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The evolution of geometric Robertson–Schrödinger uncertainty principle for spin 1 system

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Geometric Quantum Mechanics is a mathematical framework that shows how quantum theory may be expressed in terms of Hamiltonian phase-space dynamics. The states are points in complex projective Hilbert space, the observables are real valued functions on the space, and the Hamiltonian flow is specified by the Schrödinger equation in this framework. The quest to express the uncertainty principle in geometrical language has recently become the focus of significant research in geometric quantum mechanics. One has demonstrated that the Robertson–Schrödinger uncertainty principle, which is a stronger version of the uncertainty relation, can be defined in terms of symplectic form and Riemannian metric. On the basis of this formulation, we study the dynamical behavior of the uncertainty relation for the spin 1 system in this work. We show that under Hamiltonian flow, the Robertson–Schrödinger uncertainty principles are not invariant. This is because, unlike the symplectic area, the Riemannian metric is not invariant under Hamiltonian flow throughout the evolution process.

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1. Introduction

The fact that classical mechanics, general relativity and others are highly geometrical inspired some physicists to cast quantum mechanics in geometrical language [1–15] in order to better understand the quantum-classical transition. The deeper investigation shows that the Hilbert space $\mathcal{H}$ is not the true space of states, since any two state vectors $\Psi, \Phi \in \mathcal{H}$ such that $\Psi = \alpha\Phi$ are physically equivalent ($\Psi \backsim \Phi$). Thus, the proper quantum space of pure states is the set of rays through the origin in $\mathcal{H}$, i.e. $P(\mathcal{H}) := \mathcal{H}/\backsim$ which is known as the complex projective Hilbert space or the quantum phase space for both finite and infinite dimensional $\mathcal{H}$. Furthermore, the existence of Hermitian inner product in $\mathcal{H}$ endows $P(\mathcal{H})$ with the structure of Kähler manifold (ω, g, j), where ω is non-degenerate, closed symplectic two-form, g is Riemannian metric and j is the compatible complex structure satisfying $j^2 = -1$. Thus, similar to classical mechanics, the correct quantum state space is also regarded as a symplectic manifold. In term of self-adjoint operator on $\mathcal{H}$, via its expectation value, one can obtain a real valued function on $\mathcal{H}$ that has well defined projection h to $P(\mathcal{H})$. Note that every phase space function induces a flow along its Hamiltonian vector field X_h. Hence, on Hilbert space, the flow is certainly defined by Schrödinger equation of the quantum theory. In other words, Schrödinger evolution is exactly the same as the Hamiltonian flow on quantum phase space $P(\mathcal{H})$. Here, one can directly see that classical mechanics and quantum mechanics have many similarities. However, the fact that Riemannian metric in quantum phase space is closely related to the notion of probability provides us with several main features that are missing in classical mechanics such as uncertainty principle and state vector reduction in quantum measurement processes.

Despite the successful of quantum mechanics in terms of application, the true nature of this theory is still far from being understood. In other words, some of its principles and concepts are clearly counter-intuitive and very difficult to explain in simple language since most of them do not have classical analogue. One of the famous examples to describe the weirdness of quantum mechanics is the uncertainty principle. The effort to cast uncertainty principle in term of geometrical language appeared to become the subject of intense study in geometric quantum mechanics. One of earliest studies refers to the work of Anandan [19] who proposes a new geometric meaning of times-energy uncertainty principle for an arbitrary quantum system. After that Ashtekar [5] has shown that for pure quantum state, the fact that the expectation values of observables correspond to the Riemannian and symplectic structure allow one to formulate a geometric version of Robertson–Schrödinger uncertainty relation. Further study of this research line is conducted by Andersson and Heydari [26, 27] by deriving a geometric uncertainty relation for observables acting on mixed quantum states. Recently Barbara [28] extends the geometric quantum mechanics which includes elements of the symplectic topology of quantum state space by defining the Robertson–Schrödinger uncertainty relation for pure quantum states as the differential version of the energy identity in the J-holomorphic curve theory.

It is generally accepted that uncertainty principle is a purely quantum concept and cannot be described using classical mechanics. However, this statement is not entirely true when recently one had successfully shown that the uncertainty principle can naturally arise from the structure of classical mechanics [20–25]. This is achieved through a topological tool known as symplectic capacity together with the notion of quantum blob. As we know, Heisenberg uncertainty principle is a minimum for the product of the uncertainties of position and momentum measurements. This is consistent with the property of symplectic camel which asserts that it is not possible to shrink a cross-section defined by conjugate coordinates like x and p_x to zero. It means that we have a minimum cross-sectional area within a given volume which cannot shrink further. Thus, it is clear that all the uncertainty principles mention in these papers are invariant under symplectic transformation since they can be expressed in term of symplectic capacity.

In this paper, motivated by this work, the possibility of the uncertainty principle in geometric quantum mechanics is invariant under the Hamiltonian flows has been demonstrated since in this formulation the uncertainty principle is partly expressed in term of symplectic form. This research may become a significant step in order to construct a connection between geometric quantum mechanics and symplectic topology.

2. Robertson–Schrödinger uncertainty principle

Uncertainty principle, firstly discovered by the German theoretical physicist Werner Heisenberg [16] is one of the fundamental concepts that shows the weirdness of quantum mechanics. It set the limitation of complementary variables such as position and momentum to be measured simultaneously with high precision. Furthermore, Robertson [17] generalized the inequality to an arbitrary observables $\hat{A}$ and $\hat{B}$ given by

$$(\Delta\hat{A})(\Delta\hat{B}) \geqslant \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|$$

and within a year the stronger extension was proposed by Schrödinger [18] by adding covariance term to the formulation

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geqslant \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|^2 + \left|\frac{1}{2}\langle[\hat{A},\hat{B}]_+\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2.$$

In geometric quantum mechanics, Ashtekar [5] shows that the symplectic form and Riemannian metric allows one to formulate a geometric version of Roberton–Schrödinger uncertainty principle. In order to do this, let Ψ be a normalized state vector, the uncertainty of the observable $\hat{A}$ in the state

is defined as
$$(\Delta\hat{A})^2_\Psi := \langle\hat{A}^2\rangle_\Psi - \langle\hat{A}\rangle^2_\Psi.$$
Consider operators $\hat{A}_\perp$ and $\hat{B}_\perp$ as follows
$$X^\perp_{\hat{A}} = \hat{A}_\perp\Psi, \quad X^\perp_{\hat{B}} = \hat{B}_\perp\Psi,$$
where $X^\perp_{\hat{A}}$ and $X^\perp_{\hat{B}}$ are vectors orthogonal to state vector Ψ. Then these operators can be defined as
$$\hat{A}_\perp := \hat{A} - \mathbf{1}.A, \quad \hat{B}_\perp := \hat{B} - \mathbf{1}.B,$$
where A and B are the expectation values of $\hat{A}$ and $\hat{B}$ respectively and $\mathbf{1}$ is an identity operator. It is obvious that $(\Delta\hat{A})^2_\Psi = \langle\hat{A}^2_\perp\rangle_\Psi$, and therefore
$$(\Delta\hat{A})^2_\Psi(\Delta\hat{B})^2_\Psi = \langle\hat{A}^2_\perp\rangle_\Psi\langle\hat{B}^2_\perp\rangle_\Psi = \langle\Psi|\hat{A}^2_\perp|\Psi\rangle\langle\Psi|\hat{B}^2_\perp|\Psi\rangle.$$
Now, by applying the Schwartz inequality
$$\langle\Psi|\hat{A}^2_\perp|\Psi\rangle\langle\Psi|\hat{B}^2_\perp|\Psi\rangle \geqslant |\langle\Psi|\hat{A}_\perp\hat{B}_\perp|\Psi\rangle|^2$$
and define
$$\hat{A}_\perp\hat{B}_\perp = \frac{1}{2}[\hat{A}_\perp, \hat{B}_\perp] + \frac{1}{2}[\hat{A}_\perp, \hat{B}_\perp]_+,$$
where $[\hat{A}_\perp, \hat{B}_\perp]_+ = \hat{A}_\perp\hat{B}_\perp + \hat{B}_\perp\hat{A}_\perp$, then we have
$$(\Delta\hat{A})^2_\Psi(\Delta\hat{B})^2_\Psi \geqslant \frac{1}{4}\left(\left\langle[\hat{A}_\perp, \hat{B}_\perp]_+\right\rangle^2_\Psi - \left\langle[\hat{A}, \hat{B}]\right\rangle^2_\Psi\right), \tag{1}$$
which is the standard form of the uncertainty relation for two quantum observables $\hat{A}$ and $\hat{B}$ in the Hilbert space formulation. Now let rephrase the above formula in terms of the canonical geometric structure of $\mathcal{H}$ that is the symplectic form Ω and the Riemannian metric G. Both structures can be expressed as
$$\begin{aligned}\Omega(X_{\hat{A}}, X_{\hat{B}}) &= -i\hbar\left(\langle X_{\hat{A}}(\Psi)|X_{\hat{B}}(\Psi)\rangle - \langle X_{\hat{B}}(\Psi)|X_{\hat{A}}(\Psi)\rangle\right)\\ &= -\frac{i}{\hbar}\langle\Psi|\hat{A}\hat{B} - \hat{B}\hat{A}|\Psi\rangle = -\frac{i}{\hbar}\langle[\hat{A}, \hat{B}]\rangle,\\ G_\Psi(X_{\hat{A}}, X_{\hat{B}}) &= \hbar\left(\langle X_{\hat{A}}(\Psi)|X_{\hat{B}}(\Psi)\rangle + \langle X_{\hat{B}}(\Psi)|X_{\hat{A}}(\Psi)\rangle\right)\\ &= \frac{1}{\hbar}\langle\Psi|\hat{A}\hat{B} + \hat{B}\hat{A}|\Psi\rangle = \frac{1}{\hbar}\langle\Psi|[\hat{A}, \hat{B}]_+|\Psi\rangle,\end{aligned}$$
where
$$X_{\hat{A}} = -\frac{i}{\hbar}\hat{A}\Psi, \quad X_{\hat{B}} = -\frac{i}{\hbar}\hat{B}\Psi \tag{2}$$
are Schrödinger vector fields. Furthermore, since
$$[\hat{A}_\perp, \hat{B}_\perp]_+ = [\hat{A}, \hat{B}]_+ + 2(AB - A\hat{B} - B\hat{A}),$$
then
$$\langle[\hat{A}_\perp, \hat{B}_\perp]_+\rangle_\Psi = \hbar\, G_\Psi(X_{\hat{A}}, X_{\hat{B}}) - 2(AB)(\Psi).$$
Thus, we may rephrase the Robertson–Schrödinger uncertainty principle (1), without any reference to a given state vector that is
$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geqslant \left(\frac{\hbar}{2}\Omega(X_{\hat{A}}, X_{\hat{B}})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{A}}, X_{\hat{B}}) - AB\right)^2,$$
where $(\Delta\hat{A})^2$ denotes a function on $\mathcal{H}$ given by $(\Delta\hat{A})^2(\Psi) := (\Delta\hat{A})^2_\Psi$. Lastly, we may see how the Robertson–Schrödinger uncertainty principle can also be expressed on the quantum phase space $P(\mathcal{H})$. Now, let consider two quantum observables $\hat{A}$ and $\hat{B}$, and let a and b be the corresponding functions

on $P(\mathcal{H})$, i.e.

$$a \circ \Pi = \langle \hat{A} \rangle_\Psi = A(\Psi), \quad b \circ \Pi = \langle \hat{B} \rangle_\Psi = B(\Psi),$$

where Π is the canonical projection $\mathcal{H} \to P(\mathcal{H})$. Besides, for $X_a = \Pi_*(X_{\hat{A}})$ and $X_b = \Pi_*(X_{\hat{B}})$ are elements of $T_\psi P(\mathcal{H})$, one can define the so-called Poisson bracket and Reimannian bracket by

$$\{a, b\}_\omega := \omega_\psi(X_a, X_b), \quad (a, b)_g := g_\psi(X_a, X_b),$$

where ω and g represent the associated symplectic form and metric tensor on $P(\mathcal{H})$, respectively.

By doing this, one can show that

$$\omega_\psi(X_a, X_b) = \Omega_\Psi(X_{\hat{A}}^\perp, X_{\hat{B}}^\perp), \quad g_\psi(X_a, X_b) = G_\Psi(X_{\hat{A}}^\perp, X_{\hat{B}}^\perp)$$

and

$$\begin{aligned} \Omega_\Psi(X_{\hat{A}}^\perp, X_{\hat{B}}^\perp) &= \Omega_\Psi(X_{\hat{A}}, X_{\hat{B}}), \\ G_\Psi(X_{\hat{A}}^\perp, X_{\hat{B}}^\perp) &= G_\Psi(X_{\hat{A}}, X_{\hat{B}}) - \frac{2}{\hbar}(AB)(\Psi). \end{aligned}$$

Thus, the Robertson–Schrödinger uncertainty principle may be rephrased as the following equation in terms of mathematical objects define on $P(\mathcal{H})$:

$$\begin{aligned} (\Delta a)^2 (\Delta b)^2 &\geqslant \frac{\hbar^2}{4} \left(\omega(X_a, X_b)^2 + g(X_a, X_b)^2 \right) \\ &\geqslant \frac{\hbar^2}{4} \left(\{a, b\}_\omega^2 + (a, b)_g^2 \right), \end{aligned}$$

where $(\Delta a)^2(\psi) := (\Delta A)^2(\Psi)$ and $(\Delta b)^2(\psi) := (\Delta B)^2(\Psi)$.

3. The evolution of uncertainty principle in Hilbert space $\mathbb{C}^3$

Let us compute the Robertson–Schrödinger uncertainty principle for the case of spin 1 in order to compare with the results of spin $\frac{1}{2}$ particle. The corresponding self-adjoint operators are defined as

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the orthonormal basis in Hilbert space $\mathcal{H} \cong \mathbb{C}^3$ is represented by (e_1, e_2, e_3) satisfying

$$\langle e_\alpha || e_\beta \rangle = \delta_{\alpha\beta}.$$

Then, we represent the state of spin 1 by

$$|\Psi\rangle = Z_1 |e_1\rangle + Z_2 |e_2\rangle + Z_3 |e_3\rangle.$$

We begin with calculating the Schrödinger vector fields correspond to the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$. Recall that the corresponding evaluation function of $\hat{S}_x$ is

$$\begin{aligned} S_x(\Psi) &= \langle \Psi | \hat{S}_x | \Psi \rangle = \frac{\hbar}{\sqrt{2}} (Z_1 \bar{Z}_2 + \bar{Z}_1 Z_2 + Z_2 \bar{Z}_3 + \bar{Z}_2 Z_3); \\ S_y(\Psi) &= \langle \Psi | \hat{S}_y | \Psi \rangle = \frac{i\hbar}{\sqrt{2}} (Z_1 \bar{Z}_2 - \bar{Z}_1 Z_2 + Z_2 \bar{Z}_3 - \bar{Z}_2 Z_3); \\ S_z(\Psi) &= \langle \Psi | \hat{S}_z | \Psi \rangle = \hbar (|Z_1|^2 - |Z_3|^2), \end{aligned}$$

and based on the equation (2), one can define Schrödinger vector fields correspond to these operators as

$$
\begin{aligned}
X_{\hat{S}_x}|\Psi\rangle &= \frac{dZ_1}{dt}|e_1\rangle + \frac{dZ_2}{dt}|e_2\rangle + \frac{dZ_3}{dt}|e_3\rangle = -\frac{1}{i\hbar}\big(Z_1\hat{S}_x|e_1\rangle + Z_2\hat{S}_x|e_2\rangle + Z_3\hat{S}_x|e_3\rangle\big);\\
X_{\hat{S}_y}|\Psi\rangle &= \frac{dZ_1}{dt}|e_1\rangle + \frac{dZ_2}{dt}|e_2\rangle + \frac{dZ_3}{dt}|e_3\rangle = -\frac{1}{i\hbar}\big(Z_1\hat{S}_y|e_1\rangle + Z_2\hat{S}_y|e_2\rangle + Z_3\hat{S}_y|e_3\rangle\big);\\
X_{\hat{S}_z}|\Psi\rangle &= \frac{dZ_1}{dt}|e_1\rangle + \frac{dZ_2}{dt}|e_2\rangle + \frac{dZ_3}{dt}|e_3\rangle = -\frac{1}{i\hbar}\big(Z_1\hat{S}_z|e_1\rangle + Z_2\hat{S}_z|e_2\rangle + Z_3\hat{S}_z|e_3\rangle\big).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\langle e_1|X_{\hat{S}_x}|\Psi\rangle &= \frac{dZ_1}{dt} = -\frac{Z_2}{\sqrt{2}i};\\
\langle e_1|X_{\hat{S}_y}|\Psi\rangle &= \frac{dZ_1}{dt} = \frac{Z_2}{\sqrt{2}};\\
\langle e_1|X_{\hat{S}_z}|\Psi\rangle &= \frac{dZ_1}{dt} = iZ_1.
\end{aligned}
$$

and in the similar way, we find

$$
\begin{aligned}
\langle e_2|X_{\hat{S}_x}|\Psi\rangle &= \frac{dZ_2}{dt} = -\frac{(Z_1+Z_3)}{\sqrt{2}i};\\
\langle e_2|X_{\hat{S}_y}|\Psi\rangle &= \frac{dZ_2}{dt} = \frac{Z_3-Z_1}{\sqrt{2}};\\
\langle e_2|X_{\hat{S}_z}|\Psi\rangle &= \frac{dZ_2}{dt} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle e_3|X_{\hat{S}_x}|\Psi\rangle &= \frac{dZ_3}{dt} = -\frac{Z_2}{\sqrt{2}i};\\
\langle e_3|X_{\hat{S}_y}|\Psi\rangle &= \frac{dZ_3}{dt} = -\frac{Z_2}{\sqrt{2}};\\
\langle e_3|X_{\hat{S}_z}|\Psi\rangle &= \frac{dZ_3}{dt} = -iZ_3.
\end{aligned}
$$

Other than $\frac{dZ_1}{dt}$, $\frac{dZ_2}{dt}$ and $\frac{dZ_3}{dt}$, one is required to compute $\frac{d\bar{Z}_1}{dt}$, $\frac{d\bar{Z}_2}{dt}$ and $\frac{d\bar{Z}_3}{dt}$ since here we consider that $\mathbb{C}^3$ is complexification of real vector space $\mathbb{R}^6$. Hence, the complexified tangent space is spanned by 6 vectors; $\frac{\partial}{\partial Z_1}$, $\frac{\partial}{\partial Z_2}$, $\frac{\partial}{\partial Z_3}$, $\frac{\partial}{\partial \bar{Z}_1}$, $\frac{\partial}{\partial \bar{Z}_2}$, $\frac{\partial}{\partial \bar{Z}_3}$.

Let $\langle\Psi| = \bar{Z}_1\langle e_1| + \bar{Z}_2\langle e_2| + +\bar{Z}_2\langle e_2|$ be a state in dual Hilbert space $\mathcal{H}^*$. The Schrödinger vector fields with respect to operator $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are

$$
\begin{aligned}
\langle\Psi|X_{\hat{S}_x} &= \frac{d\bar{Z}_1}{dt}\langle e_1| + \frac{d\bar{Z}_2}{dt}\langle e_2| + \frac{d\bar{Z}_3}{dt}\langle e_3| = \frac{1}{i\hbar}\left(\bar{Z}_1\langle e_1|\hat{S}_x + \bar{Z}_2\langle e_2|\hat{S}_x + \bar{Z}_3\langle e_3|\hat{S}_x\right);\\
\langle\Psi|X_{\hat{S}_y} &= \frac{d\bar{Z}_1}{dt}\langle e_1| + \frac{d\bar{Z}_2}{dt}\langle e_2| + \frac{d\bar{Z}_3}{dt}\langle e_3| = \frac{1}{i\hbar}\left(\bar{Z}_1\langle e_1|\hat{S}_y + \bar{Z}_2\langle e_2|\hat{S}_y + \bar{Z}_3\langle e_3|\hat{S}_y\right);\\
\langle\Psi|X_{\hat{S}_z} &= \frac{d\bar{Z}_1}{dt}\langle e_1| + \frac{d\bar{Z}_2}{dt}\langle e_2| + \frac{d\bar{Z}_3}{dt}\langle e_3| = \frac{1}{i\hbar}\left(\bar{Z}_1\langle e_1|\hat{S}_z + \bar{Z}_2\langle e_2|\hat{S}_z + \bar{Z}_3\langle e_3|\hat{S}_z\right).
\end{aligned}
$$

Hence, it is clear that

$$
\langle\Psi|X_{\hat{S}_x}|e_1\rangle = \frac{d\bar{Z}_1}{dt} = \frac{\bar{Z}_2}{\sqrt{2}i};
$$

$$\langle\Psi|X_{\hat{S}_y}|e_1\rangle = \frac{d\bar{Z}_1}{dt} = \frac{\bar{Z}_2}{\sqrt{2}};$$
$$\langle\Psi|X_{\hat{S}_z}|e_1\rangle = \frac{d\bar{Z}_1}{dt} = -i\bar{Z}_1.$$

Besides, one finds that

$$\langle\Psi|X_{\hat{S}_x}|e_2\rangle = \frac{d\bar{Z}_2}{dt} = \frac{\bar{Z}_1+\bar{Z}_3}{\sqrt{2}i};$$
$$\langle\Psi|X_{\hat{S}_y}|e_2\rangle = \frac{d\bar{Z}_2}{dt} = \frac{\bar{Z}_3-\bar{Z}_1}{\sqrt{2}};$$
$$\langle\Psi|X_{\hat{S}_z}|e_2\rangle = \frac{d\bar{Z}_2}{dt} = 0,$$

and

$$\langle\Psi|X_{\hat{S}_x}|e_3\rangle = \frac{d\bar{Z}_3}{dt} = \frac{\bar{Z}_2}{\sqrt{2}i};$$
$$\langle\Psi|X_{\hat{S}_y}|e_3\rangle = \frac{d\bar{Z}_3}{dt} = -\frac{\bar{Z}_2}{\sqrt{2}};$$
$$\langle\Psi|X_{\hat{S}_z}|e_3\rangle = \frac{d\bar{Z}_3}{dt} = i\bar{Z}_3.$$

Therefore, the Schrödinger vector fields correspond to $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are

$$X_{\hat{S}_x} = -\frac{Z_2}{i\sqrt{2}}\frac{\partial}{\partial Z_1} - \frac{(Z_1+Z_3)}{i\sqrt{2}}\frac{\partial}{\partial Z_2} - \frac{Z_2}{i\sqrt{2}}\frac{\partial}{\partial Z_3} + \frac{\bar{Z}_2}{i\sqrt{2}}\frac{\partial}{\partial \bar{Z}_1} + \frac{(\bar{Z}_1+\bar{Z}_3)}{i\sqrt{2}}\frac{\partial}{\partial \bar{Z}_2} + \frac{\bar{Z}_2}{i\sqrt{2}}\frac{\partial}{\partial \bar{Z}_3}; \quad (3)$$
$$X_{\hat{S}_y} = \frac{Z_2}{\sqrt{2}}\frac{\partial}{\partial Z_1} + \frac{Z_3-Z_1}{\sqrt{2}}\frac{\partial}{\partial Z_2} - \frac{Z_2}{\sqrt{2}}\frac{\partial}{\partial Z_3} + \frac{\bar{Z}_2}{\sqrt{2}}\frac{\partial}{\partial \bar{Z}_1} + \frac{\bar{Z}_3-\bar{Z}_1}{\sqrt{2}}\frac{\partial}{\partial \bar{Z}_2} - \frac{\bar{Z}_2}{\sqrt{2}}\frac{\partial}{\partial \bar{Z}_3}; \quad (4)$$
$$X_{\hat{S}_z} = iZ_1\frac{\partial}{\partial Z_1} - iZ_3\frac{\partial}{\partial Z_3} - i\bar{Z}_1\frac{\partial}{\partial \bar{Z}_1} + i\bar{Z}_3\frac{\partial}{\partial \bar{Z}_3}. \quad (5)$$

The solutions of Z_1 and Z_2 according to $X_{\hat{S}_x}$ are computed as follows. From equation (3) we can show that

$$\frac{dZ_1}{dt} = -\frac{Z_2}{i\sqrt{2}}, \quad \frac{dZ_2}{dt} = -\frac{(Z_1+Z_3)}{i\sqrt{2}}, \quad \frac{dZ_3}{dt} = -\frac{Z_2}{i\sqrt{2}}.$$

Rearrange the equations, one can obtain

$$-i\sqrt{2}\frac{dZ_1}{dt} = Z_2; \quad (6)$$
$$-i\sqrt{2}\frac{dZ_2}{dt} = Z_1 + Z_3; \quad (7)$$
$$-i\sqrt{2}\frac{dZ_3}{dt} = Z_2. \quad (8)$$

From equations (6) and (8), we notice that $Z_1 = Z_3$ and equation (7) becomes

$$-\frac{i\sqrt{2}}{2}\frac{dZ_2}{dt} = Z_1 = Z_3. \quad (9)$$

Substitute equation (9) into (6)

$$-i\sqrt{2}\frac{d}{dt}\left(-\frac{i\sqrt{2}}{2}\frac{dZ_2}{dt}\right) = Z_2;$$

$$\frac{d^2 Z_2}{dt^2} + Z_2 = 0. \tag{10}$$

It is obvious that, the general solution for equation (10) is

$$Z_2(t) = Ae^{it} + Be^{-it},$$

where A and B are complex numbers.

From equation (9)

$$\begin{aligned} Z_1 = Z_3 &= -\frac{i\sqrt{2}}{2}\frac{d}{dt}(Ae^{it} + Be^{-it}) \\ &= -\frac{i\sqrt{2}}{2}(iAe^{it} - iBe^{-it}). \end{aligned}$$

Thus, one obtains

$$Z_1(t) = Z_3(t) = \frac{\sqrt{2}}{2}Ae^{it} - \frac{\sqrt{2}}{2}Be^{-it}.$$

Furthermore, we calculate the solution for Z_1 and Z_2 with respect to $X_{\hat{S}_y}$. Referring to equation (4), it is obvious that

$$\frac{dZ_1}{dt} = \frac{Z_2}{\sqrt{2}}, \quad \frac{dZ_2}{dt} = \frac{(Z_3 - Z_1)}{\sqrt{2}}, \quad \frac{dZ_3}{dt} = -\frac{Z_2}{\sqrt{2}}.$$

Rearrange the equations, we obtain

$$\sqrt{2}\frac{dZ_1}{dt} = Z_2; \tag{11}$$

$$\sqrt{2}\frac{dZ_2}{dt} = Z_3 - Z_1; \tag{12}$$

$$-\sqrt{2}\frac{dZ_3}{dt} = Z_2. \tag{13}$$

From equations (11) and (13), we notice that $Z_1 = -Z_3$ and equation (12) becomes

$$\frac{\sqrt{2}}{2}\frac{dZ_2}{dt} = -Z_1 = Z_3. \tag{14}$$

Substitute equation (14) into (13)

$$\sqrt{2}\frac{d}{dt}\left(-\frac{\sqrt{2}}{2}\frac{dZ_2}{dt}\right) = Z_2;$$

$$\frac{d^2 Z_2}{dt^2} + Z_2 = 0. \tag{15}$$

It is obvious that, the general solution for equation (15) is

$$Z_2(t) = Ce^{it} + De^{-it},$$

where C and D are complex numbers. From equation (14)

$$\begin{aligned} Z_1 = -Z_3 &= -\frac{\sqrt{2}}{2}\frac{d}{dt}(Ce^{it} + De^{-it}) \\ &= -\frac{\sqrt{2}}{2}(iCe^{it} - iDe^{-it}). \end{aligned}$$

Thus, one obtains

$$Z_1(t) = -Z_3(t) = -\frac{i\sqrt{2}}{2}Ce^{it} + \frac{i\sqrt{2}}{2}De^{-it}.$$

Lastly, we find a solution of Z_1 and Z_2 for the case of $X_{\hat{S}_z}$. According to equation (5), one can show that

$$\frac{dZ_1}{dt} = iZ_1; \tag{16}$$

$$\frac{dZ_2}{dt} = -iZ_2. \tag{17}$$

Solve the equation (16), we get

$$Z_1(t) = Ee^{it}, \quad E = \text{const}\,.$$

Solve the equation (17), one obtains

$$Z_2(t) = Fe^{-it}, \quad F = \text{const}\,.$$

Note that, in $\mathbb{C}^3$, the Schrödinger equation defines a hamiltonian system with respect to the symplectic form Ω stated as

$$\Omega = i\hbar\, dZ_1 \wedge d\bar{Z}_1 + i\hbar\, dZ_2 \wedge d\bar{Z}_2 + i\hbar\, dZ_3 \wedge d\bar{Z}_3$$

and the Riemannian metric is represented by

$$G = \hbar\, dZ_1 d\bar{Z}_1 + \hbar\, dZ_2 d\bar{Z}_2 + \hbar\, dZ_3 d\bar{Z}_3.$$

Now we are ready to calculate the Robertson–Schrödinger uncertainty principle for the case of spin 1 particle. The uncertainty principles for this case are

$$(\Delta\hat{S}_y)^2(\Delta\hat{S}_z)^2 \geqslant \left(\frac{\hbar}{2}\Omega(X_{\hat{S}_y}, X_{\hat{S}_z})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{S}_y}, X_{\hat{S}_z}) - S_y S_z\right)^2; \tag{18}$$

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_z)^2 \geqslant \left(\frac{\hbar}{2}\Omega(X_{\hat{S}_x}, X_{\hat{S}_z})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{S}_x}, X_{\hat{S}_z}) - S_x S_z\right)^2; \tag{19}$$

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 \geqslant \left(\frac{\hbar}{2}\Omega(X_{\hat{S}_x}, X_{\hat{S}_y})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{S}_x}, X_{\hat{S}_y}) - S_x S_y\right)^2, \tag{20}$$

where the contraction of these Schrödinger vector fields with symplectic form Ω are given by

$$\Omega(X_{\hat{S}_y}, X_{\hat{S}_z}) = \iota_{X_{\hat{S}_y}}\iota_{X_{\hat{S}_z}}\Omega = -\frac{\hbar}{\sqrt{2}}(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2 + Z_2\bar{Z}_3 + \bar{Z}_2 Z_3);$$

$$\Omega(X_{\hat{S}_x}, X_{\hat{S}_z}) = \iota_{X_{\hat{S}_x}}\iota_{X_{\hat{S}_z}}\Omega = \frac{i\hbar}{\sqrt{2}}(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2 + Z_2\bar{Z}_3 - \bar{Z}_2 Z_4);$$

$$\Omega(X_{\hat{S}_x}, X_{\hat{S}_y}) = \iota_{X_{\hat{S}_x}}\iota_{X_{\hat{S}_y}}\Omega = \hbar(|Z_3|^2 - |Z_1|^2),$$

and the components of Riemannian metric correspond to the vectors are

$$G(X_{\hat{S}_y}, X_{\hat{S}_z}) = \frac{i\hbar}{\sqrt{2}}(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2 + \bar{Z}_2 Z_3 - Z_2\bar{Z}_3);$$

$$G(X_{\hat{S}_x}, X_{\hat{S}_z}) = \frac{\hbar}{\sqrt{2}}(\bar{Z}_1 Z_2 - Z_2\bar{Z}_3 + Z_1\bar{Z}_2 - \bar{Z}_2 Z_3);$$

$$G(X_{\hat{S}_x}, X_{\hat{S}_y}) = i\hbar(Z_1\bar{Z}_3 - \bar{Z}_1 Z_3).$$

Then, we can express the equations (18), (19) and (20) as

$$(\Delta\hat{S}_y)^2(\Delta\hat{S}_z)^2 \geqslant \left[-\frac{\hbar^2}{2\sqrt{2}}(Z_1\bar{Z}_2+\bar{Z}_1Z_2+Z_2\bar{Z}_3+\bar{Z}_2Z_3)\right]^2$$
$$+\left[\frac{i\hbar^2}{2\sqrt{2}}(Z_1\bar{Z}_2-\bar{Z}_1Z_2+\bar{Z}_2Z_3-Z_2\bar{Z}_3)-\frac{i\hbar^2}{\sqrt{2}}(Z_1\bar{Z}_2-\bar{Z}_1Z_2+Z_2\bar{Z}_3-\bar{Z}_2Z_4)(|Z_1|^2-|Z_3|^2)\right]^2;$$
$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_z)^2 \geqslant \left[\frac{i\hbar^2}{2\sqrt{2}}(Z_1\bar{Z}_2-\bar{Z}_1Z_2+Z_2\bar{Z}_3-\bar{Z}_2Z_3)\right]^2$$
$$+\left[\frac{\hbar^2}{2\sqrt{2}}(\bar{Z}_1Z_2-Z_2\bar{Z}_3+Z_1\bar{Z}_2-\bar{Z}_2Z_3)-\frac{\hbar^2}{\sqrt{2}}(Z_1\bar{Z}_2+\bar{Z}_1Z_2+Z_2\bar{Z}_3+\bar{Z}_2Z_3)(|Z_1|^2-|Z_3|^2)\right]^2;$$
$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 \geqslant \left[\frac{\hbar^2}{2}(|Z_2|^2-|Z_1|^2)\right]^2$$
$$+\left[\frac{i\hbar^2}{2}(Z_1\bar{Z}_3-\bar{Z}_1Z_3)-\frac{i\hbar^2}{2}(Z_1\bar{Z}_2+\bar{Z}_1Z_2+Z_2\bar{Z}_3+\bar{Z}_2Z_3)(Z_1\bar{Z}_2-\bar{Z}_1Z_2+Z_2\bar{Z}_3-\bar{Z}_2Z_4)\right]^2.$$

Thus, the evolution of Robertson-Schrödinger uncertainty principles correspond to operators

1. $\hat{S}_y$ and $\hat{S}_z$ along $X_{\hat{S}_x}$ is
$$(\Delta\hat{S}_y)^2(\Delta\hat{S}_z)^2 \geqslant \left[\hbar^2(|B|^2-|A|^2)\right]^2+\left[i\hbar^2\left(A\bar{B}e^{2it}-\bar{A}Be^{-2it}\right)\right]^2;$$
2. $\hat{S}_x$ and $\hat{S}_z$ along $X_{\hat{S}_y}$ is
$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_z)^2 \geqslant \left[\hbar^2(|D|^2-|C|^2)\right]^2+\left[-i\hbar^2\left(C\bar{D}e^{2it}-\bar{C}De^{-2it}\right)\right]^2;$$
3. $\hat{S}_x$ and $\hat{S}_y$ along $X_{\hat{S}_z}$ is
$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 \geqslant \left[\frac{\hbar^2}{2}(|F|^2-|E|^2)\right]^2+\left[\frac{i\hbar^2}{2}\left(E\bar{F}e^{2it}-\bar{E}Fe^{-2it}\right)\right]^2.$$

Note that, similar to the case of $\mathbb{C}^2$, any state vector $\Psi,\Phi\in\mathcal{H}$ such that $\Psi=c\Phi$, $c\in\mathbb{C}$ has different expression of uncertainty principles although these state vector represent the same physical state. Therefore in this context, it is necessarily to find the expression of Robertson–Schrödinger uncertainty principle in $\mathbb{C}P^2$ which is the quantum phase space of spin 1 particle.

4. The evolution of uncertainty principle in projective Hilbert space $\mathbb{C}P^2$

In order to compute the Robertson–Schrödinger uncertainty principle on $\mathbb{C}P^2$, we need to find the pushforward vector field of $X_{\hat{S}_x}$, $X_{\hat{S}_y}$ and $X_{\hat{S}_z}$ under the map $\Pi_*\colon T_\Psi\mathcal{H}\to T_\psi P(\mathcal{H})$. Let $\Pi(Z_1,Z_2,Z_3)=(z_1,z_2)=\left(\frac{Z_2}{Z_1},\frac{Z_3}{Z_1}\right)$ be a local coordinate of U_1, where $Z_1\neq 0$. Firstly we compute the pushforward bases as follow

$$\Pi_*\left(\frac{\partial}{\partial Z_1}\right)=\frac{\partial z_1}{\partial Z_1}\frac{\partial}{\partial z_1}+\frac{\partial z_2}{\partial Z_1}\frac{\partial}{\partial z_2}=\frac{\partial}{\partial Z_1}\left(\frac{Z_2}{Z_1}\right)\frac{\partial}{\partial z_1}+\frac{\partial}{\partial Z_1}\left(\frac{Z_3}{Z_1}\right)\frac{\partial}{\partial z_2}=-\frac{Z_2}{Z_1^2}\frac{\partial}{\partial z_1}-\frac{Z_3}{Z_1^2}\frac{\partial}{\partial z_2};$$
$$\Pi_*\left(\frac{\partial}{\partial Z_2}\right)=\frac{\partial z_1}{\partial Z_2}\frac{\partial}{\partial z_1}+\frac{\partial z_2}{\partial Z_2}\frac{\partial}{\partial z_2}=\frac{\partial}{\partial Z_2}\left(\frac{Z_2}{Z_1}\right)\frac{\partial}{\partial z_1}+\frac{\partial}{\partial Z_2}\left(\frac{Z_3}{Z_1}\right)\frac{\partial}{\partial z_2}=\frac{1}{Z_1}\frac{\partial}{\partial z_1};$$
$$\Pi_*\left(\frac{\partial}{\partial Z_3}\right)=\frac{\partial z_1}{\partial Z_3}\frac{\partial}{\partial z_1}+\frac{\partial z_2}{\partial Z_3}\frac{\partial}{\partial z_2}=\frac{\partial}{\partial Z_3}\left(\frac{Z_2}{Z_1}\right)\frac{\partial}{\partial z_1}+\frac{\partial}{\partial Z_3}\left(\frac{Z_3}{Z_1}\right)\frac{\partial}{\partial z_2}=\frac{1}{Z_1}\frac{\partial}{\partial z_2};$$
$$\Pi_*\left(\frac{\partial}{\partial \bar{Z}_1}\right)=\frac{\partial \bar{z}_1}{\partial \bar{Z}_1}\frac{\partial}{\partial \bar{z}_1}+\frac{\partial \bar{z}_2}{\partial \bar{Z}_1}\frac{\partial}{\partial \bar{z}_2}=\frac{\partial}{\partial \bar{Z}_1}\left(\frac{\bar{Z}_2}{\bar{Z}_1}\right)\frac{\partial}{\partial \bar{z}_1}+\frac{\partial}{\partial \bar{Z}_1}\left(\frac{\bar{Z}_3}{\bar{Z}_1}\right)\frac{\partial}{\partial \bar{z}_2}=-\frac{\bar{Z}_2}{\bar{Z}_1^2}\frac{\partial}{\partial \bar{z}_1}-\frac{\bar{Z}_3}{\bar{Z}_1^2}\frac{\partial}{\partial \bar{z}_2};$$

$$\Pi_*\left(\frac{\partial}{\partial \bar{Z}_2}\right) = \frac{\partial \bar{z}_1}{\partial \bar{Z}_2}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial \bar{z}_2}{\partial \bar{Z}_2}\frac{\partial}{\partial \bar{z}_2} = \frac{\partial}{\partial \bar{Z}_2}\left(\frac{\bar{Z}_2}{\bar{Z}_1}\right)\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{Z}_2}\left(\frac{\bar{Z}_3}{\bar{Z}_1}\right)\frac{\partial}{\partial \bar{z}_2} = \frac{1}{\bar{Z}_1}\frac{\partial}{\partial \bar{z}_1};$$

$$\Pi_*\left(\frac{\partial}{\partial \bar{Z}_3}\right) = \frac{\partial \bar{z}_1}{\partial \bar{Z}_3}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial \bar{z}_2}{\partial \bar{Z}_3}\frac{\partial}{\partial \bar{z}_2} = \frac{\partial}{\partial \bar{Z}_3}\left(\frac{\bar{Z}_2}{\bar{Z}_1}\right)\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{Z}_3}\left(\frac{\bar{Z}_3}{\bar{Z}_1}\right)\frac{\partial}{\partial \bar{z}_2} = \frac{1}{\bar{Z}_1}\frac{\partial}{\partial \bar{z}_2}.$$

Then, the pushforward vector field corresponding to $X_{\hat{S}_x}$, $X_{\hat{S}_y}$ and $X_{\hat{S}_z}$ are

$$\begin{aligned}
\Pi_* X_{\hat{S}_x} =& \frac{1}{i\sqrt{2}}(z_1^2 - z_2 - 1)\frac{\partial}{\partial z_1} + \frac{1}{i\sqrt{2}}(z_1 z_2 - z_1)\frac{\partial}{\partial z_2} \\
&- \frac{1}{i\sqrt{2}}(\bar{z}_1^2 - \bar{z}_2 - 1)\frac{\partial}{\partial \bar{z}_1} - \frac{1}{i\sqrt{2}}(\bar{z}_1\bar{z}_2 - \bar{z}_1)\frac{\partial}{\partial \bar{z}_2}; \\
\Pi_* X_{\hat{S}_y} =& -\frac{1}{\sqrt{2}}(z_1^2 - z_2 + 1)\frac{\partial}{\partial z_1} - \frac{1}{\sqrt{2}}(z_1 z_2 + z_1)\frac{\partial}{\partial z_2} \\
&- \frac{1}{\sqrt{2}}(\bar{z}_1^2 - \bar{z}_2 + 1)\frac{\partial}{\partial \bar{z}_1} - \frac{1}{\sqrt{2}}(\bar{z}_1\bar{z}_2 + \bar{z}_1)\frac{\partial}{\partial \bar{z}_2}; \\
\Pi_* X_{\hat{S}_z} =& -iz_1\frac{\partial}{\partial z_1} - 2iz_2\frac{\partial}{\partial z_2} + i\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + 2i\bar{z}_1\frac{\partial}{\partial \bar{z}_1}.
\end{aligned}$$

Note that, in $\mathbb{C}P^2$, the symplectic form ω is expressed as

$$\omega = i\hbar\left[\frac{(1+|z|^2)dz_1\wedge d\bar{z}_1 - \bar{z}_1 dz_1 \wedge z_1 d\bar{z}_1}{(1+|z|^2)^2} + \frac{(1+|z|^2)dz_2\wedge d\bar{z}_2 - \bar{z}_2 dz_2\wedge z_2 d\bar{z}_2}{(1+|z|^2)^2}\right]$$

and the Riemannian metric is given by

$$g = 2\hbar\left[\frac{(1+|z_2|^2)dz_1 d\bar{z}_1 - \bar{z}_1 z_2 dz_1 d\bar{z}_2}{(1+|z|^2)^2} + \frac{(1+|z|^2)dz_2 d\bar{z}_2 - \bar{z}_2 z_1 dz_2 d\bar{z}_2}{(1+|z|^2)^2}\right]$$

where $|z|^2 = |z_1|^2 + |z_2|^2$.

Now we are ready to compute the Robertson–Schrödinger uncertainty principle for the case of spin 1 on $\mathbb{C}P^2$. Let us define the Robertson–Schrödinger uncertainty relation corresponding to the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ as

$$\begin{aligned}
(\Delta\hat{S}_y)^2(\Delta\hat{S}_z)^2 &\geqslant \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2 + g(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2]; \\
(\Delta\hat{S}_x)^2(\Delta\hat{S}_z)^2 &\geqslant \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2 + g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2]; \\
(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 &\geqslant \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2 + g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2],
\end{aligned}$$

where the contraction of pushforward vector fields with symplectic form ω are

$$\begin{aligned}
\omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z}) &= \iota_{\Pi_* X_{\hat{S}_y}}\iota_{\Pi_* X_{\hat{S}_z}}\omega = \frac{\hbar}{\sqrt{2}}\frac{(z_1 + \bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1 z_2)}{1+|z_1|^2+|z_2|^2}; \\
\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z}) &= \iota_{\Pi_* X_{\hat{S}_x}}\iota_{\Pi_* X_{\hat{S}_z}}\omega = -\frac{i\hbar}{\sqrt{2}}\frac{(z_1 - \bar{z}_1 - z_1\bar{z}_2 + \bar{z}_1 z_2)}{1+|z_1|^2+|z_2|^2}; \\
\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y}) &= \iota_{\Pi_* X_{\hat{S}_x}}\iota_{\Pi_* X_{\hat{S}_y}}\omega = \frac{\hbar(|z_2|^2 - 1)}{1+|z_1|^2+|z_2|^2},
\end{aligned}$$

and the components of Riemannian metric with respect to these vectors are

$$\begin{aligned}
g(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z}) =& -\frac{i\hbar}{\sqrt{2}}\frac{[(z_1 - \bar{z}_1 + z_1\bar{z}_2 - \bar{z}_1 z_2)|z_1|^2 + (-3z_1 + 3\bar{z}_1 + z_1\bar{z}_2 - \bar{z}_1 z_2)|z_2|^2]}{(1+|z_1|^2+|z_2|^2)} \\
&+ \frac{i\hbar}{\sqrt{2}}\frac{[(1-3\bar{z}_2)z_1 - (1-3z_2)\bar{z}_1]}{(1+|z_1|^2+|z_2|^2)};
\end{aligned}$$

$$g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z}) = \frac{\hbar}{\sqrt{2}} \frac{[(z_1 + \bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2)|z_1|^2 + (3z_1 + 3\bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1 z_2)|z_2|^2]}{(1 + |z_1|^2 + |z_2|^2)} - \frac{\hbar}{\sqrt{2}} \frac{[(1 + 3\bar{z}_2)z_1 + (1 + 3z_2)\bar{z}_1]}{(1 + |z_1|^2 + |z_2|^2)};$$

$$g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y}) = \frac{i\hbar[(z_2 - \bar{z}_2)(|z_1|^2 - |z_2|^2) + (1 - \bar{z}_2^2)z_1^2 - (1 - z_2^2)\bar{z}_1^2 - z_2 + \bar{z}_2]}{(1 + |z_1|^2 + |z_2|^2)^2}.$$

Thus, the evolution of Robertson–Schrödinger uncertainty principle for the case of

1. $\hat{S}_y$ and $\hat{S}_z$ is

$$(\Delta\hat{S}_y)^2(\Delta\hat{S}_z)^2 \geqslant \left[(\hbar^2[|B|^2 - |A|^2])^2 + (i\hbar^2[A\bar{B}e^{2it} - \bar{A}Be^{-2it}])^2\right]$$

along the projection of solution associated with Hamiltonian vector field $X_{\hat{S}_x}$

$$z_1(t) = \frac{Z_2(t)}{Z_1(t)} = \frac{Ae^{it} + Be^{-it}}{\frac{\sqrt{2}}{2}Ae^{it} - \frac{\sqrt{2}}{2}Be^{-it}};$$

$$z_2(t) = \frac{Z_3(t)}{Z_1(t)} = \frac{\frac{\sqrt{2}}{2}Ae^{it} - \frac{\sqrt{2}}{2}Be^{-it}}{\frac{\sqrt{2}}{2}Ae^{it} - \frac{\sqrt{2}}{2}Be^{-it}} = 1.$$

2. $\hat{S}_x$ and $\hat{S}_z$ is

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_z)^2 \geqslant \left[\left(\hbar^2[|C|^2 - |D|^2]\right)^2 + \left(-i\hbar^2[C\bar{D}e^{2it} - \bar{C}De^{-2it}]\right)^2\right]$$

along the projection of solution corresponding to Hamiltonian vector field $X_{\hat{S}_y}$

$$z_1(t) = \frac{Z_2(t)}{Z_1(t)} = \frac{Ce^{it} + De^{-it}}{-\frac{i\sqrt{2}}{2}Ce^{it} + \frac{i\sqrt{2}}{2}De^{-it}};$$

$$z_2(t) = \frac{Z_3(t)}{Z_1(t)} = \frac{\frac{i\sqrt{2}}{2}Ce^{it} - \frac{i\sqrt{2}}{2}De^{-it}}{-\frac{i\sqrt{2}}{2}Ce^{it} + \frac{i\sqrt{2}}{2}De^{-it}} = -1.$$

3. $\hat{S}_x$ and $\hat{S}_y$ is

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 \geqslant \left(\frac{\hbar^2}{2}\left[|F|^2 - |E|^2\right]^2\right) + \left(\frac{i\hbar^2}{2}\left[E\bar{F}e^{2it} - \bar{E}Fe^{-2it}\right]\right)^2$$

along the projection of solution corresponding to Hamiltonian vector field $X_{\hat{S}_z}$

$$z_1(t) = \frac{Z_2(t)}{Z_1(t)} = \frac{0}{Ee^{it}} = 0;$$

$$z_2(t) = \frac{Z_3(t)}{Z_1(t)} = \frac{Fe^{-it}}{Ee^{it}}.$$

5. Discussion and conclusion

In our computation previously, we start by examining the Robertson–Schrödinger uncertainty principle in Hilbert space $\mathbb{C}^3$. We show that the symplectic areas $\Omega(X_{\hat{S}_i}, X_{\hat{S}_j})$ uniquely preserved along X_{S_k}. This is based on the fact that $\Omega(X_{\hat{S}_i}, X_{\hat{S}_j}) = S_k$ and the evaluation function S_k is uniquely preserved along $X_{\hat{S}_k}$ since it satisfies the condition $\iota_{X_{\hat{S}_k}}\Omega = dS_k$.

Besides, we demonstrate that, the angles θ between $X_{\hat{S}_x}, X_{\hat{S}_y}$ and $X_{\hat{S}_z}$ vary under symplectic transformation since $G(X_{\hat{S}_i}, X_{\hat{S}_j})$ continuously change for all time t. It is clear that, the anti-commutator condition $[\hat{S}_i, \hat{S}_j]_+ \neq 0$ is manifested by non-orthogonality of vectors in geometric framework. Note that, although the angle and magnitude of $X_{\hat{S}_i}$ and $X_{\hat{S}_j}$ vary with time, they must preserve symplectic area $\Omega(X_{\hat{S}_i}, X_{\hat{S}_j})$.

According to the above equations, it is obvious that, the Robertson–Schrödinger uncertainty principles for spin 1 particle are not constant under any Hamiltonian flows. However, these results clearly show that for the case of Robertson uncertainty principle i.e.

$$(\Delta\hat{S}_i)^2(\Delta\hat{S}_j)^2 \geqslant \left(\frac{\hbar}{2}\Omega(X_{\hat{S}_i}, X_{\hat{S}_j})\right)^2,$$

it is invariant under Hamiltonian flow along $X_{\hat{S}_k}$.

Note that, although for any state vectors $\Psi, \Phi \in \mathcal{H}$ such that $\Phi = c\Psi$, $c \in \mathbb{C}$ define the same physical state, the expression of uncertainty principle for Φ is different compared to Ψ by factor $|c|^2$ as follows

$$(\Delta\hat{S}_i)^2(\Delta\hat{S}_j)^2 \geqslant \left(\frac{\hbar|c|^2}{2}\Omega(X_{\hat{S}_i}, X_{\hat{S}_j})\right)^2 + \left(\frac{\hbar|c|^2}{2}G(X_{\hat{S}_i}, X_{\hat{S}_j}) - [|c|^2 S_i][|c|^2 S_j]\right)^2$$

Thus, since the uncertainty principle's expression is not unique in $\mathbb{C}^3$, it is necessary to find the expression of Robertson–Schrödinger uncertainty principle in $\mathbb{C}P^2$ which are the quantum phase space for spin 1 particles.

In this space, the contraction of $\Pi_* X_{\hat{S}_i}$ and $\Pi_* X_{\hat{S}_j}$ with symplectic form $\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$ is invariant under projection of Hamiltonian flow induced by $X_{\hat{S}_k}$ implies that the area between vectors $\Pi_* X_{\hat{S}_i}$ and $\Pi_* X_{\hat{S}_j}$ is preserved under the transformation. This is because $\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j}) = \Omega(X_{\hat{S}_i}, X_{\hat{S}_j}) = S_k$ and the evaluation function S_k is uniquely conserved along $X_{\hat{S}_k}$ since it satisfies the condition $\iota_{X_{\hat{S}_k}}\Omega = dS_k$.

The fact that the Riemannian metric $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$ is non-zero since $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j}) = G(X_{\hat{S}_i}, X_{\hat{S}_j}) - \frac{2}{\hbar} S_i S_j$ and varies under any Hamiltonian flow shows that the magnitude and angle between $\Pi_* X_{\hat{S}_i}$ and $\Pi_* X_{\hat{S}_j}$ are changing under the transformation. However, these vectors preserve symplectic area $\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$. Besides that, unlike $\mathbb{C}^3$ the Riemannian metric $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$ represent the covariance since $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j}) = G(X_{\hat{S}_i}, X_{\hat{S}_j}) - \frac{2}{\hbar} S_i S_j$. The covariance is purely depends on $G(X_{\hat{S}_i}, X_{\hat{S}_j})$ due to the fact that $S_i S_j = 0$ along the flow.

Generally, we show that the Robertson–Schrödinger uncertainty principle in $\mathbb{C}P^2$ varies under any Hamiltonian flows. However similar to $\mathbb{C}^3$ the Robertson uncertainty principle i.e.

$$(\Delta\hat{S}_i)^2(\Delta\hat{S}_j)^2 \geqslant \left(\frac{\hbar}{2}\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})\right)^2,$$

in $\mathbb{C}P^2$ are invariant under projection of Hamiltonian flow generated by $X_{\hat{S}_k}$. This invariant property of Robertson uncertainty principle may become an excellent step to make a connection between geometric quantum mechanics and symplectic topology.

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Еволюція геометричного принципу невизначеності Робертсона–Шредінгера для системи зі спіном 1

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Геометрична квантова механіка — це математичний опис, який показує, як квантова теорія може бути виражена у термінах гамільтонової динаміки фазового простору. Стани є точками в комплексному проективному просторі Гільберта, спостережувані є дійсними функціями у цьому просторі, а гамільтоновий потік визначається рівнянням Шредінгера у цьому описі. Питання вираження принципа невизначеності на геометричній мові нещодавно стало центром значних досліджень у геометричній квантовій механіці. Було показано, що принцип невизначеності Робертсона–Шедінгера, який є більш сильною версією співвідношення невизначеності, може бути визначений з точки зору симплектичної форми та ріманівської метрики. На основі цього формулювання досліджуємо динамічну поведінку співвідношення невизначеності для системи зі спіном 1. Показуємо, що для гамільтонового потоку принципи невизначеності Робертсона–Шредінгера не є інваріантними. Це пояснюється тим, що, на відміну від симплектичної області, ріманова метрика не є інваріантною для гамільтонового потоку у процесі еволюції.

Ключові слова: *диференціальна геометрія, принцип невизначеності, геометрична квантова механіка, квантова динаміка, гамільтонова механіка.*