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INVESTIGATION OF THE ASYMMETRY OF THE EARTH'S GRAVITATIONAL FIELD USING THE REPRESENTATION OF POTENTIALS OF DISKS

The paper considers representations of the Earth external gravitational field, supplementing its traditional approximation by series in spherical functions. The necessity for additional means of describing the external potential is dictated by the need to study and use it at points in space close to the Earth's surface. It is in such areas that the need arises to investigate the convergence of series with respect to spherical functions and to adequately determine the value of the potential. The apparatus for approximating a piecewise continuous function in the middle of the ellipse is used for the representation of the Earth external gravitational field by the simple and double layer integrals. This makes it possible to expand the convergence region for the series supplying the potential to the entire space outside the integration ellipse. Therefore, as a result, the value of the gravitational potential coincides with the values of these series outside the body containing the interior masses (except for the integration ellipse). It becomes possible to evaluate the gravitational field behavior in surface areas and to carry out studies of geodynamic processes with greater reliability. Approximation of the gravitational field with the help of surface integrals also determines the geophysical aspect of the problem. Indeed, in the process of solving the problem we constructed two-dimensional integrands, which are uniquely determined by a set of Stokes constants. In this case, their expansion coefficients into series are defined by linear combinations of their function power moments. The resulting function schedules can be used to study the external gravitational field features, e.g., to study its asymmetry with respect to the equatorial plane.

Key words: the gravitational field asymmetry; Earth, potential; Bjerhamer sphere; Stokes constants.

Introduction

The history of research into the theory of attraction dates back hundreds of years and continues to develop both in theoretical [Axler et al, 2013; Landkoff, 1966] and practical directions [Kondratiev, 2007; Antonov et al., 1988]. The force of gravity of the Earth's gravitational field can be applied in different ways [Antonov et al., 1982]. The most common is the representation of the components of this force in terms of partial derivatives of a certain function called potential, which can be defined in different ways. The practical use of gravity in various fields affects the way it is specified [Antonov et al., 1988]. For example, for space geodesy purposes, it is advisable to represent the Earth's potential as the sum of the point mass potentials located in the middle of the Earth [Marchenko et al., 1985; Ostach, Ageeva, 1982]. In this case, different approaches and criteria for the spatial significance of the point mass placement are possible [Ostach, & Ageeva, 1982]. A special niche is occupied by the representation of the potential, caused primarily by the schedule of the inverse radius vector into series [Hobson, 1953]. This approach, in turn, defines the scope, determined by convergence. After all, the schedule of the inverse radius in a convergent series in spherical functions is guaranteed outside the sphere of a fixed radius and is automatically transferred to the region of convergence of the potential representation in spherical functions [Pellinen, 1978]. In other words, outside a certain sphere, which is called the convergence sphere, it is legitimate to represent the gravitational field in the form of a series in spherical functions. In the middle of this sphere, the question of convergence is open: the series can either coincide or be divergent. In the theory of the Earth figure, the concept of a minimal sphere, the “Bjerhamer” sphere, was even introduced [Pellinen, 1978]. A detailed discussion of this issue was made in [Sacerdote, & Sanso, 1991], where, along with theoretical studies, examples of the practical application of the methodology and the difficulties of its implementation are considered. Therefore, in practice, the questions of convergence of the series in areas close to the Earth’s surface are tried to be omitted, motivated by the representation of the series by a finite sum [Meshcheryakov, 1991]. Obviously, this is not entirely correct, because for large expansion orders (namely, this is what takes place in modern models of the gravitational field [Pavlis et al., 2008]), the contribution of high harmonics is determined by sensitive small numbers. Moreover, constantly updated modern potential models are time-variable [Kusche, et al 2009, Landerer, et al 2010] [Marchenko, Lopushanskyi 2018]. Therefore, when applying the potential in areas close to the convergence limit, one should, if possible, consider other ways of representing the potential, or investigate the possible convergence of series in spherical functions.

In modern research, special attention is paid to the study of the structure of the external potential [Shkodrov, & Ivanova, 1988; Fys et al, 2019]. After all, it is known to be characterized by its anisotropy, in relation to the equatorial plane, primarily due to the asymmetry of the Earth's figure. However, the three-dimensional nature of the external gravitational field is also due to the inhomogeneities of the mass distribution inside the Earth, which, unlike the deviations of the figure from the ellipsoid (sphere), are difficult to identify. It is quite obvious that it is impossible to establish an unambiguous connection between the inhomogeneities of the gravitational field and the internal structure, and approximate methods give potential approximate connections. For an unambiguous interpretation, it is necessary to put significant restrictions on the mass distribution of the Earth's interior, for example, the radial distribution of masses, etc. And even with such restrictions, the uniqueness of the solution is not always achieved. One of the possible options for describing the potential is its representation by the point mass potentials, which gives a way to localize the sources of generation of the features of the Earth's potential. However, this raises a number of additional questions: the placement of point masses and their optimal representation of the gravitational field. At its core, such a statement is presented as the main gravimetric problem [Grushinsky, 1983].

A certain step forward can be the transition from a discrete representation of the potential [Marchenko et al., 1985; Ostach, & Ageeva, 1982] to its description using them, which leads to the definition of a function of two variables. It is this approach that makes it possible to preserve the uniqueness of the problem solution (the integrand uniqueness) for a fixed placement surface. Specifically, this approach is the basis of the concept of gravitational disks proposed in [Meshcheryakov, 1991]. It was partially implemented in works [Zavision, 2000], [Zavision, 2001]. However, the lack of apparatus for approximating a function in an ellipse (circle) did not allow it to be fully realized. Therefore, the authors limited themselves to the one-dimensional case (if we consider the rod as the ultimate compression of the ellipse). If we use the classical results obtained by a number of mathematicians [Kampé de Fériet & P.E., 1926; Bateman, & Erdane, 1974], then we can construct an apparatus for approximating piecewise continuous functions in an ellipse by analogy for an ellipsoid [Fys et al, 2018]. It can be used to approximate a two-dimensional piecewise continuous function by series in biorthogonal systems. The schedule coefficients of the series are expressed in terms of the power moments of the approximating function and in the two-dimensional case are uniquely represented by a linear combination of the Stokes constants.

It is this approach that is implemented in this work, while the integration surface is chosen as one of the simplest and represents ellipses located in the equatorial plane. This allows you to compare the action of the gravitational field relative to the equator, that is, to identify symmetry or its absence during the Earth's rotation.

Methodology

When solving problems of mathematical physics, the solution is often presented in the form of integrals of a simple or double layer. The implementation of this approach is carried out in this work, while the integration surface is chosen as one of the simplest ellipses and represents ellipses located in the equatorial plane.

Such a simplification further makes it possible to compare the values of the external potential with respect to the equator, that is, to reveal symmetry or its absence.

Traditionally, the potential of the planet's gravity is given by a series of spherical functions and looks like this:

$$V(P)=\frac{GM}{R}\left(1+\sum_{n=0}^{\infty}\left(\frac{a_e}{r}\right)^n\left(C_{n,0}P_n(\cos\vartheta)+\right.\right.$$
$$\left.\left.+\sum_{k=1}^{n}P_n^r(\cos\vartheta)\left(C_{n,k}\cos k\lambda+S_{n,k}\sin\lambda\right)\right)\right), \tag{1}$$

where $C_{n,k},S_{n,k}$ – expansion coefficients (Stokes constants).

In the Earth sciences, slightly different quantities are given, the so-called normalized coefficients, related to the Stokes constants as follows:

$$C_{n,k}=A_{n,k}\sqrt{RR\frac{(n-k)!(2n+1)}{(n+k)!}}, \tag{2}$$

$$RR=\begin{cases}1,k=0\\2,k\neq 0\end{cases}. \tag{3}$$

Quantities (2) completely describe the Earth's external gravitational field in the region of convergence of series (1), defined as a sphere covering all integrating masses. However, decomposition (1) is used in practice for the inner part of the sphere, while obtaining completely adequate results. Simply put, series (1) can also coincide inside the sphere. In theoretical geodesy [Hofmann-Wellenhof, & Moritz, 2005], a special term is even introduced: “Bjerhamer sphere” – the minimum sphere outside of which the series begins to coincide.

It is also not easy to find a possible interpretation of the formation of the Stokes constants and relate it to the internal structure of the Earth. These issues were dealt with by a number of researchers (geophysicists, geodesists, gravimetrists), such as Moritz G. [Hofmann-Wellenhof, & Moritz, 2005], [Pellinen, 1978], Tarakanov Yu. O. [Tarakanov, & Cherevko, 1979].

All the above considerations point to the necessity to search for such potential representations for which the problem of convergence would be at least partially solved and would also carry some geophysical information. One of the options for solving the problem is proposed in this paper.

It was shown in [Meshcheryakov, 1991] that the external potential of the planet can, in particular, be represented by the sum of the potentials of two flat figures (ellipses) with variable density located in the equatorial plane, namely:

$$V = V' + V'', \tag{4}$$

where V' and V'' – the single-layer potential and the double-layer potential S, respectively, defined as

$$V'(P) = \iint_S \frac{\mu(\xi,\eta)}{r(Q,P)} dS_Q \tag{5}$$

$$V''(P) = \iint_S \frac{\nu(\xi,\eta) z}{r^3(Q,P)} dS_Q . \tag{6}$$

To determine the expressions (5) (the disclosure of (6) is described in [Meshcheryakov, 1991]), we establish the form of the density distribution functions of a flat figure S. Relation (5) can be represented as a sum of series in spherical functions [Hobson, 1953].

To do this, we represent the inverse distance $\frac{1}{r_{Q,P}}$ between two points $Q(r',\theta',\lambda')$, $P(R,\theta,\lambda)$ as follows:

$$\frac{1}{r_{Q,P}} = \frac{1}{R}\left(1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\psi)\right) = \frac{1}{R} + \sum_{n=1}^{\infty}\frac{r^n}{R^{n+1}} \times$$
$$\times \left(P_n(\cos\theta')P_n(\cos\theta) + \sum_{k=0}^{n}\frac{2(n-k)!}{(n+k)!}P_n^k(\cos\theta')P_n^k(\cos\theta)(\cos m\lambda \cos m\lambda' + \sin m\lambda \sin m\lambda')\right), \tag{7}$$

where ψ – the angle between vectors $\overrightarrow{OP}, \overrightarrow{OQ}$.

If the point is in the equatorial plane, then its coordinates are as follows and the radius vector has the form:

$$\frac{1}{r_{Q,P}} = \frac{1}{R}\left(1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\psi)\right) =$$
$$\frac{1}{R}\left(1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \left(P_n(0)P_n(\cos\theta) + 2\sum_{k=0}^{n}\frac{(n-k)!}{(n+k)!}P_n^k(0)P_n^k(\cos\theta)(\cos m\lambda \cos m\lambda' + \sin m\lambda \sin m\lambda')\right)\right) \tag{8}$$

The values of the attached Legendre polynomials at point 0 are as follows:

$$P_n^k(0) = L_{n,k} = \begin{cases} \frac{(n+k-1)!!(-1)^m}{m!2^m}, & 2m = n-k \\ 0, & n-k - \text{odd} \end{cases},$$

$$P_{2m}(0) = \begin{cases} \frac{(2m-1)!!(-1)^m}{m!2^m}, & n = 2m \\ 0, & n - \text{odd} \end{cases} . \tag{9}$$

Therefore,

$$\frac{1}{r_{Q,P}} = \frac{1}{R}\left(1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \left(L_{n,0}P_n(\cos\theta) + \sum_{k=0}^{n} L_{n,k}P_n^k(\cos\theta)(\cos m\lambda \cos m\lambda' + \sin m\lambda \sin m\lambda')\right)\right). \tag{10}$$

$$L_{n,0} = \begin{cases} \frac{(2m-1)!!(-1)^m}{m!2^m}, & n = 2m \\ 0, & n \neq 2m \end{cases}, \quad L_{n,k} = \begin{cases} \frac{(n+k-1)!!(-1)^m}{m!2^m}, & 2m = n-k \\ 0, & 2m \neq n-k \end{cases}, \quad m = \left[\frac{n-k}{2}\right].$$

Substitution (10) in the expression for potential (5) gives the following:

$$V' = \frac{GM}{R}\left(a_{0,0} + \sum_{n=1}^{\infty}\left(\frac{a}{R}\right)^{n}\left(a_{n,0}P_n(\cos\vartheta) + \sum_{k=0}^{n} L_{n,k}P_n^k(\cos\vartheta)\left(a_{n,k}\cos k\lambda + b_{n,k}\sin k\lambda\right)\right)\right), \quad (11)$$

where $a_{n,k} = \frac{1}{a^n M}\iint_S \mu(\xi,\eta)(r)^n \cos k\lambda' dS$, $b_{n,k} = \frac{1}{a^n M}\iint_S \mu(\xi,\eta)(r)^n \sin k\lambda' dS$ are analogues of Stokes constants.

Equating the traditional notation of the potential with the obtained expression (11) for even $n-k$, we obtain the relationship between the coefficients:

$$C_{n,k} = L_{n,k}a_{n,k},\ S_{n,k} = L_{n,k}b_{n,k}.$$

Thus, the expansion coefficients $a_{n,k}, b_{n,k}$ can be considered known if the Stokes constants are given. They can be given in a rectangular coordinate system like this:

$$a_{n,k} + ib_{n,k} = \frac{1}{a^n M}\iint_S \mu(\xi,\eta)\left(x^2+y^2\right)^m (x+iy)^k dS,$$

$$0 \le k \le n,\ m = \left[\frac{n-k}{2}\right]. \quad (12)$$

The system of equations (12) is a linear combination of the following expressions

$$I_{n,k} = \frac{1}{a^m b^n M}\iint_S \mu(\xi,\eta)\, x^m y^n dS, \quad (13)$$

which are called power moments [Ahiezer, & Crane, 1938]. Note that in the case of equality (12), they split into two subsystems with respect to $a_{n,k}$ and $b_{n,k}$, from which their moments (13) are determined. Let us touch upon the definition of the function $\mu(\xi,\eta)$, which also decomposes into two terms (corresponding to the Stokes constants $C_{n,k}$ and $S_{n,k}$ of pair orders):

$$\mu(\xi,\eta) = \mu_1(\xi,\eta) + \mu_2(\xi,\eta).$$

Since the function is piecewise continuous, it can be expanded into a series of biorthogonal polynomials $W_{m,n}(\xi,\eta), w_{m,n}(\xi,\eta)$, which can be considered as a simplified version of the corresponding spatial systems [Meshcheryakov, 1991]. Detailed calculations and justifications will be performed in a separate publication. Here we use post factum only the properties and formulas necessary for approximation:

$$w_{mn} = \frac{1}{a^m b^n}\sum_{l=0}^{\frac{N}{2}}(-1)^{N-2l}(N-l+1)!\sum_{i+j=l}\frac{\left(\frac{x}{a}\right)^{m-2i}\left(\frac{y}{b}\right)^{n-2j}}{i!\,j!\,(m-2i)!\,(n-2j)!},$$

$$W_{mn} = \frac{a^{-m}b^{-n}}{m!n!k!}\sum_{l=0}^{\left[\frac{N}{2}\right]}\frac{(-2)^l}{2^N l!}\sum_{t_1+t_2=l}^{N}\frac{(2t_1-1)!!(2t_2-1)!!}{(2t_1-m)!(2t_2-n)!}\left(\frac{x}{a}\right)^{2t_1-m}\left(\frac{y}{b}\right)^{2t_2-n} = \frac{2^{-N}}{m!n!}\frac{\partial^N}{\partial x^m \partial y^n}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^N, \quad (14)$$

$$l_{mn} = \int_\tau W_{mn}w_{m_1n_1}d\tau = \begin{cases} 0, & m \ne m_1, \text{ or } n \ne n_1 \\ \dfrac{N!S_e}{(N+1)m!n!a^{2m}b^{2n}}, & m = m_1, n = n_1 \end{cases}. \quad (15)$$

The possibility of approximating the function can be formulated by the following theorem.

Theorem. Any square-integrated function ($\mu \in l_2$) can be represented as a series

$$\mu(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_{mn}W_{mn}(\xi,\eta), \quad (16)$$

where

$$d_{mn} = \frac{\iint_S \mu W_{mn}(\xi,\eta)dS}{\iint_S w_{mn}(\xi,\eta)W_{mn}(\xi,\eta)dS} \quad (17)$$

and the series is convergent on average, i.e.:

$$\lim_{m,n\to\infty}\iint_S\left(\mu - \sum_{n=0}^{\infty}\sum_{m=0}^{N} d_{mn}W_{mn}(\xi,\eta)\right)^2 = 0. \quad (18)$$

The consequence of this theorem is the uniform convergence of the following series

$$V = G\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_{mn}\iint_S \frac{\Omega_{mn}}{\rho}dS\,, \qquad (19)$$

representing external potential V. Thus, a set of functions

$$u_{mn} = \iint_S \frac{\Omega_{mn}}{\rho}dS \qquad (20)$$

is a set of harmonic functions describing a part of the potential (1) with Stokes constants, the indices of which are $k-n$ is odd, and the sum of this series coincides with the value of the potential in the region that does not include the integrating masses. To calculate expressions (20), we perform the following transformations. We expand the reversible radius into a binomial series as follows:

$$\frac{1}{r_{Q,P}} = \frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos\psi}} = \frac{1}{R\sqrt{r^2+\rho^2-2r\rho\cos\psi \pm 1}} = \frac{1}{R\left(\sqrt{r^2+1}\right)\sqrt{1+\dfrac{\rho^2-2r\rho\cos\psi-1}{r^2+1}}}$$

$$\frac{1}{R\left(\sqrt{r^2+1}\right)}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left(\frac{\rho^2-2r\rho\cos\psi-1}{r^2+1}\right)^n\right). \qquad (21)$$

The series (21) is absolutely convergent because the modulus of its common term is less than one, ie

$$-1 < \frac{\rho^2-2r\rho\cos\psi-1}{r^2+1} < 1. \qquad (22)$$

This is easy to see by performing the following transformations:

$$1)\ \frac{\rho^2-2r\rho\cos\psi-1}{r^2+1} < 1,$$

$$\rho^2-2r\rho\cos\psi-1 < r^2+1,$$

$$(\rho-r)^2+2r\rho(1+\cos\psi)+2\left(1-\rho^2\right) =$$

$$-\rho^2+r^2+2r\rho(\cos\psi)+1+1 \geq 0$$

$$\rho^2-2r\rho(\cos\psi)-1 < r^2+1,$$

$$2)\ -r^2-1 < \rho^2-2r\rho\cos\psi-1,$$

$$(\rho-r)^2+2r\rho(1-\cos\psi) > 0.$$

Therefore, it can be differentiated term by term, including with respect to variables ξ,η. To do this, we first write its expression in a rectangular coordinate system.

Then

$$\frac{1}{r_{Q,P}} = \frac{1}{R\left(\sqrt{r^2+1}\right)}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left(\frac{\rho^2-2r\rho\cos\psi-1}{r^2+1}\right)^n\right) =$$

$$\frac{1}{R\left(\sqrt{r^2+1}\right)}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left(\rho^2-2r\rho\cos\psi-1\right)^n\right) =$$

$$\frac{1}{R\left(\sqrt{r^2+1}\right)}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n n!}{\left(r^2+1\right)^n}\sum_{l=0}^{n}\frac{(-1)^{n-l}\left(\xi^2+\eta^2-2\xi x-2\eta y\right)^l}{l!(n-l)!}\right) =$$

$$\frac{1}{R\left(\sqrt{r^2+1}\right)}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n n!}{\left(r^2+1\right)^n}\sum_{l=0}^{n}\frac{(-1)^{n-l}}{(n-l)!}\sum_{t_1+t_2+t_3+t_4=l}\frac{(-2)^{t_2+t_4}\xi^{2t_1+t_3}\eta^{2t_2+t_4}x^{t_2}y^{t_4}}{t_1!t_2!t_3!t_4!}\right) \qquad (23)$$

The functions u_{mn} can be reduced to the following form using the Stokes formula:

$$u_{mn} = \iint_S \frac{\Omega_{mn}}{\rho}dS = \frac{1}{2^N m!n!}\iint_S \frac{\frac{\partial^n}{\partial\xi^m\partial\eta^n}\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1\right)^N}{\rho}dS = \frac{(-1)^N}{2^N m!n!}\iint_S \left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1\right)^N \frac{\partial^n}{\partial\xi^m\partial\eta^n}\left(\frac{1}{\rho}\right)dS\,. \qquad (24)$$

After substitution (23) in this formula we obtain:

$$u_{mn}=\frac{\lambda_{mn}}{2^N}\iint_S\left(\rho^2-1\right)^N\frac{\partial^n}{\partial\xi^m\partial\eta^n}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n n!}{\left(r^2+1\right)^n}\sum_{l=0}^{n}\frac{(-1)^{n-l}}{(n-l)!}\sum_{t_1+t_2+t_3+t_4=l}\frac{(-2)^{t_2+t_4}\xi^{2t_1+t_3}\eta^{2t_2+t_4}x^{t_2}y^{t_4}}{t_1!t_2!t_3!t_4!}\right)dS=$$

$$\lambda_{mn}\left(\frac{1}{S}+\sum_{n=1}^{\infty}\frac{n!}{\left(r^2+1\right)^n}\sum_{l=0}^{n}\frac{(-1)^l}{(n-l)!}\sum_{t_1+t_2+t_3+t_4=l}\frac{\iint_S\left(\rho^2-1\right)^N\xi^{2t_1+t_3-m}\eta^{2t_2+t_4-n}dS\,(-2)^{t_2+t_4}x^{t_2}y^{t_4}(2t_1+t_3)!(2t_2+t_4)!}{(2t_2+t_4-n)!(2t_1+t_3-m)!t_1!t_2!t_3!t_4!}\right)$$

$$=\lambda_{mn}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^n n!}{\left(r^2+1\right)^n}\sum_{l=0}^{n}\frac{(-1)^{n-l}}{(n-l)!}\sum_{t_1+t_2+t_3+t_4=l}\frac{N!(-2)^{t_2+t_4}x^{t_2}y^{t_4}(2t_1+t_3)!(2t_2+t_4)!}{(l+N+1)!!(2t_2+t_4-n)!!(2t_1+t_3-m)!!t_1!t_2!t_3!t_4!}\right),\qquad(25)$$

where

$$\lambda_{mn}=\frac{S\left(\sqrt{r^2+1}\right)^{-1}}{m!n!\mathrm{R}}.$$

Finally, we now determine the potential V by a sequence of formulas (25), (19), (17), (11) and compare this value with the value of the potential obtained from pair powers of spherical functions in schedule (1). Outside the security sphere, these values coincide, and points with different values define the area of divergence of part of the series (1). In this case, in contrast to (1), we obtain a function that generates this part of the gravitational field value. Therefore, it is possible to draw certain conclusions on the basis of this function about the possible areas that generate the features of the gravitational field.

Results

Numerical experiments were carried out to verify the above methodology. The twenty first orders of the EGM2008 gravity model [Pavlis et al., 2008] were taken as initial data. The GR-84 ellipsoid of revolution recommended by the International Geophysical Union [NI&MATR, 1984] with parameters $a=6378137\,м$, $\alpha=1/298.257223$, $b=6356752.3142\,м.$, was taken as the figure of the Earth. Based on these data, we calculated the coefficients of the expansion of function $\mu_1(\xi,\eta)$ (6) for even powers of Stokes constants of even orders. Its visual representation is shown in Fig. 1 in two versions: a three-dimensional image and representation of a function using isolines. However, it is difficult to make any analysis from these figures, since the change in function is carried out not in a planned interpretation (in latitude and longitude), but in a combined form (in length and longitude). Therefore, for clarity, it is necessary to develop other ways of displaying such information, which is currently being done by the authors of this publication. From the figures, certain clusters can be distinguished at a depth of 6371×0.6 km = 3826 km at longitudes of 180 and 40 degrees, which in geographical location is a projection of points under Africa and in the Pacific Ocean. Without making any interpretations and conclusions, one can only notice the correlation of these placements with the recently discovered two regions of the features of the Earth`s internal structure [21]; one can also distinguish a feature at a depth of 637 km in the region of 180 degrees. At depths of 0.6-0.8, positive values of the integrand are clearly manifested, and from 0 to 0.02, a negative value falling off from the origin. However, it is difficult to make any qualitative interpretation and associate it with a geographical location, because these values are the result of action over the entire latitude.

The potential is determined from the Stokes constants up to the 10th order and takes into account the paired Stokes constants. It is given in Table 1. (columns with the name “sphere”) for different points in space in latitude and longitude for a fixed radius of the relative sphere ($R=3$). For radii that are less than unity, there is a discrepancy between the values calculated in two ways, which is not given here. This discrepancy can be explained by the small number of considerations for the terms of the sum and the way in which the coefficients of the schedule (17) are calculated. We emphasize that the linear combinations include the values of the Stokes orders of all orders to the power of the determination coefficient (17). This effect is leveled out with increasing order of summation, but for small orders it can be noticeable.

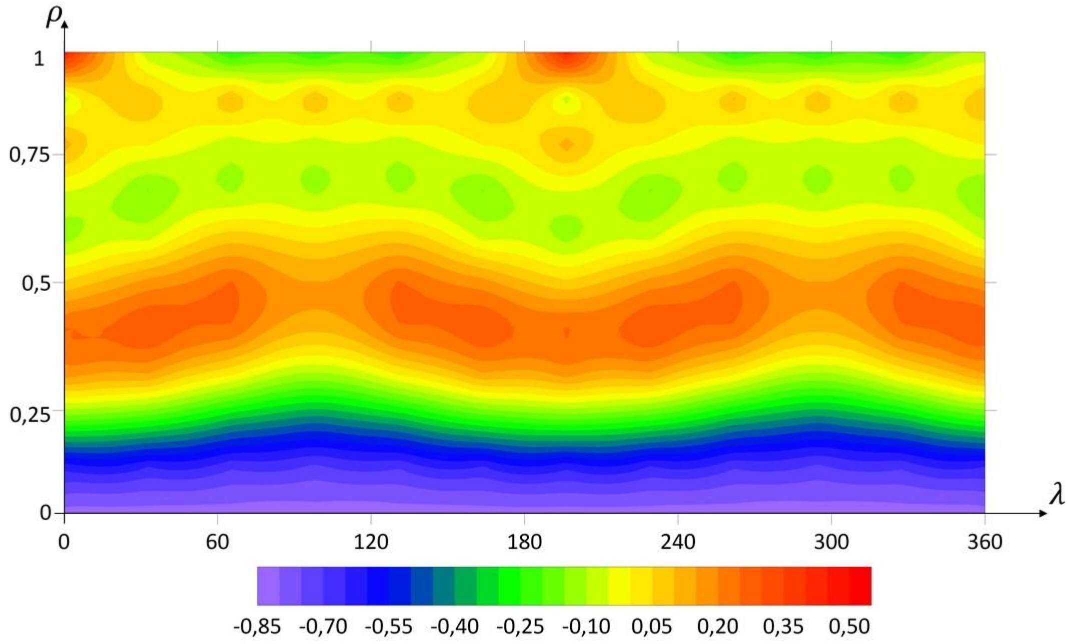

Fig. 1. Ellipse density distribution function (g/cm^2), which corresponds to the Stokes constants of paired orders, (ρ is the relative radius of the ellipse, λ is the polar angle, $0 \le \lambda \le 360°$), isolines are drawn every 0.15 g/cm^2.

Also the results of Table 2 clearly illustrate the closeness of the potential values for a relative radius greater than unity. If the relative radius is less than unity, then its gradual decrease gradually increases the difference between the potential values calculated by the two formulas (1), (19).

One more important note. Although the asymmetry is also generated by the Stokes constants $S_{n,k}$ ($n-k$-pair), their influence on the value of the total potential is much smaller and is not studied in this paper due to the need for additional study approaches (introduction of scaling factors, the possibility of performing calculations, etc.)

Table 1

The value of the potential calculated for the radius *R* = 3 by spherical functions and using the integral of a simple layer taking into account the Stokes constants $C_{n,k}$ ($n-k$-pair)

	0˚		60˚		120˚		180˚		240˚		300˚	
	$V \times 106249482.0 \times 10^8\ \frac{м^2}{с^2}$											
0˚	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005
45˚	-0.501	-0.499	-0.502	-0.502	-0.502	-0.502	-0.501	-0.5	-0.502	-0.502	-0.502	-0.502
90˚	1.003	1.005	1.002	1.001	1.002	1.001	1.003	1.006	1.002	1.001	1.002	1.001
135˚	-0.501	-0.499	-0.502	-0.502	-0.502	-0.502	-0.501	-0.5	-0.502	-0.502	-0.502	-0.502
180˚	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005	-2.005

Table 2

The value of the potential calculated for different radii at a fixed angle $\vartheta = \frac{\pi}{3}$ by spherical functions and using the integral of a simple layer taking into account the Stokes constants $C_{n,k}$ (*n* – *k*-pair)

	0˚		60˚		120˚		180˚		240˚		300˚		0˚	
	$V \times 106249482.0 \times 10^{8}\ \frac{м^{2}}{с^{2}}$													
0.7	-0.518	-0.392	-0.446	-0.395	-0.446	-0.396	-0.518	-0.395	-0.446	-0.396	-0.446	-0.395	-0.518	-0.392
0.8	-0.315	-0.263	-0.295	-0.264	-0.295	-0.265	-0.315	-0.265	-0.295	-0.265	-0.295	-0.264	-0.315	-0.263
0.9	-0.216	-0.185	-0.211	-0.186	-0.211	-0.186	-0.216	-0.186	-0.211	-0.186	-0.211	-0.186	-0.216	-0.185
1.0	-0.149	-0.135	-0.129	-0.135	-0.129	-0.136	-0.149	-0.135	-0.129	-0.136	-0.129	-0.135	-0.149	-0.135
1.1	-0.108	-0.101	-0.101	-0.102	-0.101	-0.102	-0.108	-0.102	-0.101	-0.102	-0.101	-0.102	-0.108	-0.101
1.2	-0.082	-0.078	-0.076	-0.078	-0.076	-0.079	-0.082	-0.078	-0.076	-0.079	-0.076	-0.078	-0.082	-0.078

Conclusions

1. The description by means of integrals of simple layers of the gravitational field on the Earth complements other attempts to represent it.
2. Calculation of the potentials of simple layers can be done with the help of series coinciding outside the region of integration.
3. Potential values obtained by different methods are the same in space, excluding the distribution of masses inside the planet.
4. The two-dimensional integrand of the surface integral can be used for geophysical interpretation, in particular, to reveal the asymmetry of the gravitational field.
5. It is planned to further improve the above technique with the aim of extending it to higher orders and studying the properties caused by other constant groups of Stokes.

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ДОСЛІДЖЕННЯ АСИМЕТРІЇ ГРАВІТАЦІЙНОГО ПОЛЯ ЗЕМЛІ, ПОДАНОГО ПОТЕНЦІАЛАМИ ПЛОСКИХ ДИСКІВ

У роботі розглянуто подання зовнішнього гравітаційного поля Землі, які доповнюють його традиційну апроксимацію рядами за кульовими функціями. Необхідність додаткових засобів опису зовнішнього потенціалу продиктована потребою його вивчення та використання в точках простору, що є близькими до поверхні Землі. Саме в таких областях виникає потреба дослідження збіжності рядів за кульовими функціями та адекватного визначення значення потенціалу. Представлення зовнішнього гравітаційного поля Землі інтегралами простого та подвійного прошарку із залученням апарату апроксимації кусково-неперервної функції в середині еліпса дає змогу розширити для рядів, що подають потенціал, область збіжності до всього простору поза еліпсом інтегрування. Тому, як результат, значення гравітаційного потенціалу збігається зі значеннями цих рядів поза тілом, що містить маси надр (крім еліпса інтегрування). Це дає можливість оцінювати поведінку гравітаційного поля в приповерхневих областях та виконувати з більшою достовірністю дослідження геодинамічних процесів. Апроксимація гравітаційного поля за допомогою поверхневих інтегралів окреслює також геофізичний аспект задачі. Адже під час її розв’язання здійснюється побудова двовимірних підінтегральних функцій, що однозначно визначаються набором стоксових сталих. При цьому коефіцієнти їх розкладів у ряди визначаються за лінійними комбінаціями степеневих моментів їх функцій. Отримані розклади функцій можуть бути використані для дослідження особливостей зовнішнього гравітаційного поля, наприклад, вивчення його асиметрії відносно екваторіальної площини.

Ключові слова: асиметрія гравітаційного поля; Земля; потенціал; сфера Б’єрхамера; стоксові постійні.

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