = -1, \\ \varphi \cdot \Phi = -1, \end{cases} \Rightarrow \frac{1}{\Phi} + \frac{1}{\varphi} = 1 \quad and \quad \begin{cases} \psi + \Psi = 3, \\ \psi \cdot \Psi = 1, \end{cases} \Rightarrow \frac{1}{\Psi} + \frac{1}{\psi} = 3 . \tag{22}$$

Because the divisionis accompanied by a decrease in the initial length of the segment, only solutions ψ, φ have meaning.

Taking into account that the numbers ψ,φ are expressed through one another, from the point of view of the "golden" division, both implementations with quantitative characteristics $Y_\varphi\big|_{L=1}=\varphi$ and $X_\psi\big|_{L=1}=\psi$ are independent and have a equal place to be. Their quantitative characteristics can be related to each other using the formulas:

$$\begin{cases} Y_\varphi - X_\psi = \varphi - \psi = 0.236 = \varphi^2, \\ Y_\varphi \cdot X_\psi = \varphi \cdot \psi = 0.236 = \varphi^2, \end{cases} (a) \quad \Rightarrow Y_\varphi - X_\psi = Y_\varphi \cdot X_\psi, \quad \Rightarrow \frac{1}{X_\psi} - \frac{1}{Y_\varphi} = 1.(b) \tag{23}$$

To confirm the independence and equality of both models, were constructed geometric progressions for numbers φ and ψ for positive integer values of the exponent $n \geq 0$. For number φ it is known [10, 22, 23]:

$$\{\varphi^0, \varphi^1, \varphi^2, \varphi^3, ..., \varphi^{n-1}, \varphi^n, ...\} \tag{24}$$

And for numbers ψ it is known [1]:

$$\{\psi^0 = \varphi^0 \quad, \psi^1 = \varphi^2, \quad ,..., \ \psi^{n-2} = \varphi^n, ...\}. \tag{25}$$

A graphical illustration of (24) and (25) is shown in fig. 2.

As can be seen from (25),the power series for a number ψ starts with the value $\psi = \varphi^2$.

Therefore, both diagrams are shifted one relative to the other so that the series (25) is obtained as a bisection of the classical "golden" progression (24). The density of points ψ^n and φ^n in these diagrams differ twice.

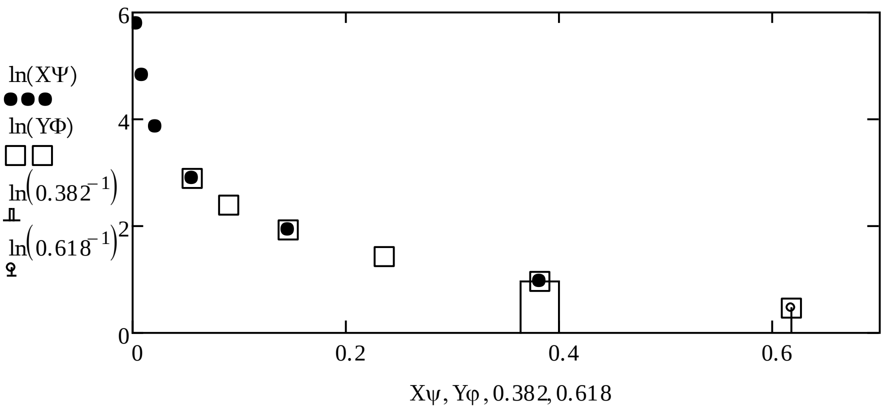


Fig.2. Geometric progressions for numbers φ and ψ for positive integer values of the exponent $n \geq 0$

In (24), (25), each of the terms ξ^n is transformed to a linear binomial $\xi^n = \alpha_n \xi + \beta_{n-1}$. For the number φ this binomial is known as:

$$\varphi^n = F_n \varphi + F_{n-1}, \tag{26}$$

were: F – Fibonacci numbers.

Multipliers (26) form a sequence $\{F_n\}$ of Fibonacci numbers. By a similar decomposition method, created a linear binomial for the number ψ:

$$\begin{cases} \psi^0 = 0 \cdot \psi + 1, \psi^1 = 1 \cdot \psi + 0, \quad (a) \\ \psi^2 = 3 \cdot \psi - 1, \ \psi^3 = 8 \cdot \psi - 3, \ \psi^4 = 21 \cdot \psi - 8, \ \psi^5 = 55 \cdot \psi - 21, \psi^6 = 134 \cdot \psi - 55, ... \end{cases} (b) \tag{27}$$

Thus, for a number ψ, the linear binomial has the form:

$$\psi^n = F_n^* \cdot \psi - F_{n-1}^*, \tag{28}$$

Multipliers (27 b) form sequence $\{F_n^*\}$:

$$F^*_n : \quad 1{,}0, \quad 1{,}3, 8, 21, 55, {,}144, \ldots, \tag{29}$$

which is fundamentally different from the Fibonacci sequence. According to the OEIS classification [30], sequence (29) is known as A001906.

"Golden" division of a unit segment with loss of "memory". In the classical "golden" division, it is assumed that after division, the parts of the segment do not change their spatial directions and they coincide with the direction of the original segment, i.e. $\alpha = 0$ [10, 22, 23]. Let's study the case $\alpha \neq 0$, when, after division, the spatial orientation of the division parts changes. For to do this, we project the projection of the difference ($1 - x_\alpha$) on the direction x_α:

$$\xi = (1 - x_\alpha)\cos\alpha, \tag{34}$$

were: $\alpha -$ angle of spatial orientation of segment parts.

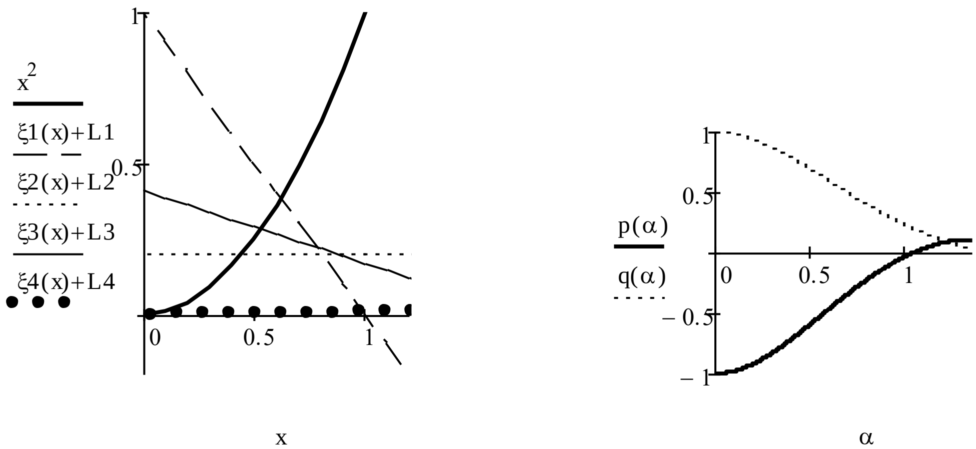
Let $\frac{1}{2} \langle\, x_\alpha \,\langle\, 1$. Then the equality of proportions between the parts x_α and ξ will be written in the form:

$$\frac{x_\alpha}{x_\alpha + \xi_\alpha} = \frac{\xi_\alpha}{x_\alpha} \Rightarrow x_\alpha{}^2 = \begin{cases} p_\alpha = L\dfrac{\cos\alpha(1 - 2\cos\alpha)}{\sin^2\alpha + \cos\alpha}, \\ q_\alpha = L^2 \dfrac{\cos^2\alpha}{\sin^2\alpha + \cos\alpha}, \\ p_\alpha x_\alpha + q_\alpha. \end{cases} \tag{35}$$

The angle dependence of the division coordinate is shown in Fig. 3(a). When the corner is:

$$\frac{\cos\alpha(1 - 2\cos\alpha)}{\sin^2\alpha + \cos\alpha} \to 0 \Rightarrow 1 - 2\cos\alpha \to 0 \Rightarrow \alpha\big|_{p \to 0} \to \frac{\pi}{3}, \tag{36}$$

graph of the straight right side of equation (35) parallel to the x-axis.



a b

Fig. 3 Angular dependence of division coordinates.

Discussion of the obtained results

There are next results for **single segment "golden" division, general model and square trinomial decomposition**. Limits (17) for recurrent sequences with constant coefficients form the content of the Poincaré theorem [22]. If roots $x(p,q)_+, x(p,q)_-$ quadratic equation $x^2 - px - q = 0$ real and unequal $|x(p,q)_+| \neq |x(p,q)_-|$, then the relation $\frac{min|x(p,q)_\mp|}{max|x(p,q)_\pm|}$ associated with the "golden" division of the unit segment by the proportion $\frac{min|x(p,q)_\mp| / max|x(p,q)_\pm|}{min|x(p,q)_\pm| / \sqrt{q}} = \frac{|min(p,q)_\pm| / \sqrt{q}}{1}$.

The general conclusion indicates that the well-known "golden" proportion between the parts of binary division is most likely a particular consequence of the Vieta and Poincaré theorems.

Golden divisionof a unit segment by a point with coordinate $x \langle \frac{1}{2}$**.** Thus, the sequence (29) is obtained by the method of bisection of the sequence $\{F_n\}$, similarly as the progression (24) is obtained by bisecting the progression (24).

In sequence (29), the first two terms (27 a) have fixed values, equal to:

$$\begin{cases} n = 1: & \psi^1 = 1 \cdot \psi + 0 \\ n = 0: & \psi^0 = 0 \cdot \psi + 1 \end{cases} \Rightarrow F^*{}_0 = 1, F^*{}_1 = 0 . \quad (30)$$

Therefore, (30) indicate the initial conditions of the sequence (29), the terms of which are calculated by the formula:

$$F_{n+2}{}^* = 3F_{n+1}{}^* - F_n{}^*, \quad n \geq 0 . \quad (31)$$

Therefore, the sequence $\{F_n{}^*\}$ has a limit $\lim_{n\to\infty} \frac{F_{n+2}{}^*}{F_{n+1}{}^*}$:

$$\lim_{n\to\infty} \frac{F_{n+2}{}^*}{F_{n+1}{}^*} = \lim_{n\to\infty} \frac{3F_{n+1}{}^* - F_n{}^*}{3F_n{}^* - F_{n-1}{}^*} = \lim_{n\to\infty} \frac{3F_{n+1}{}^* / F_n{}^* - 1}{3 - F_{n-1}{}^* / F_n{}^*} = \frac{3 \cdot 2.618 - 1}{3 - 0.38} \cong \frac{6.854}{2.618} = 2.618 = \Psi = \frac{1}{\psi} . \quad (32)$$

For the Fibonacci sequence, the limit similar to (32) is $\lim_{n\to\infty} \frac{F_{n+2}{}^*}{F_{n+1}{}^*} = \Phi$.

The numerical values of progressions (24) and (25) in Fig. 2 are shown on a semi-logarithmic scale as graphs of "golden" full geometric progressions for numbers $, \varphi, \psi$

$$...\{...\xi^{-n}, \xi^{-(n-1)}, ..., ^{-2}, \xi^{-1}, \xi^0, \xi^1, \xi^2, \xi^3, ..., \xi^{n-1}, \xi^n, ...\}. \quad (33)$$

They are aligned with each other, however, the shift of their beginnings ($\psi = \varphi^2$) causes to different values of the lower limits $ln X\Psi = ln X_\Psi$ and $ln Y\Phi = ln Y_\Phi$, equal $ln(\varphi^{-1}) = 0.481 \quad and \quad ln(\psi^{-1}) = 0.962$.

Therefore, their relationship is $\frac{ln(\psi^{-1})}{ln(\varphi^{-1})} = 2$, which agrees with the ratio of the densities of geometric progressions (24) and (25). Reverse transformations $\begin{cases} \varphi = e^{-0.481}, \\ \psi = e^{-0.962}, \end{cases}$ show that the numbers $\ln \psi$, $\ln \varphi$ can also be associated with the characteristics of the exponential relaxation process of the system parameter, which is subjected to the "golden" division.ю In this case, the exponent with a single initial value.

"Golden" division of a unit segment with loss of "memory". Given angle value $\alpha|_{p\to 0} \to \frac{\pi}{3}$ approaching the angle $\alpha \cong 63.405^o$ lateral surface relative to the vertical in the famous pyramid of Cheops

[28]. With the increase α, values $p_{k,\alpha}, q_{k,\alpha}$ decrease and at $\alpha = \frac{\pi}{2}$, straight line of the right side of equality (35) crosses the vertex of the parabola parallel to the x-axis. In this case, the proportional relationship between the division parts x and $1-x$ disappears. Instead the spatial construction is implemented in the form of the Pythagorean theorem from the constituent parts of a right-angled triangle with arbitrary values of the sides. For other points $k \neq 1$ for phase direction (6), division model (35) has the form:

$$\frac{x_{k,\alpha}}{x_{k,\alpha}+\xi_{k,\alpha}} = k\frac{\xi_{k,\alpha}}{x_{k,\alpha}} \Rightarrow x_{k,\alpha}^{2} = \begin{cases} p_{k,\alpha} = \kappa L\dfrac{\cos\alpha(1-2\cos\alpha)}{1-\kappa\cos^{2}\alpha+\kappa\cos\alpha}, \\ q_{k,\alpha} = \kappa L^{2}\dfrac{\cos^{2}\alpha}{1-\kappa\cos^{2}\alpha+\kappa\cos\alpha}, \\ p_{k,\alpha}x_{k,\alpha}+q_{k,\alpha}. \end{cases} \quad (37)$$

As a result, the transition to phase points $k \neq 1$ is not accompanied by a significant change in the angular dependences of the coefficients $p_{k,\alpha}$, $q_{k,\alpha}$ (Fig. 3b), including the angle value (36) does not change.

Conclusions

1. The binary division of a unit segment into two unequal parts from the properties of the "golden" proportion is realized at an arbitrary point of the phase plane $0pq$. Also, the decomposition of a square trinomial leads to the formation of recurrent sequences with Fibonacci properties.

2. Quantitative characteristics of the "golden" division of a single segment by two points with coordinates in intervals $x \rangle 1/2$ and $x \langle 1/2$ interconnected by a non-linear parabolic relation $\psi = \varphi^2$.

3. The angular dependence of the "golden" division of a unit segment with the loss of "memory" about the spatial orientation of its parts after division, exhibits a characteristic angle $\alpha\big|_{p\to 0} \to \frac{\pi}{3}$. At a given angle α, a spatial construction is implemented from its constituent parts, which forms a right-angled Pythagorean triangle, with arbitrary values of the sides.

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НОВІ ЗАКОНОМІРНОСТІ ПОДІЛУ СЕГМЕНТІВ ЗА ЗОЛОТИМ ПЕРЕРІЗОМ

У роботі досліджено чотири задачі про поділ одиничного відрізка за «золотою» пропорції, а саме загальна модель «золотого» поділу одиничного відрізка, декомпозиція квадратного тричлена, «золотий» поділ одиничного відрізка точкою з координатою $x < \frac{1}{2}$, «золоте» січення одиничного відрізка із втратою «пам'яті». У даній статті поняття декомпозиції використано як піднесення до ступеня квадратного тричлена. Бінарний поділ одиничного відрізка на дві нерівні частини із властивостями «золотої» пропорції реалізується у довільній точці фазової площини $0pq$, а декомпозиція квадратного тричлена призводить до формування рекурентних послідовностей зі властивостями Фібоначчі. Можна зауважити, що відома «золота» пропорція між частинами бінарного поділу, представляє швидше за все часткове наслідування теорем Вієта та Пуанкаре.

Закономірності «золотого» поділу для випадку $x > \frac{1}{2}$ добре вивчені. Тому досліджено закономірності для випадку $x < 1/2$. Незважаючи на те, що числа

ψ, φ виражаються одне через інше, з точки зору «золотого» поділу обидві реалізації з кількісними характеристиками $Y_{\varphi}\big|_{L=1} = \varphi$ та $X_{\psi}\big|_{L=1} = \psi$ незалежні та рівноправні, хоча їх кількісні характеристики можна пов'язати між собою за допомогою відповідних формул. Побудовано геометричні прогресії для чисел φ та ψ для цілих позитивних значень показника ступеня $n \geq 0$ для підтвердження незалежності та рівноправності обох моделей. Кількісні характеристики «золотого» поділу одиничного відрізка двома точками з координатами в інтервалах $x \rangle 1/2$ та $x \langle 1/2$ пов'язані між собою нелінійним співвідношенням параболічного типу $\psi = \varphi^2$. У класичному «золотому» січенні передбачається, що після розподілу частини відрізка не змінюють своїх просторових напрямів, і вони збігаються з напрямком вихідного відрізка, тобто $\alpha = 0$. У цій роботі розглянуто випадок $\alpha \neq 0$, коли після розподілу, просторова орієнтація елементів розподілу змінюється. Кутова залежність «золотого» поділу одиничного відрізка із втратою «пам'яті» його частин про просторову орієнтацію після поділу, виявляє відомий кут $\alpha\big|_{p \to 0} \to \frac{\pi}{3}$ нахилу бічної поверхні Хіопса.

Ключові слова: «золоте»січення, числа Фібоначчі, теорема Вієта, рекурентні послідовності.