MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 3, pp. 519–535 (2022)

# Anisotropic parabolic problem with variable exponent and regular data

Mecheter Rabah

*Department of Mathematics and Informatics, University of M'sila, M'sila, Algeria,*
*Laboratory of Functional Analysis and Geometry of Spaces,*
*rabah.mecheter@univ-msila.dz*

(Received 19 November 2021; Revised 6 February 2022; Accepted 7 February 2022)

In this paper, we study the existence of weak solutions for a class of nonlinear parabolic equations with regular data in the setting of variable exponent Sobolev spaces. We prove a "version" of a weak Lebesgue space estimate that goes back to "*Lions J. L. Quelques méthodes de résolution des problèmes aux limites. Dunod, Paris (1969)*" for parabolic equations with anisotropic constant exponents ($p_i(\cdot) = p_i$).

**Keywords:** *anisotropic parabolic, nonlinear parabolic equations, regular data.*
**2010 MSC:** 35K10, 35K20, 35K55 **DOI:** 10.23939/mmc2022.03.519

## 1. Introduction

Let consider Dirichlet problem for the Nonlinear anisotropic parabolic equation in the variable exponent Sobolev space of the following type

$$(P_1) \qquad \begin{cases} \partial_t u + Au + F(t,x,u) = f \ \text{ in } \ Q_T \doteq \Omega \times ]0,T[; \\ u(0,x) = u_0(x) \ \text{ in } \ \Omega; \\ u = 0 \ \text{ on } \Gamma_T = ]0,T[\times \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded open set of $\mathbb{R}^N$ ($N \geqslant 2$) with a Lipschitz boundary denoted by $\partial\Omega$, $T > 0$ a real number, $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$, and $A$ is the operator given by

$$Au = -\operatorname{div}(\widehat{a}(t,x,Du)) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(t,x,D_i u)).$$

Suppose that $\widehat{a}(t,x,Du)$ and $F(t,x,u)$ are functions satisfying the conditions:
â.1) There exist two constants $\alpha > 0$ and $\beta > 0$ such that for almost everywhere $(t,x) \in Q_T$, $\forall\, u \in \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$, the function $\widehat{a}\colon Q_T \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following growths:

$$* \quad \widehat{a}(t,x,\xi)\xi \geqslant \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \quad \widehat{a}(\cdot) = (a_1(\cdot), \dots, a_N(\cdot)), \tag{1}$$

$$* \quad |a_i(t,x,\xi)| \leqslant \beta \Big( g(t,x) + \sum_{j=1}^{N} |\xi_j|^{p_j(x)} \Big)^{1 - \frac{1}{p_i(x)}}, \tag{2}$$

where $g \in L^1(Q_T)$ is a given positive function, and the variable exponents $p_i\colon \mathbb{R}^N \to (1,\infty)$ are continuous functions.
â.2) The mapping $\widehat{a}$ is a Carathéodory function, that is to say, the function $(t,x,\xi) \mapsto \widehat{a}(t,x,\xi)$ is measurable in $(t,x)$ for all $\xi \in \mathbb{R}^N$, and continuous in $\xi$ for a.e. $(t,x) \in Q_T$.
â.3) For a.e. $(t,x) \in Q_T$, and for all $\xi \neq \xi'$,

$$(\widehat{a}(t,x,\xi) - \widehat{a}(t,x,\xi'))(\xi - \xi') > 0, \tag{3}$$

This research is supported by the Ministry of Higher Education and Scientific Research of Algeria, Algiers University, Faculty of Sciences, Department of Mathematics, PRFU project C00L03UN160120190001.

Let $F\colon Q_T \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following conditions:

$$\sup_{|\sigma|\leqslant\lambda} |F(t,x,\sigma)| \in L^1(0,T;L^1(\Omega)), \quad \forall\lambda > 0, \tag{4}$$

$$F(t,x,u)\,\mathrm{sign}(u) \geqslant 0, \text{ a.e. } (t,x) \in Q_T, \tag{5}$$

for all $u \in \mathbb{R}$.

**Example 1.** The example model for the operator $A$ satisfies the conditions $(\widehat{a}.1-3)$, is given by

$$Av = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial v}{\partial x_i}\right), \quad v \in W_0^{1,p_i(\cdot)}(\Omega).$$

## 2. Preliminary work

We present the anisotropic Sobolev space with variable exponent which is used for the study of problem $(P1)$. For further details see for example [5, 9]

Let $p_i(\cdot)\colon \Omega \to [1,\infty)$ be a continuous function for all $i = 1,\ldots,N$. We note by

$$p_i^- = \min_{x\in\overline{\Omega}}\{p_i(x)\}, \qquad p_i^+ = \max_{x\in\overline{\Omega}}\{p_i(x)\}.$$

The anisotropic variable exponent Sobolev space $W^{1,p_i(\cdot)}(\Omega)$ is defined by

$$W^{1,p_i(\cdot)}(\Omega) = \left\{u \in L^{p_i(\cdot)}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_i = \|u\|_{L^{p_i^-}(\Omega)} + \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad i = 1,\ldots,N \tag{6}$$

and the variable exponent Sobolev space $W_0^{1,p_i(\cdot)}(\Omega)$ is introduced:

$$W_0^{1,p_i(\cdot)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega)\right\},$$

with respect to the norm (6).

**Theorem 1 (Ref. [8]).** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p_i(\cdot) > 1$ are continuous functions. Suppose that*

$$p_i(x) < \overline{p}^*(x),$$

*where*

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{if } \overline{p}(x) < N, \\ +\infty, & \text{if } \overline{p}(x) \geqslant N \end{cases}$$

*and $\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}$. Then the following Poincaré-type inequality holds:*

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leqslant C\sum_{i=1}^{N}\|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \qquad \forall u \in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega).$$

*where $C$ is a positive constant independent on $u$ and $p_+(\cdot) = \max\{p_1(x),\ldots,p_N(x)\}$, $x \in \overline{\Omega}$. Thus, $\sum_{i=1}^{N}\|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega)$.*

**Remark 1.** Remark that if $\|u\|_i$ is finite, then $\|D_i u\|_{L^{p_i(\cdot)}(\Omega)} \leqslant C$ and using the theorem 1, we have $\|u\|_{L^{p_+(\cdot)}(\Omega)} \leqslant C$ and therefore $\|u\|_{L^{p_i(\cdot)}(\Omega)} \leqslant C$.

Suppose that

$$1 + \frac{N}{N+1} < p_i(\cdot) < \overline{p}^*(x), \quad i = 1,\ldots,N. \tag{7}$$

Anisotropic spaces (see [18]) are defined as,

$$\bigcap_{i=1}^{N} L^{p_i^-}\left(0,T;W_0^{1,p_i(\cdot)}(\Omega)\right) \doteq \Bigg\{v\colon [0,T]\to \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega) \quad \text{measurable} \quad \sum_{i=1}^{N}\int_0^T \|v\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)}\,dt<\infty,\ i=1,\dots,N\Bigg\},$$

the norm in this space is given by

$$\|v\| = \sum_{i=1}^{N}\Big(\int_0^T \|v\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)}dt\Big)^{1/p_i^-}.$$

**Lemma 1 (Ref. [14]).** *Let $f$ be a strictly positive measurable function. Then for all $\varepsilon>0$ it exists $\delta>0$ such that for all measurable $A\subset\Omega$,*

$$\int_A f\,dx<\delta \Rightarrow \operatorname{meas}(A)<\varepsilon. \tag{8}$$

**Lemma 2 (Ref. [1]).** *Let $p(\cdot)\in C^+(\overline{\Omega})$, then for every $f\in L^{p(\cdot)}(\Omega)$*

$$\min\left\{\|f\|^{p^-}_{p(\cdot)},\|f\|^{p^+}_{p(\cdot)}\right\}\leqslant \rho_{p(\cdot)}(f)\leqslant \max\left\{\|f\|^{p^-}_{p(\cdot)},\|f\|^{p^+}_{p(\cdot)}\right\}. \tag{9}$$

$$\min\left\{\rho_{p(\cdot)}(f)^{1/p^-},\rho_{p(\cdot)}(f)^{1/p^+}\right\}\leqslant \|f\|_{p(\cdot)}\leqslant \max\left\{\rho_{p(\cdot)}(f)^{1/p^-},\rho_{p(\cdot)}(f)^{1/p^+}\right\}. \tag{10}$$

**Remark 2 (Ref. [4]).** Let $\Omega\subseteq\mathbb{R}^N$, $Q=(0,T)\times\Omega$, and $p_i\colon\Omega\to(1,\infty)$ be a continuous function. The following continuous dense embeddings are true

$$L^{p_i^+}\left(0,T;L^{p_i(\cdot)}(\Omega)\right)\hookrightarrow L^{p_i(\cdot)}(Q)\hookrightarrow L^{p_i^-}\left(0,T;L^{p_i(\cdot)}(\Omega)\right).$$

### 2.1. The anisotropic spaces $W(Q_T)$

By $\mathbf{W}(Q_T)$ we denote the Banach space, see [2]

$$\mathbf{W}(Q_T)=\left\{u\colon[0,T]\to\bigcap_{i=1}^{N}W_0^{1,p_i(\cdot)}(\Omega)\mid u\in L^{p_i(\cdot)}(Q_T), D_iu\in L^{p_i(\cdot)}(Q_T),\ u=0\ \text{ on }\ \Gamma_T\right\}.$$

quipped with the following norm

$$\|u\|_{\mathbf{W}(Q_T)}=\sum_i\left(\|u\|_{L^{p_i(\cdot)}(Q_T)}+\|D_iu\|_{L^{p_i(\cdot)}(Q_T)}\right).$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$v\in\mathbf{W}'(Q_T)\Leftrightarrow\begin{cases} v=\sum_i(v_i+D_iv_i), & v_i\in L^{p_i'(\cdot)}(Q_T),\\ \forall\phi\in\mathbf{W}(Q_T), & \langle v,\phi\rangle=\int_{Q_T}\sum_i(v_i\phi+v_iD_i\phi)\,dx\,dt.\end{cases}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)}=\sup_{\substack{\phi\in\mathbf{W}(Q_T)\\ \|\phi\|\leqslant1}}|\langle v,\phi\rangle|$$

**Proposition 1.** Let $p_i(\cdot)\colon \Omega \to (1,\infty)$ be a continuous function. We have the following continuous embedding

$$\mathbf{W}(Q_T) \hookrightarrow \bigcap_{i=1}^{N} L^{p_i^-}\left(0,T; W_0^{1,p_i(\cdot)}(\Omega)\right).$$

**Proof.** Let $v \in \mathbf{W}(Q_T)$, using (10) and Hölder inequality, one can find the estimate

$$\begin{aligned}\int_0^T \|v(t,\cdot)\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)} dt &= \int_0^T \left(\|v\|_{L^{p_i^-}(\Omega)} + \|D_i v\|_{L^{p_i(\cdot)}(\Omega)}\right)^{p_i^-} dt \\ &\leqslant C \int_0^T \|v\|^{p_i^-}_{L^{p_i^-}(\Omega)} + \left[\max\left(\rho_{p_i(\cdot)}(D_i v)^{1/p_i^-}, \rho_{p_i(\cdot)}(D_i v)^{1/p_i^+}\right)\right]^{p_i^-} dt \\ &\leqslant C \int_0^T \int_\Omega |v|^{p_i^-} dx + \max\left(\rho_{p_i(\cdot)}(D_i v), \rho_{p_i(\cdot)}(D_i v)^{p_i^-/p_i^+}\right) dt.\end{aligned}$$

So,

$$\begin{aligned}&\int_0^T \|v(t,\cdot)\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)} dt \\ &\qquad \leqslant \int_{Q_T} |v|^{p_i(x)} dx\,dt + \int_{Q_T} |D_i v|^{p_i(x)} dx\,dt + T^{1-p_i^-/p_i^+}\left(\int_{Q_T} |D_i v|^{p_i(x)} dx\,dt\right)^{p_i^-/p_i^+} < \infty.\end{aligned}$$

■

**Lemma 3.** *The operator $A$ maps $W(Q_T)$ into $W'(Q_T)$.*

**Proof.** In fact, if for $u \in \mathbf{W}(Q_T)$, we put

$$Au = -\operatorname{div}(\widehat{a}(t,x,Du),$$

then

$$\begin{aligned}\|Au\|_{\mathbf{W}'(Q_T)} &= \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} |\langle Au,\varphi\rangle| = \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \left|\int_{Q_T} \hat{a}(t,x,Du) D\varphi\,dx\,dt\right| \\ &= \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \left|\int_{Q_T} \sum_{i=1}^N a_i(t,x,Du) D_i\varphi\,dx\,dt\right| \leqslant \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \int_{Q_T} |a_i(t,x,Du)||D_i\varphi|\,dx\,dt.\end{aligned}$$

Using Hölder's inequality,

$$\|Au\|_{\mathbf{W}'(Q_T)} \leqslant 2 \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)} \|D_i\varphi\|_{L^{p_i(\cdot)}(Q_T)}.$$

Let recall that

$$\forall a_i, b_i > 0, \quad \sum_{i=1}^N a_i \cdot b_i \leqslant \sum_{i=1}^N a_i \sum_{i=1}^N b_i,$$

then

$$\begin{aligned}\|Au\|_{\mathbf{W}'(Q_T)} &\leqslant 2 \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)} \times \sum_{i=1}^N \|D_i\varphi\|_{L^{p_i(\cdot)}(Q_T)} \\ &\leqslant 2\sum_{i=1}^N \|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)}.\end{aligned}$$

In fact that

$$\int_{Q_T} |a_i(t,x,Du)|^{p_i'(\cdot)} dx\,dt \leqslant \int_{Q_T} \left(g(t,x) + \sum_{j=1}^{N} |D_j u|^{p_j(x)}\right) dx\,dt$$
$$\leqslant \int_{Q_T} g(t,x)\,dx\,dt + \int_{Q_T} \sum_{j=1}^{N} |D_j u|^{p_j(x)}\,dx\,dt,$$

because that $g \in L^1(Q_T)$, and $u \in \mathbf{W}(Q_T)$,

$$\int_{Q_T} |a_i(t,x,Du)|^{p_i'(\cdot)}\,dx\,dt \leqslant C.$$

So, using (10), one can obtain

$$\|Au\|_{\mathbf{W}'(Q_T)} \leqslant 2\sum_{i=1}^{N} \|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)} \leqslant C.$$

■

**Remark 3 (see [6]).** Since $p_i(\cdot) > 1 + \frac{N}{N+1}$, then

$$1 + \frac{N}{N+1} - \frac{2N}{N+2} = \frac{3N+2}{(N+1)(N+2)} > 0.$$

So, $p_i(\cdot) > \frac{2N}{N+2}$ which implies

$$W_0^{1,p_i(\cdot)}(\Omega) \subset L^2(\Omega) \subset \left(W_0^{1,p_i(\cdot)}(\Omega)\right)', \quad \forall i \in \{1,2,\ldots,N\},$$

where these injections are continuous and dense.

The dual of $W_0^{1,p_i(\cdot)}(\Omega)$ is denoted by $(W_0^{1,p_i(\cdot)}(\Omega))' = W^{-1,p_i'(\cdot)}(\Omega)$, where $1/p_i(\cdot) + 1/p_i'(\cdot) = 1$.

**Definition 1.** *Let $X$ be a reflexive Banach space. A single-valued operator $A\colon X \to X^*$ is called*

— *bounded, if $A$ maps bounded subsets of $X$ into bounded subsets of $X^*$;*
— *hemicontinuous, if $t \mapsto \langle A(u+tv), w\rangle_{X^* \times X}$ is continuous for all $u, v, w \in X$.*

**Lemma 4.** *Let $A_i$ be nonlinear operators of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$ such that*

$$\langle A_i u, \varphi\rangle = \int_{\Omega} a_i(t,x,D_i u) D_i\varphi\,dx, \quad \forall \varphi \in W_0^{1,p_i(\cdot)}(\Omega).$$

*Then, the operator $A_i$ satisfies the next*

1. *$A_i$ is hemicontinuous and bounded of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$.*
2. *$A_i$ is monotone of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$.*
3. *$\langle A_i(v), v\rangle \geqslant \alpha_i \|v\|_i^{p_i}$, $\alpha_i > 0$, $\forall v \in W_0^{1,p_i(\cdot)}(\Omega)$ or ($v \in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega)$), $1 < p_i < \infty$.*

**Remark 4 (Ref. [10]).** If $[v]_i = \|\frac{\partial v}{\partial x_i}\|_{L^{p_i(\cdot)}}$ is quasi-norm in $W_0^{1,p_i(\cdot)}(\Omega)$, then for all $\lambda_i$ suitable,

$$[v]_i + \lambda_i |v| \text{ is equivalent to } \|v\|_i$$

and if, instead of the condition 3 in lemma 4,

$$\langle A_i(v), v\rangle \geqslant \alpha_i [v]_i^{p_i}.$$

**Proof.** [Proof of lemma 4]

**Firstly.** Let $r$ is strictly positive and let $u \in B_{W_0^{1,p_i(\cdot)}(\Omega)}(0,r)$ then $\|u\|_{W_0^{1,p_i(\cdot)}(\Omega)} \leqslant r$, one can get

$$\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} = \sup_{\substack{\varphi\in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\|\leqslant 1}} |\langle A_i u, \varphi\rangle| = \sup_{\substack{\varphi\in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\|\leqslant 1}} \left|\int_\Omega a_i(t,x,D_i u) D_i\varphi \, dx\right|.$$

Using the Hölder's inequality and (10),

$$\begin{aligned}
\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} &\leqslant 2 \sup_{\substack{\varphi\in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\|\leqslant 1}} \|a_i(t,x,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \|D_i\varphi\|_{L^{p_i(\cdot)}(\Omega)} \\
&\leqslant 2 \sup_{\substack{\varphi\in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\|\leqslant 1}} \|a_i(t,x,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \Big(\|D_i\varphi\|_{L^{p_i(\cdot)}(\Omega)} + \|\varphi\|_{L^{p_i^-}(\Omega)}\Big) \\
&\leqslant 2 \|a_i(t,x,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \\
&\leqslant 2 \max\left\{ \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} dx\right)^{1/p_i^{+\prime}}, \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} dx\right)^{1/p_i^{-\prime}} \right\} \\
&\leqslant \left( \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)}\right)^{1/p_i^{+\prime}} + \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)}\right)^{1/p_i^{-\prime}} \right)
\end{aligned}$$

because that $p_i^{-\prime} \geqslant p_i^{+\prime} \Rightarrow 1/p_i^{-\prime} \leqslant 1/p_i^{+\prime}$ and we recall that

$$\forall a \geqslant 0, \quad \alpha \leqslant \beta \Rightarrow a^\alpha \leqslant a^\beta + 1 \tag{11}$$

and using (2),

$$\begin{aligned}
\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} &\leqslant 2 \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} dx\right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant 2\beta^{p_i'(\cdot)/p_i^{+\prime}} \left(\int_\Omega \Big(g(t,x) + \sum_{j=1}^N |D_j u|^{p_j(x)}\Big) dx\right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant C \left(\|g(t,x)\|_{L^1(\Omega)} + \sum_{j=1}^N \rho_{p_j(\cdot)}(D_j u)\right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant C \left(\|g(t,x)\|_{L^1(\Omega)} + 2\sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}}^{p_i^+} + N\right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant C' \left(1 + \sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}}^{p_i^+}\right)^{1/p_i^{+\prime}} + 1,
\end{aligned}$$

where $C' = C\left(\max\left\{\|g(t,x)\|_{L^1(\Omega)} + N, 2\right\}\right)^{1/p_i^{+\prime}}$.

Because of $p_i^+ \leqslant p_+^+ \Rightarrow 1/p_i^{+\prime} \leqslant 1/p_+^{+\prime} = 1 - 1/p_+^+$,

$$\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} \leqslant K\left(1 + \|u\|_{W_0^{1,p_i(\cdot)}(\Omega)}\right)^{p_+^+ - 1} + 1.$$

So, we have

$$\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} \leqslant K(1+r)^{p_+^+ - 1} + 1 = r'.$$

Which implies the boundedness of $A_i$.

In fact, let $u, v, w \in W_0^{1,p_i(\cdot)}(\Omega)$ and $\lambda \in \mathbb{R}$. Let's show that the function of $\mathbb{R}$ in $\mathbb{R}$:

$$\lambda \mapsto \langle A_i(u+\lambda v), w\rangle = \int_\Omega a_i\big(t,x,D_i(u+\lambda v)\big)\cdot D_i w$$

is continuous.

Let $\lambda \in \mathbb{R}$ be fixed and let $\{\lambda_n\}$ be a sequence of $\mathbb{R}$ converging to $\lambda$. Since $a_i$ is from Carathéodory function and $\lambda_n \to \lambda$ in $\mathbb{R}$,

$$a_i\big(t,x,D_i(u+\lambda_n v)\big) \to a_i\big(t,x,D_i(u+\lambda v)\big) \text{ a.e. in } \Omega.$$

Using (2) and applying Young's inequality,

$$\begin{aligned}
a_i\big(t,x,D_i(u+\lambda_n v)\big)\cdot D_i w &\leqslant \beta\Big(g(t,x)+\sum_{j=1}^N |D_j(u+\lambda_n v)|^{p_j(x)}\Big)^{1-\frac{1}{p_i(x)}}\cdot |D_i w| \\
&\leqslant \frac{|D_i w|^{p_i(x)}}{p_i(x)} + \beta\frac{g(t,x)+\sum_{j=1}^N |D_j(u+\lambda_n v)|^{p_j(x)}}{p_i'(x)} \\
&\leqslant \frac{|D_i w|^{p_i(x)}}{p_i(x)} + \beta\frac{g(t,x)+\sum_{j=1}^N |D_j u+\lambda_n D_j v)|^{p_j(x)}}{p_i'(x)} \\
&\leqslant \frac{1}{p_i(x)}|D_i w|^{p_i(x)} + \frac{\beta}{p_i'(x)} g(t,x) \\
&\quad + \frac{\beta}{p_i'(x)}\sum_{j=1}^N \big(2^{p_j(x)-1}\big)\left|D_j u^{p_j(x)} + \lambda_n^{p_j(\cdot)} D_j v^{p_j(x)}\right|.
\end{aligned}$$

Since the sequence $\{\lambda_n\}$ is bounded, it follows then from the dominated convergence theorem of Lebesgue that

$$\lim_{n\to\infty}\langle A_i(u+\lambda_n v), w\rangle = \langle A_i(u+\lambda v), w\rangle,$$

hence the hemicontinuity of $A_i$.

**Secondly.** Indeed, for $u, v \in W_0^{1,p_i(\cdot)}(\Omega)$, and by the condition â.3):

$$\begin{aligned}
\langle A_i u - A_i v, u-v\rangle &= \int_\Omega \big(a_i(t,x,D_i u) - a_i(t,x,D_i v)\big)\frac{\partial}{\partial x_i}(u-v)dx \\
&= \int_\Omega \big(a_i(t,x,D_i u) - a_i(t,x,D_i v)\big)(D_i u - D_i v)\,dx > 0.
\end{aligned}$$

So $A_i$ is monotone of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$.

**Thirdly.** For $v \in W_0^{1,p_i(\cdot)}(\Omega)$, and by the condition (1),

$$\langle A_i(v), v\rangle = \int_\Omega a_i(t,x,D_i v)D_i v dx \geqslant \alpha\int_\Omega \|D_i v\|^{p_i(x)}dx \geqslant \alpha\rho_{p_i(\cdot)}(D_i v).$$

By lemma (2) and remark 4, one can obtain

$$\langle A_i(v), v\rangle \geqslant \alpha\min\{[v]_i^{p_i^-}, [v]_i^{p_i^+}\} \geqslant \alpha[v]_i^{\eta_i},$$

where

$$\eta_i = \begin{cases} p_i^+, & \text{if } [v]_i \leqslant 1; \\ p_i^-, & \text{if } 1 \leqslant [v]_i < \infty. \end{cases}$$

■

## 3. Statement of main results

Given a real positive number $k$, we will define, for $r$ in $\mathbb{R}$, the functions

$$T_k(r) = \begin{cases} k, & \text{if } r \geqslant k, \\ r, & \text{if } |r| < k, \\ -k, & \text{if } r \leqslant -k. \end{cases}$$

and its primitive $\Theta_k \colon \mathbb{R} \to \mathbb{R}^+$ defined by

$$\Theta_k(r) = \int_0^r T_k(t)\,dt = \begin{cases} \frac{r^2}{2}, & \text{if } |r| \leqslant k, \\ k|r| - \frac{k^2}{2}, & \text{if } |r| > k. \end{cases}$$

We will then use the following results

$$\int_0^T \langle \partial_t v, T_k(v) \rangle\,dt = \int_\Omega \Theta_k(v(T)) - \int_\Omega \Theta_k(v(0)). \tag{12}$$

Before stating our main results, we define a weak solution of the problem $(P_1)$.

**Definition 2.** *A function $u(t,x) \in \mathbf{W}(Q_T)$ is called weak solution of problem $(P_1)$ if for every test-function*

$$\zeta \in \mathbf{Z} \equiv \left\{\varphi(z) \colon \varphi \in \mathbf{W}(Q_T) \cap L^\infty(Q_T), \varphi_t \in \mathbf{W}'(Q_T)\right\} \tag{13}$$

*and the following identity holds:*

$$\int_0^T \langle \partial_t u, \varphi \rangle\,dt + \sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,D_i u) D_i \varphi\,dx\,dt + \int_0^T \int_\Omega F(t,x,u)\varphi\,dx\,dt = \int_0^T \int_\Omega \varphi(t,x) f\,dx\,dt, \tag{14}$$

*where $a_i(t,x,\xi) \in L^{p_i'(\cdot)}(Q_T)$, $F(t,x,\xi) \in L^1(Q_T)$.*

Now, we announce our main results.

**Theorem 2.** *Let $p_i(\cdot)$ be such that*

$$p_i(\cdot) > 2 - \frac{1}{N+1} = 1 + \frac{N}{N+1}.$$

*Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1-3)$ and let $F$ be satisfying (4)–(5). Then the problem $(P_1)$ has at least one weak solution*

$$u \in \bigcap_{i=1}^N L^{p_i^-}\left(0,T; W_0^{1,p_i(\cdot)}(\Omega)\right).$$

**Proposition 2 (the case $F = 0$).** Let $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$ assume that $p_i(\cdot)$, $i = 1,\dots,N$ are defined as in (7) and $A_i$ is an operator which verifies all the hypotheses of Lemma 4. Then the problem $(P_1)$ has at least one weak solution

$$u \in \bigcap_{i=1}^N L^{p_i^-}\left(0,T; W_0^{1,p_i(\cdot)}(\Omega)\right).$$

**Proof.** The proof of this Proposition in lemma 4. ∎

### 3.1. Proof of Theorem 2

The proof of this theorem is realized in three steps.

**Step 1: Approximate problems.** We consider the following approximate problems

$$(P_{1n}) \qquad \begin{cases} \partial_t u_n - \operatorname{div}(\widehat{a}(t,x,Du_n)) + F_n(t,x,u_n)) = f \ \text{ in } \ Q_T \doteq \Omega\times]0,T[; \\ u_n(0,x) = u_0(x) \ \text{ in } \ \Omega; \\ u_n = 0 \ \text{ on } \Gamma_T = ]0,T[\times\partial\Omega, \end{cases}$$

where, for each $n>0$, $F_n(t,x,\xi) = \frac{F(t,x,\xi)}{1+\frac{1}{n}|F(t,x,\xi)|}$ a.e. $(t,x)\in Q_T$, $\forall\xi\in\mathbb{R}$, note that $F_n(t,x,\xi)$ satisfies the following conditions,

$$|F_n(t,x,\xi)| \leqslant |F(t,x,\xi)| \quad \text{and} \quad |F_n(t,x,\xi)| \leqslant n.$$

**Remark 5.** Under the conditions $(\widehat{a}.1-3)$, there exists at least one solution $u_n \in \bigcap_{i=1}^N L^{p_i^-}\left(0,T;W_0^{1,p_i(\cdot)}(\Omega)\right)$ of the problem $(P_{1n})$ (see the Proposition 2).

**Remark 6.** From classical results (see [10]), there exists a solution $u_n$ of such a problem, moreover $u_n$ is in $C\left([0,T];L^2(\Omega)\right)$.

**Step 2: Uniform estimates.**

**Lemma 5.** *Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1-3)$ and let $F$ be satisfying (4)–(5). Then the sequence $(u_n)$ is bounded in $\bigcap_{i=1}^N L^{p_i^-}\left(0,T;W_0^{1,p_i(\cdot)}(\Omega)\right)$.*

**Proof.** Taking $\varphi = I_d(u_n) = u_n$ as a test function in $(P_{1n})$, one can obtain

$$\int_0^T \langle\partial_t u_n, u_n\rangle\,dt + \int_0^T\!\!\int_\Omega \widehat{a}(t,x,Du_n)Du_n\,dx\,dt + \int_0^T\!\!\int_\Omega F_n(t,x,u_n)u_n\,dx\,dt = \int_0^T\!\!\int_\Omega f u_n\,dx\,dt. \tag{15}$$

For all $n\in\mathbb{N}$ and with $u_n(0,x)=u_0$ on $\Omega$,

$$\int_0^T \langle\partial_t u_n, u_n\rangle\,dt = \frac{1}{2}\int_\Omega u_n^2(t,x)\,dx - \frac{1}{2}\int_\Omega u_0^2\,dx.$$

Then,

$$\frac{1}{2}\int_\Omega u_n^2(t,x)\,dx + \sum_{i=1}^N \int_0^T\int_\Omega a_i(t,x,D_iu_n)D_iu_n\,dx\,dt + \int_0^T\int_\Omega F_n(t,x,u_n)u_n\,dx\,dt$$
$$= \int_0^T\int_\Omega f u_n\,dx\,dt + \frac{1}{2}\int_\Omega u_0^2\,dx.$$

According to the two conditions (1), (5) and after dropping the non-negative term, we derive

$$\alpha\sum_{i=1}^N\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}\,dx\,dt \leqslant \int_0^T\int_\Omega |f||u_n|\,dx\,dt + \frac{1}{2}\int_\Omega u_0^2\,dx.$$

Using the Hölder's inequality, and $f\in L^\infty(Q_T)$, $u_0\in L^\infty(\Omega)$,

$$\alpha\sum_{i=1}^N\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant 2\int_0^T \|f\|_{L^{p_i^-{}'}(\Omega)}\|u_n\|_{L^{p_i^-}(\Omega)}dt + C_0$$
$$\leqslant 2C\|f\|_{L^\infty}\int_0^T \|u_n\|_{L^{p_i^-}(\Omega)}dt + C_0.$$

By lemma 1.1 of [12],

$$\alpha\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant C_i\int_0^T \|D_iu_n\|_{L^{p_i^-}(\Omega)}dt + C_0, \quad C_i>0$$
$$\leqslant C\int_0^T \|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}dt + C_0.$$

By the inequality (10),

$$\int_0^T \|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}dt \leqslant \int_0^T \max\left\{\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}, \left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^+}\right\}dt$$
$$\leqslant \int_0^T \left(2\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}+1\right)dt$$
$$\leqslant 2\int_0^T \left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}dt + T.$$

Using Young's inequality for all $\varepsilon>0$, we obtain

$$\int_0^T \left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}dt \leqslant \varepsilon\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + C_1$$
$$\leqslant \varepsilon\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + C_1.$$

Then,

$$\int_0^T \|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}dt \leqslant 2\varepsilon\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + 2C_1 + T.$$

Finally,

$$\alpha\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant 2C\varepsilon\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + C_T,$$

where $C_T = C(2C_1+T)+C_0$. Now, we choose $\varepsilon=\alpha/4C$, then

$$\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant \sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant \frac{2}{\alpha}C_T. \tag{16}$$

So, the sequence $(D_iu_n)$ is bounded in $L^{p_i(\cdot)}(Q_T)$, that is to say $\|D_iu_n\|_{L^{p_i(\cdot)}(Q_T)} \leqslant C$. From the embedding in Remark 2,

$$\int_0^T \|D_iu_n\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}dt \leqslant C. \tag{17}$$

Now, it remains to prove that $\int_0^T \|u_n\|^{p_i^-}_{L^{p_i^-}(\Omega)}dt \leqslant C$. By lemma 1.1 of [12],

$$\|u_n\|_{L^{p_i^-}(\Omega)} \leqslant c_i\|D_iu_n\|_{L^{p_i^-}(\Omega)} \leqslant C\|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}, \quad \forall u_n\in W_0^{1,p_i(\cdot)}(\Omega).$$

So,

$$\|u_n\|^{p_i^-}_{L^{p_i^-}(\Omega)} \leqslant C\|D_iu_n\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)} \leqslant C\sum_{i=1}^{N}\|D_iu_n\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}, \quad \forall u_n\in\bigcap_{i=1}^{N}W_0^{1,p_i(\cdot)}(\Omega).$$

We integrate on $[0,T]$ and use (17) to get

$$\int_0^T \|u_n\|^{p_i^-}_{L^{p_i^-}(\Omega)}\,dt \leqslant C.$$

Therefore, the sequence $u_n$ is bounded in $\bigcap_{i=1}^N L^{p_i^-}\big(0,T;W_0^{1,p_i(\cdot)}(\Omega)\big)$. ■

**Lemma 6.** *Let*

$$1<r<\min_{1\leqslant i\leqslant N}\min_{x\in\overline{\Omega}}\left\{\frac{p_i(x)}{p_i(x)-1}\right\}. \tag{18}$$

*The sequence $(u_n'=\partial_t u_n)$ remains in a bounded set of $L^r\big(0,T;(W_0^{1,r'}(\Omega))'\big)+L^1\big((0,T)\times(\Omega)\big)$ where $r'$ is the conjugate $r$.*

**Proof.** For all $n\geqslant 1$,

$$u_n'=\operatorname{div}\big(\widehat{a}(t,x,Du_n)\big)+f-F_n$$

as $f-F_n$ is a bounded sequence in $L^1((0,T)\times(\Omega))$, we still have to show that

$$v_n=\operatorname{div}(\hat{a}(t,x,Du_n))\subset\Big(\text{bounded in}\quad L^r\big(0,T;(W_0^{1,r'}(\Omega))'\big)\quad\text{whith}\quad r>1\Big).$$

For $r>1$

$$\begin{aligned}\|v_n\|_{(W_0^{1,r'}(\Omega))'}&=\sup_{\substack{\varphi\in W_0^{1,r'}(\Omega)\\ \|\varphi\|\leqslant 1}}\Big|\int_\Omega\sum_{i=1}^N a_i(t,x,D_iu_n)D_i\varphi\,dx\Big|\\&\leqslant\beta\sup_{\substack{\varphi\in W_0^{1,r'}(\Omega)\\ \|\varphi\|\leqslant 1}}\sum_{i=1}^N\int_\Omega\Big(g+\sum_{j=1}^N|D_ju_n|^{p_j(\cdot)}\Big)^{1-\frac{1}{p_i(\cdot)}}|D_i\varphi|\,dx.\end{aligned}$$

Using the Hölder inequality, we see that

$$\begin{aligned}\|v_n\|_{(W_0^{1,r'}(\Omega))'}&\leqslant\beta\sup_{\substack{\|\varphi\|_{W_0^{1,r'}(\Omega)}\\ \|\varphi\|\leqslant 1}}\sum_{i=1}^N\Big(\int_\Omega|D_i\varphi|^{r'}dx\Big)^{1/r'}\Big(\int_\Omega\Big(g+\sum_{j=1}^N|D_ju_n|^{p_j(\cdot)}\Big)^{\left(1-\frac{1}{p_i(\cdot)}\right)r}dx\Big)^{1/r}\\&\leqslant C\sum_{i=1}^N\Big(\int_\Omega\Big(g+\sum_{j=1}^N|D_ju_n|^{p_j(\cdot)}\Big)^{\left(1-\frac{1}{p_i(\cdot)}\right)r}dx\Big)^{1/r},\end{aligned}$$

where again

$$\|v_n\|^r_{\left(W_0^{1,r'}(\Omega)\right)'}\leqslant C\sum_{i=1}^N\Big(\int_\Omega(g+\sum_{j=1}^N|D_ju_n|^{p_j(.)})^{(1-\frac{1}{p_i(.)})r}dx\Big). \tag{19}$$

By (18), $0<\big(1-\frac{1}{p_i(\cdot)}\big)r<1$. Integrating relation (19) on $[0,T]$ and applying Young's inequality,

$$\int_0^T\|v_n\|^r_{\left(W_0^{1,r'}(\Omega)\right)'}dt\leqslant C'\sum_{i=1}^N\left(\int_0^T\int_\Omega\Big(g+\sum_{j=1}^N|D_ju_n|^{p_j(\cdot)}\Big)dx\,dt+C_T\right).$$

Using (16) and $g\in L^1(Q_T)$, we obtain

$$\int_0^T\|v_n\|^r_{\left(W_0^{1,r'}(\Omega)\right)'}dt\leqslant C.$$

This finishes the proof of Lemma 6. ■

**Step 3: Passage to the limit.**

**Lemma 7.** *There exists a subsequence (still denoted by $(u_n)$) which converges a.e. in $(0,T)\times\Omega$ to a function $u\in L^1((0,T)\times\Omega)$. Therefore*

$$F_n(t,x,u_n)\to F(t,x,u) \quad \text{a.e. in} \quad (0,T)\times\Omega.$$

**Proof.** By lemma 6, the sequence $(u_n')$ remains in a bounded set of $L^r\big(0,T;W^{-1,r}(\Omega)\big)+L^1(Q)$ according to Rellich–Kondrachov's theorem, it comes that

$$L^1(\Omega)\subset W^{-1,r}(\Omega) \quad \text{if} \quad r'>N, \quad r<\frac{N}{N-1}.$$

So that

$$L^1(Q)\subset L^1\big(0,T;W^{-1,r}(\Omega)\big) \quad \text{and} \quad L^r\big(0,T;W^{-1,r}(\Omega)\big)\subset L^1\big(0,T;W^{-1,r}(\Omega)\big)$$

so

$$L^1(Q)+L^r\big(0,T;W^{-1,r}(\Omega)\big)\subset L^1\big(0,T;W^{-1,r}(\Omega)\big).$$

Therefore, the sequence $(u_n')$ remains bounded in $L^1\big(0,T;W^{-1,r}(\Omega)\big)$.

So, we can use Corollary 4 of [16], to see that $u_n$ is relatively compact in $L^1((0,T)\times\Omega)$.

This implies that we can extract a subsequence (denote again by $(u_n)$) such

$$u_n\to u \quad \text{strongly in} \quad L^1((0,T)\times\Omega) \quad \text{and a.e. in} \quad (0,T)\times\Omega. \tag{20}$$

Furthermore, we have $F_n(t,x,u_n)\to F(t,x,u)$ a.e. in $(0,T)\times\Omega$. ■

**Lemma 8.** *Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1-3)$ and let $F$ be satisfying (4)–(5). Then*

$$F_n(t,x,u_n)\to F(t,x,u) \quad \textit{strongly in} \quad L^1(0,T;L^1(\Omega)).$$

**Proof.** We shall first obtain local-integrability of $F_n(t,x,u_n)$ on $(0,T)\times\Omega$. Observe that: if $|u_n|\geqslant\gamma$ then $|T_\gamma(u_n)|=\gamma$, where $T_\gamma$ the truncation function at height $\gamma$ ($\gamma>0$).

So,

$$\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt=\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\frac{\gamma}{\gamma}dx\,dt \leqslant\frac{1}{\gamma}\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)T_\gamma(u_n)|\,dx\,dt$$

and because $F_n(t,x,u_n)T_\gamma(u_n)=\frac{F(t,x,u_n)}{1+\frac{1}{n}|F(t,x,u_n)|}T_\gamma(u_n)>0$, we find

$$\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt\leqslant\frac{1}{\gamma}\int_0^T\int_\Omega F_n(t,x,u_n)T_\gamma(u_n)\,dx\,dt.$$

We choose $\varphi=T_\gamma(u_n)$ as a test function in problems $(P_{1n})$, then

$$\int_\Omega dx\int_0^{u_n(T,x)}T_\gamma(\sigma)\,d\sigma+\sum_{i=1}^N\int_0^T\int_\Omega a_i(t,x,D_iu_n)D_iu_nT_\gamma'(u_n)\,dx\,dt + \int_0^T\int_\Omega F_n(t,x,u_n)T_\gamma(u_n)\,dx\,dt=\int_0^T\int_\Omega fT_\gamma(u_n)\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}T_\gamma(\sigma)\,d\sigma.$$

After dropping the non-negative term, we derive

$$\int_0^T\int_\Omega F_n(t,x,u_n)T_\gamma(u_n)\,dx\,dt\leqslant\int_0^T\int_\Omega|f||T_\gamma(u_n)|\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}|T_\gamma(\sigma)|\,d\sigma.$$

So,

$$\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt\leqslant\frac{1}{\gamma}\left(\int_0^T\int_\Omega|f||T_\gamma(u_n)|\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}|T_\gamma(\sigma)|\,d\sigma\right).$$

Now, for any $M > 0$, $0 \leqslant |T_\gamma(s)| \leqslant M + \gamma \mathbf{1}_{|s|>M}$ for any $s \in \mathbb{R}$,

$$\int_0^T \int_\Omega |f||T_\gamma(u_n)|\,dx\,dt \leqslant M \int_0^T \int_\Omega |f|\,dx\,dt + \gamma \int_0^T \int_{|u_n|>M} |f|\,dx\,dt$$
$$\leqslant C_1 \cdot M \|f\|_{L^\infty(Q_T)} + \gamma \int_0^T \int_{|u_n|>M} |f|\,dx\,dt.$$

Because $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$, we conclude that

$$\int_0^T \int_{|u_n|\geqslant\gamma} |F_n(t,x,u_n)|\,dx\,dt \leqslant \frac{1}{\gamma}\left(C_1 \cdot M \|f\|_{L^\infty(Q_T)} + \gamma \int_0^T \int_{|u_n|>M} |f|\,dx\,dt + \int_\Omega dx \int_0^{u_n(0,x)} |T_\gamma(\sigma)|\,d\sigma\right)$$
$$\leqslant C_1 \frac{M}{\gamma} + \int_0^T \int_{|u_n|>M} |f|\,dx\,dt + \frac{C_0}{\gamma}, \quad \text{where} \quad C_0 = \frac{1}{2}\int_\Omega u_0^2\,dx$$
$$\leqslant \frac{K}{\gamma} + \int_0^T \int_{|u_n|>M} |f|\,dx\,dt, \quad \text{where} \quad K = C_1 \cdot M + C_0$$
$$\leqslant \frac{K}{\gamma} + \int_0^T \int_\Omega \chi_{\{|u_n|>M\}} |f|\,dx\,dt.$$

Taking $M = \sqrt{\gamma}$, we conclude that

$$\int_0^T \int_{|u_n|\geqslant\gamma} |F_n(t,x,u_n)|\,dx\,dt \overset{\gamma\to+\infty}{\longrightarrow} 0 \quad \text{uniformly with respect to } n.$$

For any measurable subset $E \subset \Omega$ and the fact that $|F_n(t,x,u_n)| \leqslant |F(t,x,u_n)|$,

$$\int_0^T \int_E |F_n(t,x,u_n)|\,dx\,dt = \int_0^T \int_{E\cap\{|u_n|\geqslant\gamma\}} |F_n(t,x,u_n)|\,dx\,dt + \int_0^T \int_{E\cap\{|u_n|\leqslant\gamma\}} |F_n(t,x,u_n)|\,dx\,dt$$
$$\leqslant \int_0^T \int_{E\cap\{|u_n|\geqslant\gamma\}} |F_n(t,x,u_n)|\,dx\,dt + \int_0^T \int_E |F(t,x,u_n)|\,dx\,dt$$
$$\leqslant \int_0^T \int_{E\cap\{|u_n|\geqslant\gamma\}} |F_n(t,x,u_n)|\,dx\,dt + \int_0^T \int_E \chi_E \sup_{|\sigma|\leqslant\gamma} |F(t,x,\sigma)|\,dx\,dt.$$

By (4),

— $\chi_E \sup_{|\sigma|\leqslant\gamma} |F(t,x,\sigma)| \to 0$ a.e. in $Q_T$;

— $|\chi_E \sup_{|\sigma|\leqslant\gamma} |F(t,x,\sigma)| \leqslant \sup_{|\sigma|\leqslant\gamma} |F(t,x,\sigma)| \in L^1(Q_T)$

and using Lebesgue's dominated convergence theorem, we find that

$$\int_0^T \int_E \chi_E \sup_{|\sigma|\leqslant\gamma} |F(t,x,\sigma)|\,dx\,dt \to 0 \quad \text{as} \quad |E| \to 0.$$

We deduce that $F_n(t,x,u_n)$ is equi-integrable in $Q_T$, then by Lemma 7, and Vitali's theorem convergence,

$$F_n(t,x,u_n) \to F(t,x,u) \quad \text{strongly in} \quad L^1(Q_T). \qquad \blacksquare$$

**Lemma 9.** *Let $\widehat{a}$ be an operator satisfying ($\widehat{a}$.1 − 3) and let $F$ be satisfying* (4)–(5)*. Then, the sequence $(Du_n)$ converges a.e. in $(0,T)\times\Omega$ to a sequence $(Du) \in L^1((0,T)\times\Omega)$, that is*

$$Du_n \to Du \quad \text{a.e. in} \quad (0,T)\times\Omega. \tag{21}$$

**Proof.** We will show that the sequence $(Du_n)$ is a Cauchy sequence in measure on $\Omega$. This is to show that

$$\forall\delta>0,\ \forall\varepsilon>0,\ \exists n_0 \text{ such as } \forall p,q \geqslant n_0 \quad \operatorname{meas}\{(t,x)\in(0,T)\times\Omega \,|\, |(Du_p - Du_q)(t,x)| \geqslant \delta\} \leqslant \varepsilon.$$

For that, let us fix $\delta > 0$ and $\varepsilon > 0$, and notice that for $\lambda > 0$ and $\eta > 0$,

$$\{(t,x) \in (0,T) \times \Omega | \, |(Du_p - Du_q)(t,x)| \geqslant \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$E_1 = \{(t,x) \in (0,T) \times \Omega | \, |Du_p| \geqslant \lambda\}, \quad E_2 = \{(t,x) \in (0,T) \times \Omega | \, |Du_q| \geqslant \lambda\}$$
$$E_3 = \{(t,x) \in (0,T) \times \Omega | \, |u_p - u_q| \geqslant \eta\}$$

and

$$E_4 = \{|Du_p - Du_q| \geqslant \delta, |Du_p| \leqslant \lambda, |Du_q| \leqslant \lambda, |u_p - u_q| \leqslant \eta\}.$$

In view of Lemma 5, by choosing $\lambda$ large we can make $\text{meas}(E_1)$ and $\text{meas}(E_2)$ arbitrarily small. For example

$$\text{meas}(E_1) = \int_{E_1} 1 \, dx \, dt = \frac{1}{\lambda} \int_{E_1} \lambda \, dx \, dt \leqslant \frac{1}{\lambda} \int_{E_1} |Du_p| \, dx \, dt \leqslant \frac{1}{\lambda} \int_{Q_T} |Du_p| \, dx \, dt \leqslant \frac{C}{\lambda}.$$

Then,

$$\text{meas}(E_1) \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

For $\text{meas}(E_3)$,

$$\int_0^T \int_\Omega |u_p - u_q| \, dx \, dt \geqslant \int_{E_3} |u_p - u_q| \, dx \, dt \geqslant \varepsilon \, \text{meas}(E_3).$$

Since $(u_n)$ is a Cauchy sequence in $L^1(Q_T)$, then, for $\varepsilon > 0$ fixed, we see that

$$\text{meas}(E_3) \to 0 \quad \text{as} \quad p, q \to +\infty.$$

It remains to control $\text{meas}(E_4)$. Because the set $\{(\xi_1, \xi_2)|, |\xi_1| \leqslant \lambda, |\xi_2| \leqslant \lambda, |\xi_1 - \xi_2| \leqslant \delta\}$ is a compact set and $\xi \to \widehat{a}(t,x,\xi)$ is continuous for a.e. $(t,x) \in Q_T$, the quantity

$$\big(\widehat{a}(t,x,\xi_1) - \widehat{a}(t,x,\xi_2)\big)(\xi_1 - \xi_2) > 0$$

reaches its minimum value on this compact set, and we will denote it by $\mu(t,x)$ such that

$$\big(\widehat{a}(t,x,\xi_1) - \widehat{a}(t,x,\xi_2)\big)(\xi_1 - \xi_2) \geqslant \mu(t,x) > 0.$$

Consequently, by (8) for any $\tau > 0$ there exists $\tau' > 0$ such that

$$\int_{E_4} \mu(x) \, dx < \tau' \Rightarrow \text{meas}(E_4) < \tau. \tag{22}$$

To get $\text{meas}(E_4) < \tau$, it suffices to show that $\int_{E_4} \mu(x) \, dx < \tau'$. By the definitions of $\mu(t,x)$ and $E_4$, we can write

$$\int_{E_4} \mu(t,x) \, dx \, dt \leqslant \int_{E_4} \big[\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big] D(u_p - u_q) \mathbf{1}_{\{|u_p - u_q| \leqslant \varepsilon\}} \, dx \, dt$$

moreover the integral term is positive and $DT_\varepsilon(u_p - u_q) = D(u_p - u_q)\mathbf{1}_{\{|u_p - u_q| \leqslant \varepsilon\}}$, so

$$\int_{E_4} \mu(t,x) \leqslant \int_{E_4} [\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)] DT_\varepsilon(u_p - u_q), \tag{23}$$

where $T_\varepsilon$ the truncation at level $-\varepsilon$ and $\varepsilon$, and $T'_\varepsilon$ are defined as

$$T'_\varepsilon(\sigma) = \begin{cases} 1, & |\sigma| \leqslant \varepsilon; \\ 0, & |\sigma| > \varepsilon. \end{cases}$$

Specifying $T_\varepsilon(u_p - u_q)$ as test function in (15) for $u_p$ and $u_q$,

$$\int_0^T \langle \partial_t u_p, T_\varepsilon(u_p - u_q) \rangle + \int_0^T \int_\Omega \widehat{a}(t,x,Du_p) DT_\varepsilon(u_p - u_q) \\ + \int_0^T \int_\Omega F_n(t,x,u_p) T_\varepsilon(u_p - u_q) = \int_0^T \int_\Omega f T_\varepsilon(u_p - u_q) \tag{24}$$

and

$$\int_0^T \langle \partial_t u_q, T_\varepsilon(u_p - u_q)\rangle + \int_0^T \int_\Omega \widehat{a}(t,x,Du_q) DT_\varepsilon(u_p - u_q) + \int_0^T \int_\Omega F_n(t,x,u_q) T_\varepsilon(u_p - u_q) = \int_0^T \int_\Omega f T_\varepsilon(u_p - u_q). \quad (25)$$

Then subtracting the resulting inequality (24) and (25), we find

$$\int_0^T \langle \partial_t (u_p - u_q), T_\varepsilon(u_p - u_q)\rangle + \int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q) = \int_0^T \int_\Omega (F_n(t,x,u_q) - F_n(t,x,u_p)) T_\varepsilon(u_p - u_q).$$

The fact that $|T_\varepsilon| \leqslant \varepsilon$, then

$$\int_\Omega \Theta_\varepsilon(u_p - u_q)(T)dx - \int_\Omega \Theta_\varepsilon(u_p - u_q)(0)dx + \int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q)\,dx\,dt = \int_0^T \int_\Omega (F_n(t,x,u_q) - F_n(t,x,u_p)) T_\varepsilon(u_p - u_q)\,dx\,dt.$$

The first term is positive ($\Theta_\varepsilon(x) \geqslant 0$) and ($\Theta_\varepsilon(x) \leqslant \varepsilon|x|$). So

$$\int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q)\,dx\,dt \leqslant \varepsilon \int_0^T \int_\Omega |F_n(t,x,u_q) - F_n(t,x,u_p)|\,dx\,dt + \varepsilon \int_\Omega |u_0^p - u_0^q| dx.$$

The fact that $F_n \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega)$, we obtain

$$\int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q)\,dx\,dt \leqslant C\varepsilon \to^{\varepsilon\to 0} 0 \text{ (uniformly in } p \text{ and } q). \quad (26)$$

For $\varepsilon$ small enough, (23) and (26) imply

$$\int_{E_4} \mu(x) dx < \tau',$$

and also by (22) we have $meas(E_4) \leqslant \tau$. Thus, we have the convergence of $Du_n$ to $Du$ in measure, as well as the property (after extracting a subsequence).

$$Du_n \to Du \text{ a.e. in } (0,T) \times \Omega.$$

■

## 4. End of the proof of Theorem 2

For $\varphi \in \mathbf{Z}$ (see (13)),

$$\int_0^T \langle \partial_t u_n, \varphi\rangle\,dt + \int_{Q_T} \widehat{a}(t,x,Du_n) D\varphi\,dx\,dt + \int_{Q_T} F_n(t,x,u_n)\varphi\,dx\,dt = \int_{Q_T} \varphi(t,x) f\,dx\,dt. \quad (27)$$

**1) Passage to the limit in $\int_0^T \langle \partial_t u_n, \varphi\rangle dt$.**

$$\int_0^T \langle \partial_t u_n, \varphi\rangle\,dt = -\int_{Q_T} u_n \partial_t \varphi\,dx\,dt - \int_\Omega \varphi(0,x)\,u_0\,dx$$

The sequence $u_n \rightharpoonup u$ in $\mathbf{W}(Q_T)$ and $\partial_t \varphi \in \mathbf{W}'(Q_T)$. Then,

$$\lim_{n\to+\infty} \int_{Q_T} u_n \partial_t \varphi\,dx\,dt = \int_{Q_T} u \partial_t \varphi\,dx\,dt.$$

**2) Passage to the limit in $\int_{Q_T} F_n(t,x,u_n)\varphi\,dx\,dt$.**

By lemma 8,

$$\Big|\int_{Q_T} F_n(t,x,u_n)\varphi - \int_{Q_T} F(t,x,u)\varphi\Big| = \Big|\int_{Q_T} (F_n(t,x,u_n) - F(t,x,u))\varphi\Big|$$
$$\leqslant C\|F_n(t,x,u_n) - F(t,x,u)\|_{L^1(Q_T)} \to 0, \text{ as } n \to +\infty.$$

Ensure that

$$\lim_{n\to+\infty} \int_{Q_T} F_n(t,x,u_n)\varphi\,dx\,dt = \int_{Q_T} F(t,x,u)\varphi\,dx\,dt.$$

**3) Passage to the limit in $\int_{Q_T} \widehat{a}(t,x,Du_n)D\varphi\,dx\,dt$.**

Using the convergence (21) and the condition ($\widehat{a}$.2),

$$\widehat{a}(t,x,Du_n) \to \widehat{a}(t,x,Du) \text{ a.e. in } (0,T)\times\Omega$$

and by (2), we find

$$\widehat{a}(t,x,Du_n) \text{ is bounded in } L^{p_i'(\cdot)}(Q_T).$$

So, we deduce

$$\widehat{a}(t,x,Du_n) \rightharpoonup \widehat{a}(t,x,Du) \text{ in } L^{p_i'(\cdot)}(Q_T).$$

Because $D\varphi \in L^{p_i(\cdot)}(Q_T)$ then,

$$\lim_{n\to+\infty} \int_{Q_T} \widehat{a}(t,x,Du_n)D\varphi\,dx\,dt = \int_{Q_T} \widehat{a}(t,x,Du)D\varphi\,dx\,dt.$$

We therefore have to prove that $u$ is a solution to problem $(P_1)$. This finishes the proof of theorem 2.

## Acknowledgment

The author is grateful to the editor and anonymous reviewer for their constructive comments and valuable suggestions which certainly improved the presentation and quality of the paper.

[1] Almeida A., Harjulehto P., Hästö P., Lukkari T. Riesz and Wolff potentials and elliptic equations in variable exponent weak Lebesgue spaces. Annali di Matematica Pura ed Applicata. **194** (4), 405–424 (2015).

[2] Antontsev S., Shmarev S. Anisotropic Parabolic Equations with variable non linearity. Publicacions Matemàtiques. **53** (2), 355–399 (2009).

[3] Atik Y. Introduction aux problèmes elliptiques quasi-linéaires a donnée mesure. Cours spéciaux de l'ENS-Kouba, Alger (1998).

[4] Bendahmane M., Wittbold P. Renormalized solutions for nonlinear elliptic equations with variable exponents and $L^1$-data. Nonlinear Analysis: Theory, Methods & Applications. **70** (2), 567–583 (2009).

[5] Boureanu M.-M., Vélez-Santiago A. Fine regularity for elliptic and parabolic anisotropic Robin problems with variable exponents. Journal of Differential Equations. **266** (12), 8164–8232 (2019).

[6] Brezis H. Analyse fonctionnelle: Théorie et applications. Masson, Paris (1983).

[7] Brezis H. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011).

[8] Fan X. Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p(x)}$-Laplacian equations. Complex Variables and Elliptic Equations. **56** (7–9), 623–642 (2011).

[9] Fan X. Local boundedness of quasi-minimizers of integral functionals with variable exponent anisotropic growth and applications. Nonlinear Differential Equations and Applications. **17** (5), 619–637 (2010).

[10] Lions J. L. Quelques méthodes de résolution des problèmes aux limites. Dunod, Paris (1969).

[11] Mokhtari F. Regularity of the Solution to Nonlinear Anisotropic Elliptic Equations with Variable Exponents and Irregular Data. Mediterranean Journal of Mathematics. **14**, 141 (2017).

[12] Mokhtari F. Anisotropic parabolic problems with measur data. Differential Equations and Applications. **2**, 123–150 (2010).

[13] Mokhtari F. Problèmes paraboliques anisotropes a données dans un espace d'Orlicz ou mesures. Thèse de doctorat. (2011).

[14] Prignet A. Problèmes elliptiques et paraboliques dans un cadre non variationnel. UMPA-ENS Lyon France (1997).

[15] Rakotoson J. M. A compactness lemma for quasilinear problems: application to parabolic equations. Journal of Functional Analysis. **106** (2), 358–374 (1992).

[16] Simon J. Compact sets in the space $L^p(0,T;B)$. Annali di Matematica Pura ed Applicata. **146**, 65–96 (1987).

[17] Stampacchia G. Le problème de Dirichlet pour les èquations elliptiques du seconde ordre à coefficientes discontinus. Annales de l'Institut Fourier. **15** (1), 189–258 (1965).

[18] Troisi M. Theoremi di inclusione per Spazi di Sobolev non isotropi. Ricerche di Matematica. **18**, 3–24 (1969).


## Анізотропна параболічна задача зі змінним показником і регулярними даними


Мечетер Р.

*Кафедра математики та інформатики, Університет Мсіла, Мсіла, Алжир,*
*Лабораторія функціонального аналізу та геометрії просторів*



У цій роботі досліджується існування слабких розв'язків для класу нелінійних параболічних рівнянь із регулярними даними у просторах Соболєва зі змінною експонентою. Доводиться "версія" слабкої оцінки простору Лебега, яка сходить до "*Lions J. L. Quelques méthodes de résolution des problèmes aux limites. Dunod, Paris (1969)*", для параболічних рівнянь з анізотропними постійними показниками $(p_i(\cdot) = p_i)$.



**Ключові слова:** *анізотропні параболічні, нелінійні параболічні рівняння, регулярні дані.*