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APPROXIMATE CALCULATION OF NATURAL FREQUENCIES OF OSCILLATIONS OF THE PLATE WITH VARIABLE CROSS-SECTION OF THE DISCRETE-CONTINUOUS INTER-RESONANCE VIBRATING TABLE

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Abstract. *Problem statement.* To ensure highly efficient inter-resonance modes of operation of vibrating equipment, the oscillating masses of the system must have certain inertia-rigid parameters, as well as a certain frequency of natural oscillations. The disadvantage of highly efficient inter-resonance oscillatory systems is that the third reactive mass must be small, and therefore the use of complex and large structures is impossible. Therefore, it is best to use the reactive mass as a continuous section. The continuous section, which is a flexible body, optimally combines inertial and rigid parameters. Scientific works have already considered the design of the vibrating table, in which the continuous section is an ordinary rectangular plate hinged in the intermediate mass. This decision looks quite promising. However, likely, the rectangular shape of the plate is not the best option to ensure maximum energy efficiency. *Purpose.* Extend the method of calculating the natural frequency of oscillations of the plates by the approximate Rayleigh-Ritz method using the general hyperboloid equation to plates with variable cross-section for the proposed types of plates and check the results with the calculation in Ansys software. *Methodology.* The calculations of the plates were performed using the basic principles of the theory of oscillations, in particular the Rayleigh-Ritz method in the software product MathCAD. *Findings (results) and originality (novelty).* Two types of elastic plates with variable cross-sections are considered. In the first case, the shape of the plate was given by quadratic functions, in the second case, it was described by trigonometric functions of cosine. In both cases, the same conditions of attachment in the intermediate mass were observed. The calculation of the first natural frequency of oscillations of the considered plates was performed using the approximate Rayleigh-Ritz method with the assumption that the deflection of the plates occurs on the surface of the hyperboloid. The reliability of the obtained results was verified by numerical calculation in the software product Ansys. *Practical value.* It is assumed that the proposed types of plates can increase the dynamic potential of the vibrating machine. *Scopes of further investigations.* For further study of the considered types of plates as a continuous section of the inter-resonance vibrating machine, it is necessary to calculate their deflections at forced oscillations.

Keywords: inter-resonance vibrating machine, continuous member, elastic plate, the natural frequency of body oscillations, Rayleigh-Ritz method.

Introduction

Today in many branches of machine-building, chemical, construction, and mining industries vibration equipment is used to perform technological tasks. The most common type of force perturbation of oscillating masses of such vibrating machines is an electromagnetic drive. It provides the required perturbation force with relative ease of manufacture, simplicity, and reliability. However, typical designs of electromagnetic vibrating machines do not fully use the potential of energy efficiency. They were designed and put into production in the 80s of the last century, and therefore at the moment, with the rapid development of technology, have become a relatively energy-intensive type of equipment. Current trends in industrial development require the creation of energy-saving technological equipment because high competition encourages reducing the cost of production. Therefore, there is a need to create such technological equipment (including vibrating), which would have significant productivity and be energy efficient.

Problem Statement

Since the existing one- and two-mass designs of vibrating machines with electromagnetic drive have no prospects for significant improvements in energy efficiency, three-mass solutions have recently been developed [1]. This applies not only to vibrating machines with electromagnetic drive but also to vibrating equipment with inertial [2] and crankshaft [3] drives.

To ensure highly efficient inter-resonance modes of operation of vibrating equipment, the oscillating masses of the system must have certain characteristics. Such characteristics include inertia-rigid parameters, as well as the frequency of natural oscillations. The disadvantage of highly efficient inter-resonance oscillating systems is that the third reactive mass must be small, and therefore the use of complex and large structures is impossible. Recently, the use of not only classical discrete models of multi-mass mechanical oscillating systems (MOS) but also the synthesis of continuous sections in them is considered [3]. It is best to use continuous sections as a reactive mass. The continuous section, which is a flexible body, optimally combines inertial and rigid parameters. In the article [4] the design of the vibrating table is considered, in which the continuous section is an ordinary rectangular plate, hinged in the intermediate mass. This decision looks quite promising. However, likely, the rectangular shape of the plate is not the best option to ensure maximum energy efficiency.

Analysis of Modern Information Sources on the Subject of the Article

The elastic plate has many eigenforms and eigenfrequencies. However, in the future, we will focus on the first natural frequency. Vibrational analysis of functionally graduated plates (FG) is often found in the literature [5-7]. In particular, various boundary conditions for fixing FG plates are considered, both simple conditions such as SSSS or CCCC, and more complex conditions such as CCSS, CSSS, CFSS, SFSS, SSSF, and others. More complex conditions are often solved using the Levy equations [8]. S. P. Tymoshenko contributed to the improvement of the method of calculation of shells and elastic plates [9].

Plates with variable cross-sections are also investigated. In [10] was investigated the bending performance of sandwich variable cross-section plate. The authors have found the most adverse regional limit bearing capacity, stress-strain relationship, and stress three-dimensions degree in the ultimate state. M. Ece et al [11] have investigated the vibration of an isotropic beam that has a variable cross-section. In their article, the governing equation is reduced to an ordinary differential equation in spatial coordinate for a family of cross-section geometries with exponentially varying width. Analytical solutions of the vibration of the beam are obtained for three different types of boundary conditions associated with simply supported, clamped, and free ends. Natural frequencies and mode shapes are determined for each set of boundary conditions. In the article [12], the free vibration of the variable cross-section (non-uniform) single-layered graphene nano-ribbons (SLGNRs) is investigated by using the Differential Quadrature Method (DQM). The authors have also assumed that the width of the cross-section is vary exponentially along the length of the ribbon. Euler–Bernoulli beam theory is considered in conjunction with the nonlocal elasticity theory of

Eringen. Boiangiu et al [13] have solved the differential equations for free bending vibrations of straight beams with variable cross-section using Bessel's functions. The general equations for one-step conical beams were used together with corresponding equations of cylindrical beams to model multi-step beams with various boundary conditions. E. Demir et al. [14] have investigated the free vibration behavior of a multilayered symmetric sandwich beam made of functionally graded materials (FGMs) with variable cross-section resting on variable Winkler elastic foundation. S. Zolkiewski in his work [15] has been focused on the problem of vibrating beams with a variable cross-section fixed on a rotational rigid disk. The beam was loaded by a transversal time-varying force orthogonal to an axis of the beam and simultaneously parallel to the disk's plane. In the paper [16], a general analytical method, based on segmentation view and iteration calculation, is proposed to obtain the modal function and natural frequency of the beam with an arbitrary variable section. In the calculation, the section function of the beam is considered as an arbitrary function. In addition, the total amount of calculation caused by high-order Taylor expansion is reduced greatly by comparing with the original Adomian decomposition method (ADM).

Statement of Purpose and Tasks of Research

Given the prospects of highly efficient discrete-continuous inter-resonance vibrating machines and the advantages of using electromagnets as a drive for this type of equipment, this work aims to improve the methodology for designing inter-resonance vibrating tables with electromagnetic drive. Namely:

– to extend the known [4] method of calculating the natural frequency of oscillations of plates by the approximate Rayleigh-Ritz method using the general equation of hyperboloid on plates with variable cross-section;

– check the results obtained by comparing them with the data obtained by calculating the natural frequencies of oscillations of similar plates in the software product Ansys.

Main Material Presentation

The continuous section, which is the object of research in this paper, is synthesized into a discrete system of the inter-resonance vibrating table with electromagnetic drive, the scheme of which is shown in Fig. 1. The vibrating table consists of active m_1, intermediate m_2, and reactive m_3 masses moving in generalized coordinates $z_1(t)$, $z_2(t)$ and $z_3(t)$ accordingly. The vibrational masses of the vibrating table are connected in series by means of elastic node systems (marked c_{12} and c_{23}). The active mass is attached to the foundation by means of a system of vibration isolators with total rigidity c_{is}. It should be noted that the reactive mass in the form of a plate is a body with distributed parameters, and therefore combines inertial and rigid parameters. These parameters are relevant only in dynamic processes. In such cases, they appear as abstract values.

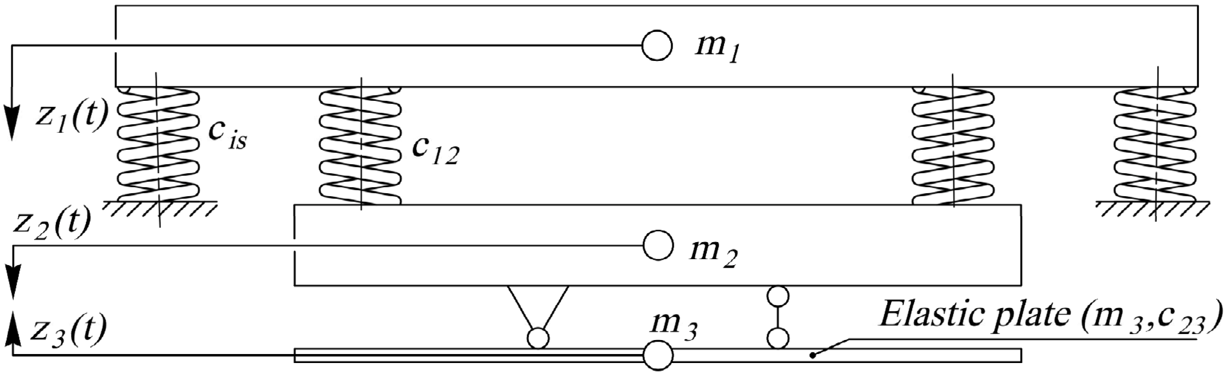

Fig. 1. Schematic diagram of the inter-resonance vibrating table

Based on existing experience in the design of high-efficiency three-mass inter-resonance vibrating machines [1], it is known that the reactive mass in this type of vibrating equipment should be quite light (tens of times lighter than the active and intermediate masses) and have the necessary values of inertial stiffness. It is accepted [4] to use elastic plates as a reactive mass. The material of the elastic plate is structural steel. This material has a low cost and is endowed with magnetic properties. In our case, the elastic plate is fixed in the intermediate mass through a hinged connection. The described scheme of fixing the plate is shown in Fig. 2.

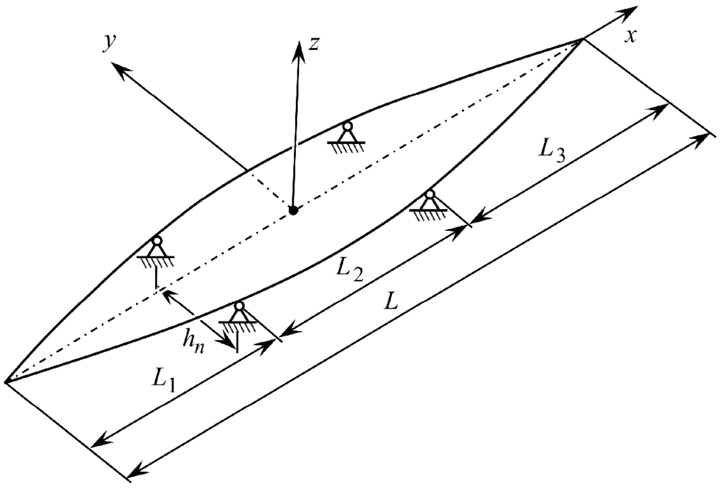

Fig. 2. Estimated plate fastening scheme,
where $L_1 = L_3 = 182\,mm,\ L_2 = 176\,mm,\ L = 540\,mm,\ h_n = 87{,}5\,mm$

The rigid parameters of the plate and its mass are related to the natural oscillation frequency parameter ω. Therefore, to synthesize a continuous section into a discrete mechanical oscillation system (MOS), it is necessary to set the natural frequency of the plate. The correct selection of the parameters of the plate, in particular its natural oscillation frequency, contributes to the implementation of highly efficient modes of operation between resonances of vibrating machines.

To calculate the natural frequency of a plate with a variable cross-section and this type of mounting, use the approximate Rayleigh-Ritz method. It was studied [4] that the deflection of such plates with hinged fastening at four points occurs on the surface describing the hyperboloid. As a result, it is assumed that when oscillating at the first natural frequency, the plate forms a surface area of the hyperboloid. The calculation scheme for the description of this process is shown in Fig. 3.

To describe the oscillations of the plate, we use the general equation of the hyperboloid, which according to the calculation scheme in Fig. 3, will look like:

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1. \tag{1}$$

To establish the displacement ε the relative coordinate system along the $0z$ – axis, use one of the four hinges. At the point with the coordinates $x = L_2/2,\ y = h_n/2$, the deflection along the $0z$ – axis will be equal to ε. Substituting the above conditions in equation (1), we obtain the following expression:

$$\frac{(h_n/2)^2}{b^2} + \frac{\varepsilon^2}{c^2} - \frac{(L_2/2)^2}{a^2} = 1. \tag{2}$$

Denote k the ratio of the sides b/c. From Fig. 3 shows that the parameter a is equal to half the length of the plate. Denoting $a = L/2$, we obtain the equality:

$$\varepsilon = \sqrt{\left(\frac{b}{k}\right)^2 \cdot \left[1 + \frac{(L_2/2)^2}{(L/2)^2} - \frac{(h_n/2)^2}{b^2}\right]}. \tag{3}$$

It is known [4] that the coefficient k does not affect the first natural frequency of the rectangular plate. This statement is also true for plates with variable cross-sections. Therefore, for further calculations, we take the value $k=1$. We also accept the value of the hyperboloid parameter $b=0,6$.

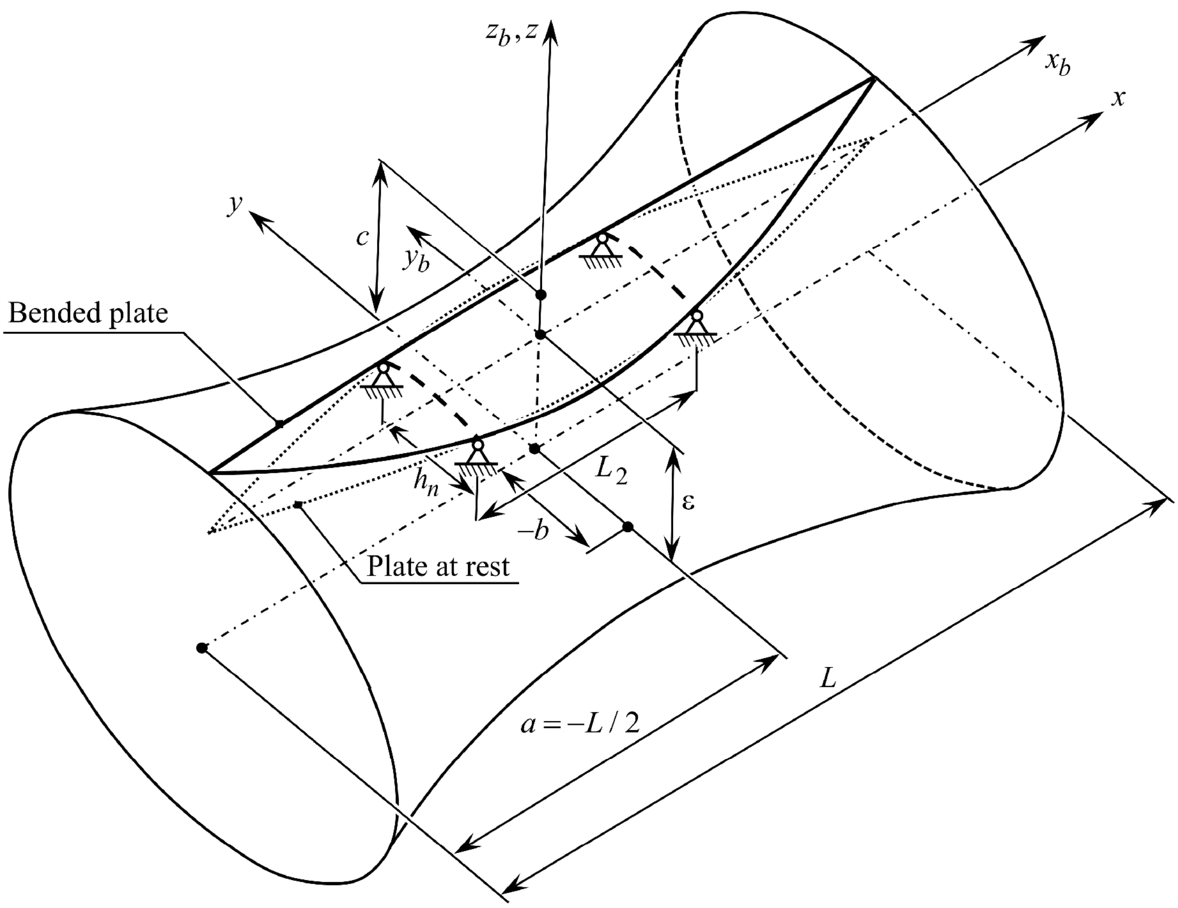

Fig. 3. Calculation scheme for determining the natural frequency of oscillations of the plate with the assumption that the oscillations occur on the hyperboloid surface, where x, y, z – the coordinates of the absolute frame of reference; x_b, y_b, z_b – coordinates of the relative frame of reference; ε – displacement of the relative coordinate system along the axis; a, b, c – hyperboloid parameters

In Fig. 3 it can be seen that the deflection of the plate along the $0z$ – axis at any point will be:

$$W(x,y)=z-\varepsilon. \quad (4)$$

Given expressions (2)–(4), the equation of deflection of the plate surface will look like:

$$W(x,y)=\sqrt{\left(\frac{a}{k}\right)^2\cdot\left[1+\frac{x^2}{(L/2)^2}-\frac{y^2}{b^2}\right]}-\sqrt{\left(\frac{a}{k}\right)^2\cdot\left[1+\frac{(L_2/2)^2}{(L/2)^2}-\frac{(h_n/2)^2}{b^2}\right]}. \quad (5)$$

When calculating the plate by the Rayleigh-Ritz method, we assume that the kinetic energy of the plate oscillations is equal to the potential. Therefore, the equality will be fair:

$$K=P. \quad (6)$$

Consider two cases of plates with variable cross-sections. In both cases, the change in the shape of the plate will be described by a continuous function at some interval. At the same time the conditions of fastening of a plate shown in Fig. 2.

In the first case, the shape of the plate on the $0xy$ plane is described by quadratic equations $y_1(x) = -\frac{2 \cdot h_n \cdot x^2}{L^2 - L_2^2} + \frac{h_n \cdot L^2}{2 \cdot \left(L^2 - L_2^2\right)}$ and $y_2(x) = \frac{2 \cdot h_n \cdot x^2}{L^2 - L_2^2} - \frac{h_n \cdot L^2}{2 \cdot \left(L^2 - L_2^2\right)}$. The graph of these functions is presented in Fig. 4.

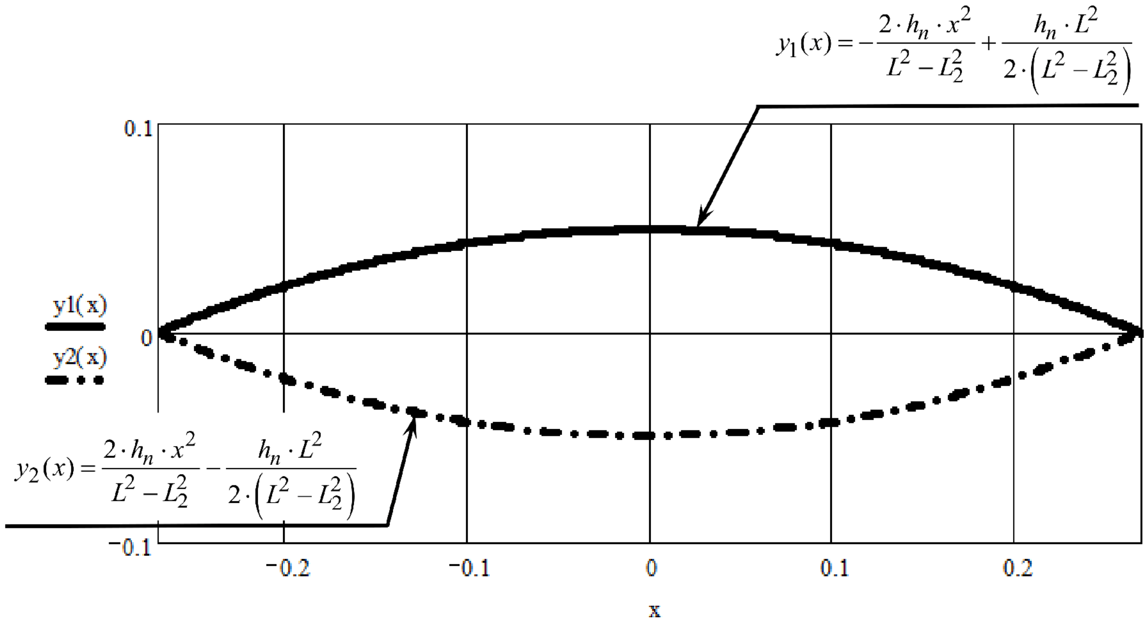

Fig. 4. The first case of describing a plate with a variable cross-section using the quadratic equations $y_1(x)$ and $y_2(x)$

As can be seen from Fig. 4, the shape of the plate is limited by two parabolas. This type of continuous section can be considered a parabolic plate.

For a parabolic plate, the ratio of kinetic energy K_1 to the square of the circular frequency of oscillations ω_1^2 during deflection on the hyperboloid surface at the first natural frequency of oscillations can be determined from equation [4]:

$$\frac{K_1}{\omega_1^2} = \frac{1}{2} \int_{-L/2}^{L/2} \int_{\frac{2 \cdot h_n \cdot x^2}{L^2 - L_2^2} - \frac{h_n \cdot L^2}{2 \cdot \left(L^2 - L_2^2\right)}}^{-\frac{2 \cdot h_n \cdot x^2}{L^2 - L_2^2} + \frac{h_n \cdot L^2}{2 \cdot \left(L^2 - L_2^2\right)}} \rho\, b_n W(x,y)^2 \,\mathrm{d}x\mathrm{d}y, \tag{7}$$

where – $\rho = 7800\ kg/m^3$ the density of steel; $b_n = 3{,}4\,mm$ – the thickness of the plate.

To find the potential deflection energy at the first natural frequency of the plate on the hyperboloid surface, we use the dependence [4]:

$$P_1 = \frac{1}{2} \int_{-L/2}^{L/2} \int_{\frac{2 \cdot h_n \cdot x^2}{L^2 - L_2^2} - \frac{h_n \cdot L^2}{2 \cdot \left(L^2 - L_2^2\right)}}^{-\frac{2 \cdot h_n \cdot x^2}{L^2 - L_2^2} + \frac{h_n \cdot L^2}{2 \cdot \left(L^2 - L_2^2\right)}} D \left(\left(\frac{\mathrm{d}^2 W(x,y)}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 W(x,y)}{\mathrm{d}y^2} \right)^2 + 2(1-\mu) \left(\left(\frac{\mathrm{d}^2 W(x,y)}{\mathrm{d}x\mathrm{d}y} \right)^2 - \frac{\mathrm{d}^2 W(x,y)}{\mathrm{d}x^2} \times \frac{\mathrm{d}^2 W(x,y)}{\mathrm{d}y^2} \right) \right) \mathrm{d}x\mathrm{d}y, \tag{8}$$

where the parameter D can be determined from the equation:

$$D = E \cdot b_n^3 / \left(12\left(1 - \mu^2\right)\right), \qquad (9)$$

$\mu = 0,26$ – Poisson's ratio; $E = 2,1 \cdot 10^{11}\,Pa$ - modulus of elasticity of the first type.

Determining the ratio of kinetic energy to the square of the circular frequency K_1 / ω_1^2 and the potential deflection energy P_1 of the parabolic plate on the hyperboloid surface according to (8) and (9), we establish the value of its first natural frequency [4]:

$$\nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{P_1}{K_1}} = \frac{1}{2\pi}\sqrt{\frac{551,7}{1,78 \cdot 10^{-3}}} = 88,6\,Hz. \qquad (10)$$

In the second case, the shape of the plate on the plane is described by trigonometric functions from cosine $y_3(x) = \frac{h_n \cdot \cos(\pi \cdot x / L)}{2 \cdot \cos\left(\pi \cdot L_2 / 2L\right)}$ and $y_4(x) = -\frac{h_n \cdot \cos(\pi \cdot x / L)}{2 \cdot \cos\left(\pi \cdot L_2 / 2L\right)}$. The graph of trigonometric functions $y_3(x)$ and $y_4(x)$ is presented in Fig. 5.

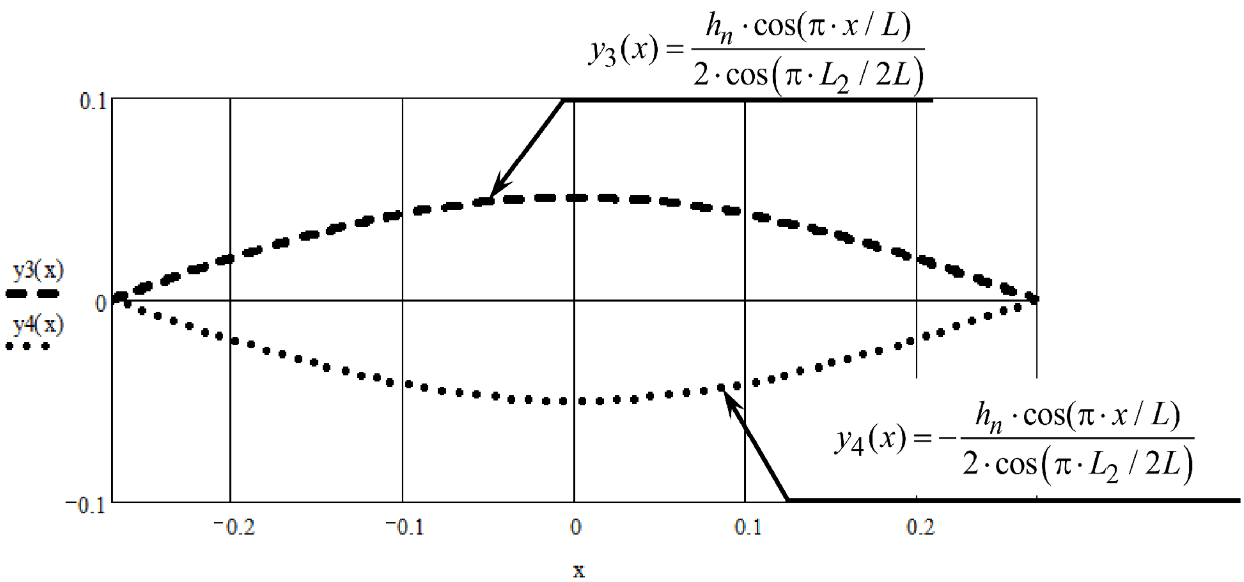

Fig. 5. The second case is the description of a plate with a variable cross-section using trigonometric functions $y_3(x)$ and $y_4(x)$

The shape of the plates in Fig. 5, described by trigonometric functions is similar to the shape of a plate bounded by two parabolas (Fig. 4). This similarity is due to the same conditions for fastening the plates in the intermediate mass of the vibrating machine.

For a plate described by trigonometric equations, the ratio of kinetic energy K_2 to the square of the circular frequency of oscillations ω_2^2 can be determined from equation [4]:

$$\frac{K_2}{\omega_2^2} = \frac{1}{2}\int_{-L/2}^{L/2}\int_{-\frac{h_n \cdot \cos(\pi \cdot x/L)}{2 \cdot \cos(\pi \cdot L_2/2L)}}^{\frac{h_n \cdot \cos(\pi \cdot x/L)}{2 \cdot \cos(\pi \cdot L_2/2L)}} \rho\, b_n W(x,y)^2\, dx dy. \qquad (11)$$

The potential energy of the deflection at the first natural frequency of the plate described by trigonometric functions is [4]:

$$P_2 = \frac{1}{2}\int_{-L/2}^{L/2}\int_{-\frac{h_n\cdot\cos(\pi\cdot x/L)}{2\cdot\cos(\pi\cdot L_2/2L)}}^{\frac{h_n\cdot\cos(\pi\cdot x/L)}{2\cdot\cos(\pi\cdot L_2/2L)}} D\left(\left(\frac{\partial^2 W(x,y)}{\partial x^2}+\frac{\partial^2 W(x,y)}{\partial y^2}\right)^2 + 2(1-\mu)\left(\left(\frac{\partial^2 W(x,y)}{\partial x\partial y}\right)^2 - \frac{\partial^2 W(x,y)}{\partial x^2}\cdot\frac{\partial^2 W(x,y)}{\partial y^2}\right)\right)dxdy, \qquad (12)$$

Having obtained the value of the ratio of kinetic energy to the square of the circular frequency K_2/ω_2^2 and the potential energy P_2 of the deflection of the plate of the second type from dependences (11) and (12), determine its first natural frequency [4]:

$$\nu_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{P_2}{K_2}} = \frac{1}{2\pi}\sqrt{\frac{550,2}{1,6\cdot 10^{-3}}} = 93,4\,Hz. \qquad (13)$$

As can be seen from equations (10) and (13), the difference in the values of the first natural frequencies of oscillations of two different types of plates with the same mounting conditions is relatively small.

The obtained values of the first natural frequencies of oscillations of the plates will be verified using the finite element method (FEM) using the software module Modal, which works in the shell of the product Ansys. To do this, build three-dimensional models of the plates described above. The results of the calculation of the first natural frequency of the parabolic plate are shown in Fig. 6, and for the plate described by trigonometric cosine functions - in Fig. 7.

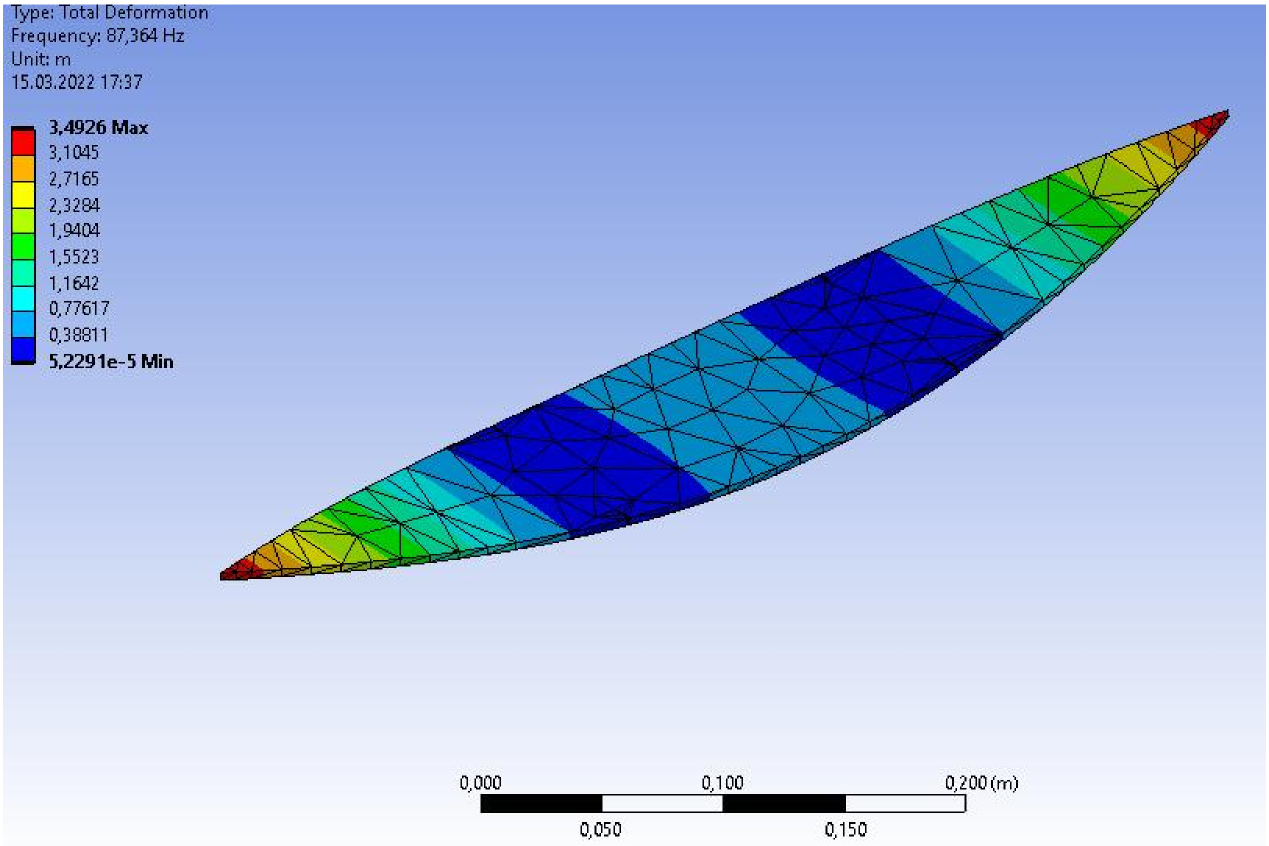

Fig. 6. The result of calculating the first natural frequency of the parabolic plate

Thus, according to numerical calculations in the software product Ansys, the first natural frequency of the parabolic plate is $87.364\,Hz$, and the plate is described by trigonometric equations – $92.757\,Hz$. The values obtained by this method are close to those determined by the approximate Rayleigh-Ritz method using the hyperboloid equation. This indicates the correctness of the calculations.

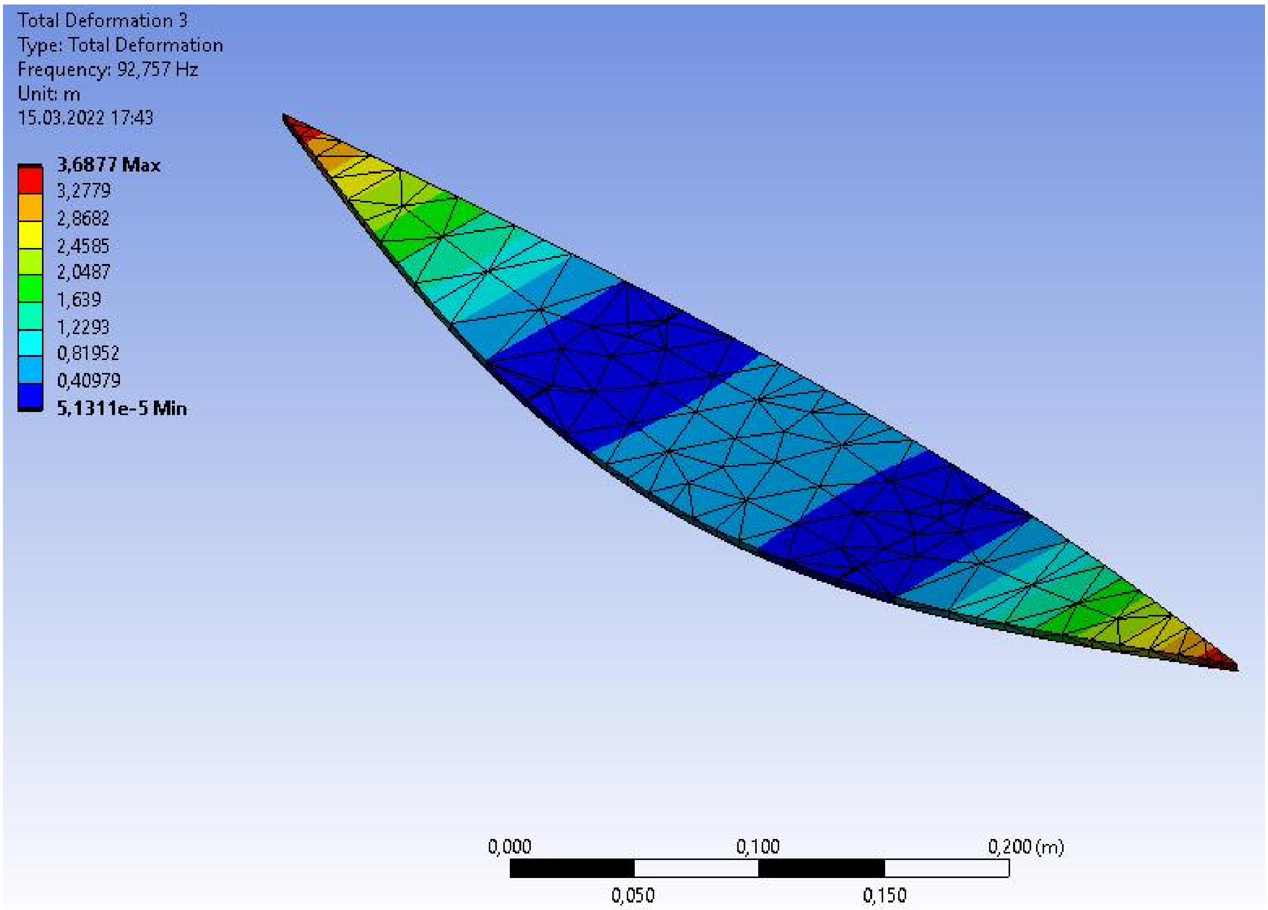

Fig. 7. The result of calculating the first natural frequency of the plate described by trigonometric cosine function

Comparing the results obtained in this article with the value of the first natural frequency of the rectangular plate [4] with similar lengths of sections and fastening conditions in the intermediate mass, we can conclude that the considered structures of plates with variable cross-section are more rigid. Therefore, to ensure their inter-resonance mode of operation of the vibrating machine at oscillation frequency $50\,Hz$, it is necessary to increase the length L_1 and L_3.

Conclusions

To ensure the energy efficiency of vibrating technological equipment, there is a tendency to introduce additional oscillating masses, as well as the creation of discrete-continuous oscillating systems. The synthesis of the continuous section in the form of a flexible body is a promising direction in the development of vibrating machines. In this paper, two types of elastic plates with variable cross-sections were proposed. In the first case, the shape of the plate was given by quadratic functions, in the second case, it was described by trigonometric functions of cosine. In both cases, the same conditions of attachment in the intermediate mass were observed. The calculation of the first natural frequency of oscillations of the considered plates was performed using the approximate Rayleigh-Ritz method with the assumption that the deflection of the plates occurs on the surface of the hyperboloid. The reliability of the obtained results was verified by numerical calculation in the software product Ansys. Compared with a rectangular plate of similar length, the natural frequency of the considered plates with variable cross-sections is significantly higher. To ensure their inter-resonance mode of operation of the vibrating machine at the frequency of oscillations $50\,Hz$ it is necessary to increase their total length. It is assumed that this may increase the amplitude of the oscillations of the ends, as well as increase the dynamic potential of the vibrating machine.

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