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A backward difference formulation for analyzing the dynamics of capital stocks

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The current study provides a numerical method that is derived in a backward difference formulation for ordinary differential equations. The proposed method employs a constant step size algorithm of order 12. The backward difference formulation serves as a competitive algorithm for solving ordinary differential equations. In the current study, the backward difference method is used to analyze the dynamics of capital stocks in terms of depreciation rate for the capital–labor ratio. Results provided in this study will validate the accuracy of the backward difference algorithm hence proving it as a viable alternative for analyzing economic problems in the form of ordinary differential equations.

Keywords: *applied mathematics, backward difference, ODEs, multistep.*

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1. Introduction

In the pursuit of extending Euler's method, Bashforth and Adams [1] established a method currently known as the Adams–Bashforth method. The Adams–Bashforth method proposed in [1] was designed to obtain the approximated solution using solutions from multiple previous steps, hence inspiring the original multistep method. Since then, many variation of the multistep method were established including the well known Adams-Moulton method. The Adams–Moulton method is an implicit method that was actually conceived by Adam in [1]. Moulton's name was associated with the Adams formulae after his observation revealed in [2] that the Adams explicit (Adams–Bashforth) and implicit pair could be used in tandem to obtain a more accurate approximation. His revelation of a Adam–Bashforth–Moulton approach prompted the predictor–corrector method.

Inspired by works of Suleiman [3], a series of multistep method were established by various authors such as [4–8]. Omar [4] extended the works of [3] by developing a block algorithm for Suleiman's divided difference formulation. Abdul Majid then established a fully implicit method for non-stiff solving higher order ordinary differential equations (ODEs) in [5]. In [6], Ibrahim formulated an algorithm for solving stiff ODEs using a backward differentiation approach. This area of research was then continued by [8] with a backward difference predictor–corrector algorithm.

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The current research adopts a backward difference formulation in variable order constant step size mode of order 12 (BDO12). Similar to the Adams–Bashforth–Moulton formulation, the proposed method implements a predict-correct algorithm with explicit and implicit integration coefficients. The backward difference method formulated in this study is used to analyze the dynamics of capital which can be found in the field of economics. Results provided also include verification of the method's accuracy. For complete error estimation, we refer readers to the works of [9]. Latest study involving multistep methods for solving ordinary differential equations can be found in [10–19].

2. Backward difference method

The current research considers the ordinary differential equation

$$\dot{y} = f(t, y), \tag{1}$$

with the initial condition,

$$y(a) = \eta$$

in the interval $t \in [a, b]$. The proposed method implements back values up to the order 12. Euler's method is initially used to obtain the require back values before continuing with the backward difference algorithm. Derivation of the algorithm begins by constructing the predictor formula. Firstly consider the ordinary differential equation (1) which is then integrated once, from 0 to 1 yielding

$$y_p(t_{n+1}) = y(t_n) + h \int_0^1 f(t, y)\, dt. \tag{2}$$

Next by substituting the Newton–Gregory backward difference formula:

$$P_n(t) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n, \qquad s = \frac{t - t_n}{h}, \tag{3}$$

we are able to approximate $f(t, y)$ in Eq. (2) thus establishing the folllowing:

$$y_p(t_{n+1}) = y(t_n) + h \sum_{j=0}^{k-1} {}_p\alpha_{1,j} \nabla^j f_n, \qquad {}_p\alpha_{1,j} = (-1)^j \int_0^1 \binom{-s}{j} ds. \tag{4}$$

The next step is to obtain the corrector formula. Again, by integrating (1) once but with the subtle difference of changing the limit of integration from -1 to 0 as follows:

$$y_c(t_{n+1}) = y(t_n) + h \int_{-1}^0 f(t, y)\, dt. \tag{5}$$

Subsequently, by approximating $f(t, y)$ with the following implicit version of the polynomial in (3):

$$P_{n+1}(t) = \sum_{j=0}^{k} (-1)^j \binom{-s}{j} \nabla^j f_{n+1}, \qquad s = \frac{t - t_{n+1}}{h}, \tag{6}$$

provides the following first order corrector formula

$$y_c(t_{n+1}) = y(t_n) + h \sum_{j=0}^{k-1} {}_c\alpha_{1,j} \nabla^j f_{n+1}, \qquad {}_c\alpha_{1,j} = (-1)^j \int_{-1}^0 \binom{-s}{j} ds. \tag{7}$$

3. Integration coefficients

In this section, we will derive the explicit and implicit integration coefficients formulae. This begins by denoting the first order generating functions in terms of integration coefficients as follows:

$$\Gamma_1^p(t)=\sum_{i=0}^{\infty}{}_p\alpha_{1,j}t \quad\text{and}\quad \Gamma_1^c(t)=\sum_{i=0}^{\infty}{}_c\alpha_{1,j}t. \tag{8}$$

By solving the integral

$${}_p\alpha_{1,j}=(-1)^j\int_0^1\binom{-s}{j}\,ds \tag{9}$$

establishes the first order explicit generating function in the following form

$$\Gamma_1^p(t)=\frac{-t}{(1-t)\log(1-t)}. \tag{10}$$

Next, the first order implicit generating function is obtained by integrating

$${}_c\alpha_{1,j}=(-1)^j\int_{-1}^0\binom{-s}{j}\,ds. \tag{11}$$

which yields the following generating function

$$\Gamma_1^c(t)=\frac{-t}{\log(1-t)}.$$

Next, through mathematical deduction, the first order explicit and implicit set of coefficients can be expressed in the following recursive relationship

$${}_p\alpha_{1,k}=\sum_{j=0}^{k}{}_c\alpha_{1,j}. \tag{12}$$

The following Tables 1 and 2 respectively consist of coefficients used for both predictor and corrector which was extended from [20].

Table 1. Predictor Coefficients for $k=0,1,\ldots,12$.

${}_p\alpha_{1,k}$	0	1	2	3	4	5	6
	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$	$\frac{19087}{60480}$
	7	8	9	10	11	12	
	$\frac{5257}{17280}$	$\frac{1070017}{3628800}$	$\frac{25713}{899600}$	$\frac{26842253}{95800320}$	$\frac{4777223}{17418240}$	$\frac{703604254357}{2615348736000}$	

Table 2. Corrector Coefficients for $k=0,1,\ldots,12$.

${}_c\alpha_{1,k}$	0	1	2	3	4	5	6
	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$	$-\frac{863}{60480}$
	7	8	9	10	11	12	
	$-\frac{275}{24192}$	$-\frac{33953}{3628800}$	$-\frac{8183}{1036800}$	$-\frac{3250433}{479001600}$	$-\frac{4671}{788480}$	$-\frac{13695779093}{261534873600}$	

Because, “Acceptance criteria is crucial because implementing variable order in a multistep method relies on the back values stored”, (refer to [21]) the error estimate E_k will used in determining order increment or reduction.

1. The algorithm will reduce the order of the method by 1 when $k > 2$ and $\max(|E_{(k-1)}|, |E_{(k-2)}|) \leqslant |E_k|$.
2. In the case when $E_{(k+1)}$ is available, the order will be reduced again by 1 for $k > 1$ if $|E_{(k-1)}| \leqslant \min(|E_k|, |E_{(k+1)}|)$.
3. Only when $k + 1$ successful step have been achieved, the order will be increased by 1 if for $k > 1$, $|E_{(k+1)}| < |E_k| < \max(|E_{(k-1)}|, |E_{(k-2)}|)$ and in the case of $k = 1$, $|E_{(k+1)}| < 0.5|E_k|$.
4. The order of the method is restricted to the range $1 \leqslant k \leqslant 12$.

4. Results, analysis and conclusion

The growth model examined in this research is a simplified variation which excludes markets and firms such that the production function takes the form of

$$Y(t) = F\big[K(t), L(t), T(t)\big],$$

where $Y(t)$ is the flow of output produced given three inputs: physical capital $K(t)$, labor $L(t)$, and knowledge $T(t)$ in respect to time, t. Let the capital be a homogeneous good with a constant with a depreciation rate $\delta > 0$ which implies that a constant portion of the capital stock diminishes at each point of time until it is no longer viable for production. Then under the assumption that all capital are equally productive, the net increase in physical capital at a point in time equals the gross investment less depreciation give by the following

$$\dot{K}(t) = s\,F\big[K(t), L(t), T(t)\big] - \delta K(t),$$

with the saving rate s.

In this research, we analyze the dynamic behavior in economics from the perspective as characterized by Solow [22] and Swan [23] production model by considering the Cobb–Douglas function

$$Y = A\,K^{\alpha} L^{1-\alpha}$$

The Cobb–Douglas production function can then expressed in form of steady–state capital–labor ratio as

$$k^* = \big[s\,A/(n+\delta)\big]^{(1/(1-\alpha)}.$$

Note that k^* increases together with the saving rate s and the level of technology A, and decreases with the increment of population growth rate, n and depreciation rate, δ, thus provides the following steady-state level of output per capita:

$$y^* = A^{1/(1-\alpha)}\big[a/(n+\delta)\big]^{\alpha/(1-\alpha)}.$$

As discussed in [24], because the transition of an economy’s per capita income converges toward its own steady–state value and to the per capita incomes of other economies, the growth rate of k can be written as

$$\dot{k}/k = s\,A\,k^{-(1-\alpha)} - (n+\delta).$$

The parameter $0 < \alpha < 1$ denotes the output elasticity with respect to the capital and we let the capital stock per effective labor be $k = K/(A\,L)$. Next assume that the growth rate of population is constant which is denoted by $\dot{L}/L = n$, the growth rate of technological progress, $\dot{A}/A = \gamma > 0$ and the rate of capital depreciation represented by δ, where $\delta \in [0, 1]$. Thus, the dynamics per-capita capital k with respect to time t, and governed by the Cobb–Douglas production function associated with the Solow–Swan model yields:

$$\dot{k} = s\,k^{\alpha} - \delta k,$$

with an analytical solution of

$$k_t = \left[\left(k_0^{(1-\alpha)} - \frac{s}{\delta}\right) e^{-(1-\alpha)\delta t} + \frac{s}{\delta}\right]^{1/(1-\alpha)}.$$

As $t \to +\infty$, $k_t \to (s/\delta)^{1/(1-\delta)}$ (source: [25]).

Tables 3–5 are provided to highlight the accuracy of the method. Results displayed in the tables are the average error per step between the approximated solution and the exact solution with a step size of $h = 0.01$. We would like to note that when using a finer step size, the accuracy of the BDO12 increases substantially. For purpose of the current research, a larger step size is selected to present the ability of the BDO12 algorithm under dire circumstances. Table 3 shows the error when s and α are fixed but δ increases, Table 4 shows the error when s and δ are fixed but α increases and Table 5 shows the error when α and δ are fixed but s increases. Each table provides two circumstances, one if the fixed parameters are low, the second when the fixed parameters are of high value. As illustrated in Tables 3–5, the BDO12 method provides accurate approximation at almost every condition. The result would be more efficient if a smaller step size was selected. In Table 3, with selected parameter of $s = 0.9$, $\alpha = 0.9$ and $\delta \in [0.1, 0.3]$ the results provided are less accurate. This is contributed to the element of stiffness cause by the selected parameters. Because the BDO12 method is developed to handle non-stiff ODEs, even the mildest of stiffness may affect its accuracy.

Table 3. Average error of the BDO12 for variable δ.

	$\delta = 0.1$	$\delta = 0.3$	$\delta = 0.5$	$\delta = 0.7$	$\delta = 0.9$
$s = 0.1$, $\alpha = 0.1$	$5.37861e-02$	$1.63504e-01$	$1.72888e-01$	$1.53322e-01$	$1.30603e-01$
$s = 0.9$, $\alpha = 0.9$	$3.26335e+00$	$1.64265e+00$	$7.19939e-01$	$2.04440e-01$	$5.93420e-02$

Table 4. Average error of the BDO12 for variable α.

	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$s = 0.1$, $\delta = 0.1$	$5.37861e-02$	$4.68832e-02$	$3.80785e-02$	$2.64712e-02$	$1.04855e-02$
$s = 0.9$, $\delta = 0.9$	$5.04973e-02$	$5.93991e-02$	$7.01513e-02$	$7.84862e-02$	$5.93420e-02$

Table 5. Average error of the BDO12 for variable s.

	$s = 0.1$	$s = 0.3$	$s = 0.5$	$s = 0.7$	$s = 0.9$
$\alpha = 0.1$, $\delta = 0.1$	$5.37861e-02$	$6.05377e-02$	$1.46807e-01$	$2.14518e-01$	$2.69348e-01$
$\alpha = 0.9$, $\delta = 0.9$	$1.53213e-01$	$1.82670e-01$	$1.99897e-01$	$1.72165e-01$	$5.93420e-02$

The selected saving rate, $s > 0$ because when $s = 0$ the dynamics per-capita capital equation will be left with only a multiplicity of capital depreciation δk, hence merely providing the depreciated value of k. Figures 1–6 are the approximated solutions of k by the proposed backward difference method of order 12 (BDO12). Figures 1 and 2 illustrate the dynamics per capital when savings rate, s and output elasticity, α are fixed but capital depreciation varies. Figure 1 considers a minimal savings and elasticity whereas Figure 1 highlights the effect of the latter. On the other hand, Figures 3–6 analyze the effects of fixed depreciation rate with different savings and output elasticity. In Figures 3 and 4, a fixed depreciation rate of $\delta = 0.1$ is selected and in Figures 5 and 6, a depreciation rate of $\delta = 0.9$ is selected to show the effect caused by different δ's.

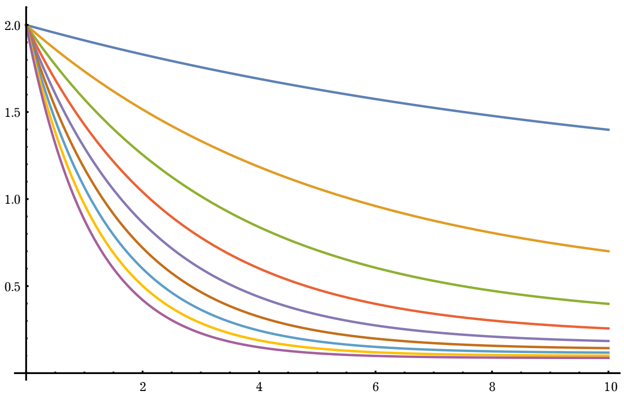

Fig. 1. Numerical approximation of capital stock per effective labor given $s = 0.1$ and $\alpha = 0.1$ with $0 < \delta < 1$.

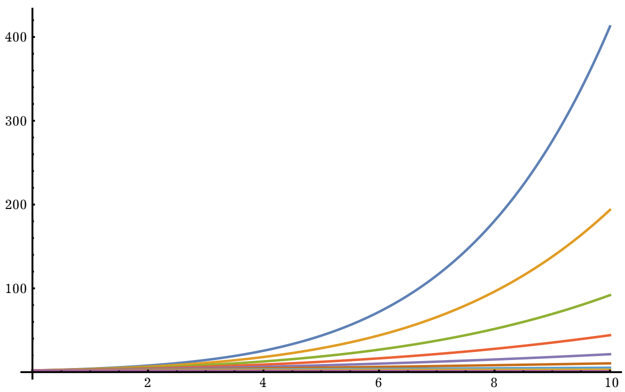

Fig. 2. Numerical approximation of capital stock per effective labor given $s = 0.9$ and $\alpha = 0.9$ with $0 < \delta < 1$.

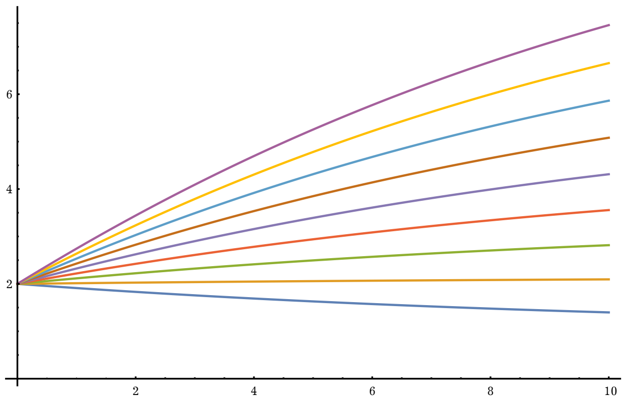

Fig. 3. Numerical approximation of capital stock per effective labor given $0 < s < 1$ and $\alpha = 0.1$ with $\delta = 0.1$.

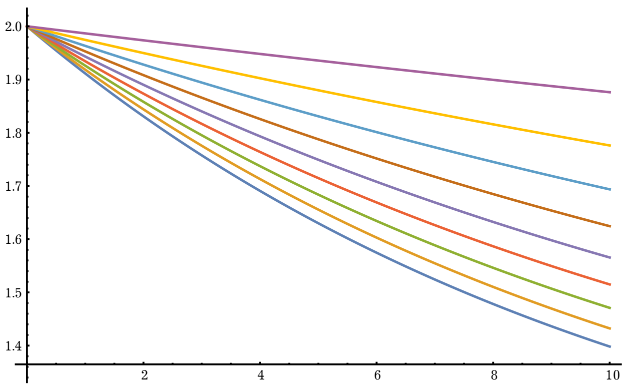

Fig. 4. Numerical approximation of capital stock per effective labor given $s = 0.1$ and $0 < \alpha < 1$ with $\delta = 0.1$.

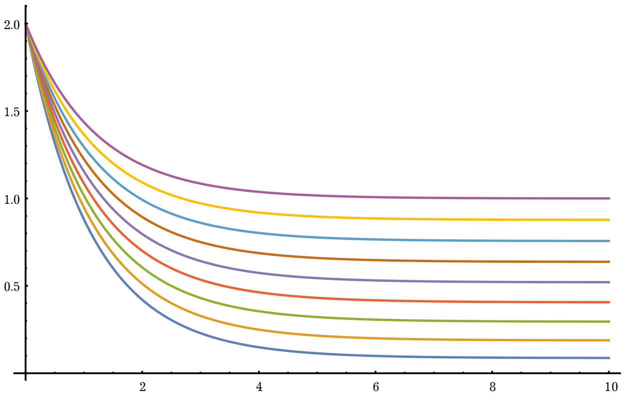

Fig. 5. Numerical approximation of capital stock per effective labor given $0 < s < 1$ and $\alpha = 0.1$ with $\delta = 0.9$.

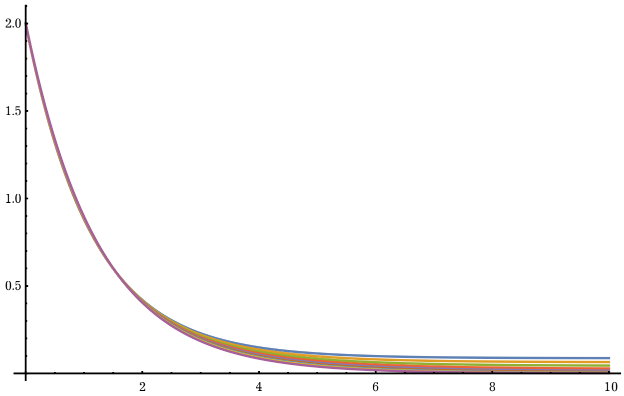

Fig. 6. Numerical approximation of capital stock per effective labor given $s = 0.1$ and $0 < \alpha < 1$ with $\delta = 0.9$.

As presented in Figure 1, when dealing with small δ and α, the increase of depreciation rate from 0 to 1 shows a rapid exponential decrease of k. Dissimilar to a large δ and α, the increase of δ reduces an exponential increase of k to a static k. Figures 3 and 5 detail the effect of variable savings on the dynamics per-capita capital. When a small δ is selected (Figure 3), the increase of s is able to change a reducing pattern of dynamics per-capita capital to an increasing pattern. In contrast to a high δ, increasing the value of s is only able to minimize the reduction of dynamics per-capita capital (Figure 5). Next, Figures 4 and 6 exemplify the effects of a variable α. Unlike the effect of s as shown in Figure 3, the effects of an increasing α is only able to minimize the reduction of dynamics per-capita capital when dealing with a small depreciation rate but barely shows any advantage with a high depreciation rate (Figure 6).

Hence, it can be concluded that to provide a positive dynamics per-capita capital, both savings and output requires a higher rate, where the increase of s provides a higher dynamics per-capita capital and the increase α offers a better increment rate.

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Зворотні різницеві форми для аналізу динаміки запасів капіталу

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Це дослідження описує чисельний метод, виведений у зворотній різницевій формі для звичайних диференціальних рівнянь. У запропонованому методі використовують алгоритм сталого розміру кроку 12-го порядку. Зворотна різницева форма слугує конкурентноздатним алгоритмом для розв'язування звичайних диференціальних рівнянь. У цьому дослідженні метод зворотної різниці використовують для аналізу динаміки основних фондів у величинах норми амортизації для співвідношення капіталу та праці. Отримані результати підтверджують точність зворотного різницевого алгоритму, доводячи його альтернативність для аналізу економічних проблем у вигляді звичайних диференціальних рівнянь.

Ключові слова: *прикладна математика, зворотна різниця, звичайні диференціальні рівняння, багатокроковість.*