MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 2, pp. 226–240 (2022)

Mathematical modeling of mixed-traffic in urban areas

Pradhan R. K.[1,2], Shrestha S.[2], Gurung D. B.[2]

[1]*Department of Mathematics, Khwopa College, Bhaktapur, Nepal*
[2]*Department of Mathematics, School of Science, Kathmandu University, Nepal*

(Received 7 July 2021; Revised 28 October 2021; Accepted 13 November 2021)

Transportation is the means of mobility. Due to the growth in the population, rising traffic on road, delay in the movement of vehicles and traffic chaos could be observed in urban areas. Traffic congestion causes many social and economic problems. Because of the convenience and the quickness, motor-bikes gradually become the main travel mode of urban cities. In this paper, we extend the Lighthill–Whitham–Richards (LWR) traffic flow model equation into the mixed-traffic flow of two entities: car and motor-bike in a unidirectional single-lane road segment. The flow of cars is modeled by the advection equation and the flow of motor-bikes is modeled by the advection–diffusion equation. The model equations for cars and motor-bikes are coupled based on total traffic density on the road section, and they are non-dimensionalized to introduce a non-dimensional number widely known as Péclet number. Explicit finite difference schemes satisfying the CFL conditions are employed to solve the model equations numerically to compute the densities of cars and motor-bikes. The simulation of densities over various time instants is studied and presented graphically. Finally, the average densities of cars and motor-bikes on the road section are calculated for various values of Péclet numbers and mixed-traffic behavior are discussed. It is observed that the mixed-traffic behavior of cars and motor-bikes depends upon the Péclet number. The densities of motor-bikes and cars in the mixed-traffic flow approach the equilibrium state earlier in time for smaller values of Péclet number whereas densities take longer time to approach the equilibrium for the greater values of Péclet number.

Keywords: *mixed-traffic, LWR traffic model, Péclet number, CFL condition.*

2010 MSC: 35F25, 35F30, 35L65, 65M06, 90B20 **DOI:** 10.23939/mmc2022.02.226

1. Introduction

The growth of the world's population causes the increase in a number of road users accordingly. Due to the increasing number of road users, many big cities in the world are suffering from traffic congestion. Rising traffic on the road networks, vehicular delays, and traffic chaos have become a normal in urban areas. Nowadays, traffic flow and congestion are one of the major societal and economical problems related to transportation in developing countries. Traffic congestion has become one of the most significant problems in urban areas. The number of motor-bike users and car users is also increasing in urban areas, which also reinforces the congestion makeup on the road. In this respect, a traffic management system in congested networks requires a clear understanding of traffic flow operations. We need the minimization of traffic congestion to maximize the vehicular flow. One should have knowledge of traffic variables and the behavior of vehicles to understand traffic congestion. This paper aims to study and analyze the mixed traffic flow in emerging urban cities where vehicles and motor-bikes share the single traffic lane by using a macroscopic flow model. Macroscopic models are presented by the Partial Differential Equations (PDEs) that describe the evolution of traffic flow over time and space on an aggregated scale using such macroscopic state variables as density, average speed, and vehicular flux. There are other models that describe the movement of individual traffic, widely known as microscopic models, given in the form of Ordinary Differential Equations (ODEs). The fundamental difference between microscopic and macroscopic models are based on the level of detail. Macroscopic models use an equilibrium relationship between speed and flow. This relationship is generally known as the fundamental diagram [1, 2].

The kinematic wave model, which is also known as the Lighthill–Whitham–Richards (LWR) model [3, 4] has been used to describe traffic flow. This model uses a continuum approach. Thus, it is regarded as a macroscopic traffic flow model. This indicates that the average number of vehicles per time unit and space is taken under consideration. In a microscopic approach, different behavioral rules and parameters such as longitudinal and lateral movement rule [5], speed choice, headway [6], reaction time, etc. are defined to model the heterogeneity of driver and vehicles. The parameters and driver behaviors are described by Shiomi et al. [7] differently depending on the interacting vehicle classes. Space discretization methods are also introduced by Chen et al. [8] to accommodate lateral movement within a lane and the variation in vehicle size. Recently much attention goes to multi-class models. Instead of considering traffic flow as a homogenous flow with homogeneous vehicles and drivers (mixed class), the heterogeneity of vehicles and drivers is taken into account, for example, see detail in references [9–12]. Multi-class traffic flows are usually evaluated following a metric called Passenger Car Equivalent (PCE), which reports the impact of a given class of traffic on traffic flow variables. Multi-class models are able to describe real-world phenomena such as capacity drop and hysteresis [11, 13]. Heterogeneous models capture the characteristics of different traffic types and the effect of their interaction on the overall flow. Slow vehicles, such as buses and lorries are considered as moving bottlenecks for cars [14], whereas Daganzo and Knoop [15] have addressed the impact of pedestrians on car traffic. The model of Fan and Work [16] has included the characteristic trait of small vehicles (e.g. motor cyclists), to manoeuvre through congestion, maintaining a higher speed than cars. Gashaw et al. [17] have developed the interaction between cars and powered two-wheelers, whose model also takes into account that a higher share of two-wheelers results in a lower speed at similar road occupancy. A first-order multi-class macroscopic flow model is presented by Wierbos et al. [2] to describe mixed bicycle-car traffic. The model uses class-specific speed functions, enabling each class to be the fastest moving one depending on density.

In this paper, we extend the LWR traffic model equation to the mixed-traffic flow of two entities: car and motor-bike in a unidirectional single lane road section. The flow of car is modeled by advection equation and the flow of motor-bike is modeled by advection-diffusion equation. The model equations for car and motor-bikes are coupled based on total traffic density on the road segment and they are non-dimensionalized to introduce a non-dimensional number widely known as Péclet number. The paper continues with mixed-traffic flow modeling with explanation of car and motor-bikes flow model in Section 2, followed by the numerical techniques and the implementation of the model in Section 3 and the results and discussion are presented in the Section 4. Finally, Section 5 presents the conclusion of the study.

2. Mix-traffic model equations

The first order LWR model was developed by Lighthill and Whitham [3] and P. I. Richards [4] independently is one of the most used macroscopic model. In this model, traffic flow was considered to be analogous to one-directional fluid flow, where macroscopic traffic state variables were described as a function of time and space. The fundamental relationship of speed, density and flow so called macroscopic state variables and mass conservation law are the basic elements for LWR formulation. It describes the flow based on the assumption that traffic is a continuum and obeys the mass conservation law. The conservation law says that the number of vehicles between any two points is conserved. Denoting the number density ρ [#vehicles/km], flow q [#vehicles/hr], average speed u [km/hr] at time [hr] and position x [km], the first order PDE based on the conservation law takes the form

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}q(t,x) = 0 \tag{1}$$

with

$$q(t, x) = \rho(t, x)u(t, x). \tag{2}$$

The average speed $u(t, x)$ depends on the density and a unique speed value corresponds to a specific traffic density $\rho(t, x)$. In the LWR model, all vehicles in a traffic stream were considered to exhibit similar characteristics and no classification between vehicle classes was made. Multi-class extensions of the LWR model emerge to accommodate the heterogeneity in many aspects of road users. Vehicles with identical characteristics are grouped into a class and a conservation law is applied to each class in multi-class modeling.

We extend the LWR traffic model equation to mixed-traffic of two entities: car and motor-bike. The cars and motor-bikes are moving in a single lane without counter movement which are the typical scenario of Asian big cities with narrow single lane road as in Fig. 1. The motor-bikers are driving in a very haphazard manner passing through every small gaps between the vehicles, however, the car-drivers have to follow the very specific lane because of the narrow road structure. The very special phenomenon of motor-bike could be seen when there is a congestion caused either by bad condition of road or traffic light signal. The motor-bikes try to fill the empty spaces in the road by adjusting itself going back and forth. This causes the bikes to diffuse through high to low density to come to an equilibrium state. Thus, the motor-bikes flow shows the diffusive phenomenon when there is congestion due to increase in traffic density. Let the symbols $\rho_c = \rho_c(t, x)$, $u_c = u_c(t, x)$, $q_c = q_c(t, x)$, $u_{c,max}$ denote the density, velocity, flow in the position x at time t and maximum allowed speed of the car and $\rho_b = \rho_b(t, x)$, $u_b = u_b(t, x)$, $q_b = q_b(t, x)$, $u_{b,max}$ denote the density, velocity, flow in the position x at time t and maximum allowed speed of the motor-bike. If we assume that the space occupied by a single car is approximately equal to η numbers of motor-bikes, the combined traffic density $\rho = \rho(t, x)$ is mentioned as in [9, 18] and given by

$$\rho = \rho_c + \frac{1}{\eta}\rho_b. \tag{3}$$

a Traffic flow in a street of Honoi city. (Source: Photo by AFP/Hoang Dinh Nam).

b Traffic flow in a street of Taipei city (Source: https://en-academic.com-Motorcycles-in-Taipei.JPG).

Fig. 1. Cars and motor-bikes in a one-way road segment.

2.1. Car flow model

The traffic flow of car is simply modeled by a continuity equation as given by LWR model without the diffusive as they follow the lane of cars:

$$\frac{\partial \rho_c}{\partial t} + \frac{\partial q_c}{\partial x} = 0 \tag{4}$$

with flow

$$q_c = \rho\, u_c. \tag{5}$$

To close the equation (4) together with flow (5) we propose the linear relationship of total traffic density ρ and car velocity u_c given by

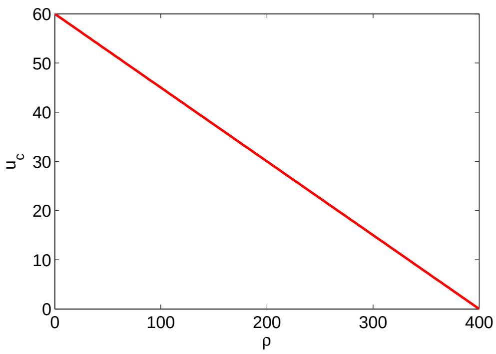
$$u_c = u_{c,max}\left(1 - \frac{\rho}{\rho_{max}}\right), \tag{6}$$

where ρ_{max} is the maximum density of the traffic that the road can hold. Following are the qualitative analysis of the car velocity u_c that can be observed from the equation (6):

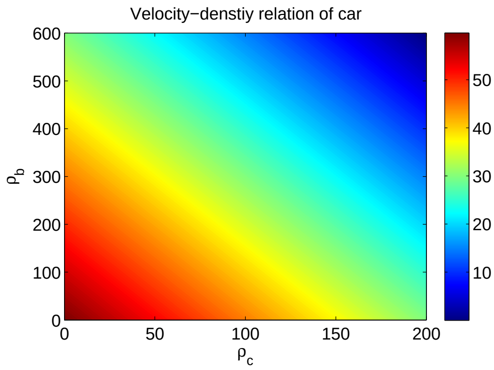
I. $u_c \to u_{c,max}$ as $\rho \to 0$ and $u_c \to 0$ as $\rho \to \rho_{max}$. This is the natural effect that when there is almost no traffic, the car moves with its maximum intended speed as per the traffic rule and when the road is full of traffic, the car has to come to the rest.

II. $\frac{du_c}{d\rho} < 0$, $\frac{\partial u_c}{\partial \rho_c} < 0$ and $\frac{\partial u_c}{\partial \rho_b} < 0$. This tells that the velocity of car decreases with increases of total traffic density or density of cars only or density of motor-bikes only.

Assuming the space occupied by cars and motor-bikes are in equal proportion, that means, $\rho_c = \frac{1}{\eta}\rho_b$ and taking $\eta = 3$, $\rho_{max} = 400$ vehicles/km, $v_{c,max} = 60$ km/hr, the Fig. 2 presents the plots of the velocity of the car versus total density (left figure) and the densities of car and motor-bike (right figure) that clearly verifies the previous qualitative analysis in the points I and II.



a Velocity of car versus total density.



b Contour plot of velocity of car.

Fig. 2. Velocity-density relation of car from equation (6).

Now, we perform the qualitative analysis of the flow q_c of car which is given by combining the equations (5) and (6) to obtain

$$q_c = u_{c,max}\left(\rho - \frac{\rho^2}{\rho_{max}}\right), \tag{7}$$

where ρ is given in the equation (3). Now, differentiating equation (7) with respect to ρ_c and ρ_b partially, we get

$$\frac{\partial q_c}{\partial \rho_c} = u_{c,max}\left(1 - \frac{2\rho}{\rho_{max}}\right);$$
$$\frac{\partial q_c}{\partial \rho_b} = \frac{u_{c,max}}{\eta}\left(1 - \frac{2\rho}{\rho_{max}}\right).$$

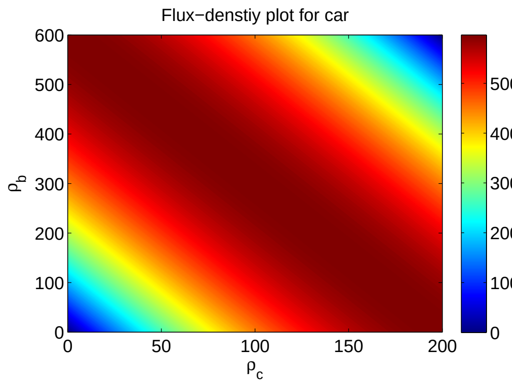


Fig. 3. The flow versus density of the car and motor-bike.

It can be seen that partial derivatives $\frac{\partial q_c}{\partial \rho_c} = 0$ and $\frac{\partial q_c}{\partial \rho_c} = 0$ at $\rho = \frac{\rho_{max}}{2}$. Also, the maximum flow of the car is $q_{c,max} = \frac{\rho_{max} u_{c,max}}{4}$ at $\rho = \frac{\rho_{max}}{2}$. The Fig. 3 presents the contour plot of flux of car versus densities of car and motor-bike.

2.2. Motor-bike flow model

The traffic flow of motor-bikes is modeled by a continuity equation as given by LWR model with the diffusive term D as motor-bikes have some random nature to move back and forth when there is an increase in the traffic density and diffusive nature disappears when there is a free flow in traffic. Thus, the diffusivity D of the motor-bike depends on the total density ρ of the traffic, that means, $D = D(\rho)$ [km^2/hr]. In our study, we assume that the the diffusivity D follows the parabolic rule such that it is proportional to $\frac{\rho}{\rho_{max}}\left(1-\frac{\rho}{\rho_{max}}\right)$. That means, $D(\rho) = \alpha\frac{\rho}{\rho_{max}}\left(1-\frac{\rho}{\rho_{max}}\right)$ where α [km^2/hr] is positive constant. Here, $D(\rho) = 0$ when $\rho = 0$ because motor-bike can run with its maximum speed. Similarly, when $\rho = \rho_{max}$, the diffusivity $D(\rho) = 0$ because motor-bikes could not move back and forth due jam condition. This physical phenomenon is depicted in the following Fig. 4.

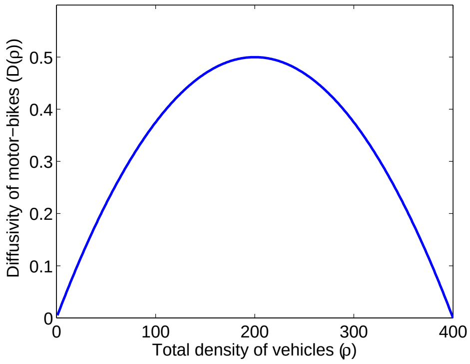


Fig. 4. Diffusivity of motor-bikes as a function of total density.

Finally, the traffic density of motor-bikes is then modeled by

$$\frac{\partial\rho_b}{\partial t} + \frac{\partial q_b}{\partial x} = \frac{\partial}{\partial x}\left(D(\rho)\frac{\partial\rho_b}{\partial x}\right), \tag{8}$$

which can be expressed as

$$\frac{\partial\rho_b}{\partial t} + \frac{\partial}{\partial x}(Q_b) = 0, \tag{9}$$

where

$$Q_b = q_b - D(\rho)\frac{\partial\rho_b}{\partial x} \tag{10}$$

with flow

$$q_b = \rho\, u_b. \tag{11}$$

To close the equation (8) together with flow equation (11), we propose the linear relationship of total traffic density ρ and motor-bike velocity u_b given by

$$u_b = u_{b,max}\left(1-\frac{\rho}{\rho_{max}}\right), \tag{12}$$

where ρ_{max} is the maximum density of the traffic that the road can hold. Following are the qualitative analysis of the motor-bike velocity u_b that can be observed from the equation (12):

I. $u_b \to u_{b,max}$ as $\rho \to 0$ and $u_b \to 0$ as $\rho \to \rho_{max}$. This is the natural effect that when there is almost no traffic, the motor-bike moves with its maximum intended speed as per the traffic rule and when the road is full of traffic, the motor-bike has to come to the rest.

II. $\frac{du_b}{d\rho} < 0$, $\frac{\partial u_b}{\partial\rho_b} < 0$ and $\frac{\partial u_b}{\partial\rho_c} < 0$. This tells that the velocity of motor-bike decreases with increases of total traffic density or density of motor-bike only or density of car only.

Now, we perform the qualitative analysis of the flow Q_b of motor-bike which is given by combining the equations (10), (11) and (12) to obtain

$$Q_b = u_{b,max}\left(\rho - \frac{\rho^2}{\rho_{max}}\right) - D(\rho)\frac{\partial\rho_b}{\partial x}, \tag{13}$$

where ρ is given in the equation (3). Now, differentiating equation (13) with respect to ρ_b and ρ_c partially, we get

$$\frac{\partial Q_b}{\partial\rho_b} = \frac{1}{\eta}\left(u_{b,max} - \frac{\alpha}{\rho_{max}}\frac{\partial\rho_b}{\partial x}\right)\left(1-\frac{2\rho}{\rho_{max}}\right);$$
$$\frac{\partial Q_b}{\partial\rho_c} = \left(u_{b,max} - \frac{\alpha}{\rho_{max}}\frac{\partial\rho_b}{\partial x}\right)\left(1-\frac{2\rho}{\rho_{max}}\right).$$

It can be seen that partial derivatives $\frac{\partial Q_b}{\partial\rho_b} = 0$ and $\frac{\partial Q_b}{\partial\rho_c} = 0$ at $\rho = \frac{\rho_{max}}{2}$. The maximum flow of the motor-bike is $q_{b,max} = \frac{\rho_{max}u_{b,max}}{4}$ at $\rho = \frac{\rho_{max}}{2}$.

3. Numerical techniques

The analytical solution of PDEs is not always possible. In such case one should solve the PDEs numerically. There are many methods of solving PDEs numerically. Finite Difference Method (FDM) is one of the simplest method among them. In order to solve the simultaneous partial differential equations with an initial boundary value problem (IBVP) of mixed traffic flow model, an explicit finite difference scheme is used. Basically, we use a centered difference technique with the mid grid points. Since the car-drivers have to follow the very specific lane because of the narrow road structure, the car flow is modeled by advection equation (4) however motor-bike flow is modeled by advection diffusion equation (8) because motor-bikers are driving in a very haphazard manner passing through every small gaps between the vehicles. We consider the flow of cars and motor-bikes in the extended LWR model in a road segment of single lane. The model equations for cars and motor-bikes are coupled based on total traffic density on the road segment and they are non-dimensionalized which introduces a non-dimensional number widely known as Péclet number. Péclet number is defined as the ratio of advective transport to diffusive transport. We have used explicit finite difference schemes satisfying the CFL-conditions to solve the model equations numerically to compute the densities of cars and motor-bikes.

We have considered the flow of cars and motor-bikes in a single lane road segment $a \leqslant x \leqslant b$ at the time $t \geqslant t_0$ represented by

$$\begin{aligned} &\frac{\partial \rho_c}{\partial t} + \frac{\partial q_c}{\partial x} = 0, \quad a \leqslant x \leqslant b, \quad t_0 \leqslant t \leqslant T, \\ &\frac{\partial \rho_b}{\partial t} + \frac{\partial q_b}{\partial x} = \frac{\partial}{\partial x}\left(D(\rho)\frac{\partial \rho_b}{\partial x}\right), \quad a \leqslant x \leqslant b, \quad t_0 \leqslant t \leqslant T. \end{aligned} \tag{14}$$

Here $\rho_c(t,x)$ and $\rho_b(t,x)$ represent the density of cars and motor-bikes respectively on the road section x at time t. The terms q_c and q_b denote the flux of cars and motor-bikes respectively.

For the dimensionless form, we take

$$R = \frac{\rho}{\rho_{max}}, \quad R_c = \frac{\rho_c}{\rho_{max}}, \quad \tau = \frac{v_{max}}{L}t, \quad y = \frac{x}{L}, \quad R_b = \frac{\rho_b}{\rho_{max}},$$

where $v_{max} = \max\{v_{c,max}, v_{b,max}\}$.

The dimensionless form of the above equation is

$$\begin{aligned} &\frac{\partial R_c}{\partial \tau} + \frac{\partial}{\partial y}(R(1-R)) = 0, \\ &\frac{\partial R_b}{\partial \tau} + \frac{\partial}{\partial y}(R(1-R)) = \frac{\alpha}{L\, v_{b,max}}\frac{\partial}{\partial y}\left(R(1-R)\frac{\partial R_b}{\partial y}\right). \end{aligned} \tag{15}$$

Non-dimensionalization introduces, so called Péclet number and denoting it by $pe = L\, v_{b,max}/\alpha$, the above equation can be written as

$$\begin{aligned} &\frac{\partial R_c}{\partial \tau} + \frac{\partial Q}{\partial y} = 0, \\ &\frac{\partial R_b}{\partial \tau} + \frac{\partial Q}{\partial y} = \frac{1}{pe}\frac{\partial}{\partial y}\left(Q\frac{\partial R_b}{\partial y}\right), \end{aligned} \tag{16}$$

where $Q = R(1-R)$.

The initial condition for density of cars and bikes on the road segment at initial time $\tau = 0$ are

$$\begin{aligned} &R_c(0,y) = R_{c,0}(y), \quad 0 \leqslant y \leqslant 1, \\ &R_b(0,y) = R_{b,0}(y), \quad 0 \leqslant y \leqslant 1. \end{aligned} \tag{17}$$

Also, for the left boundary conditions, the density of cars and motorbikes at $y = 0$ on the road segment for $\tau \geqslant 0$ are

$$\begin{aligned} R_c(\tau, 0) &= R_{c,0}(\tau), \quad 0 \leqslant \tau \leqslant \mathrm{T_f}, \\ R_b(\tau, 0) &= R_{b,0}(\tau), \quad 0 \leqslant \tau \leqslant \mathrm{T_f}, \end{aligned} \tag{18}$$

where $\mathrm{T_f} = \frac{\tau L}{v_{max}}$.

The time independent Dirichlet boundary conditions are used for both cars and motor-bikes.

We discretize the dimensionless spatial domain $0 \leqslant y \leqslant 1$ to approximate the solution of the model equations as $0 = y_0 < y_1 < y_2 < \ldots < y_M = 1$ and the temporal domain $0 \leqslant \tau \leqslant \mathrm{T_f}$ as $0 = \tau^0 < \tau^1 < \tau^2 < \ldots < \tau^N = \mathrm{T_f}$ with the spatial grid size $\Delta y_i = y_i - y_{i-1}$ and temporal grid size $\Delta\tau^n = \tau^n - \tau^{n-1}$ for $i = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$. We take the equal grid size, that is, $\Delta y_i = \Delta y$ and $\Delta\tau^n = \Delta\tau$ for all $i = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$. The temporal grid size is selected to satisfy the CFL-condition for both cars and motor-bikes. The time step for the advection equation of car is chosen to satisfy $\frac{\Delta\tau}{\Delta y}|\frac{dQ}{dR}|_{max} \leqslant 1$ and the time step for the advection diffusion equation of motor-bikes is chosen to satisfy $\frac{1}{pe}\frac{\Delta\tau}{\Delta y^2}|Q|_{max} \leqslant 0.5$ and $\frac{\Delta y}{|Q|_{min}}|\frac{dQ}{dR}|_{max} pe < 1$. Finally, $\Delta\tau$ is chosen minimum of advective time step for car and diffusive time step for motor-bikes. We discretize $\frac{\partial R}{\partial \tau}$ by the first order forward difference formula in time, $\frac{\partial Q}{\partial y}$ by the first order backward difference formula in space at any point (τ^n, y_i) for $i = 1, \ldots, M$, $n = 0, 1, \cdots, N-1$ and considering $R(\tau^n, y_i) = R_i^n$. Basically, we use a centered difference technique with the mid grid points $i + \frac{1}{2}$ and $i - \frac{1}{2}$. The studies are also carried out on the average density of the vehicles on the domain which are given by $\int_0^1 R_c\,dy$ and $\int_0^1 R_b\,dy$ for cars and motor-bikes, respectively. The $\frac{1}{3}$-Simpson's rule is employed to approximate the average densities. The numerical scheme for the non-dimensional equation (16) is

$$R_{c_i}^{n+1} = R_{c_i}^n - \frac{\Delta\tau}{\Delta y}\left(Q_i^n - Q_{i-1}^n\right), \tag{19}$$

$$R_{b_i}^{n+1} = R_{b_i}^n - \frac{\Delta\tau}{2\Delta y}\left(Q_{i+1}^n - Q_{i-1}^n\right) + \frac{\Delta\tau}{pe\,(\Delta y)^2}\left[Q_{i+\frac{1}{2}}\left(R_{b_{i+1}}^n - R_{b_i}^n\right) - Q_{i-\frac{1}{2}}\left(R_{b_i}^n - R_{b_{i-1}}^n\right)\right], \tag{20}$$

where

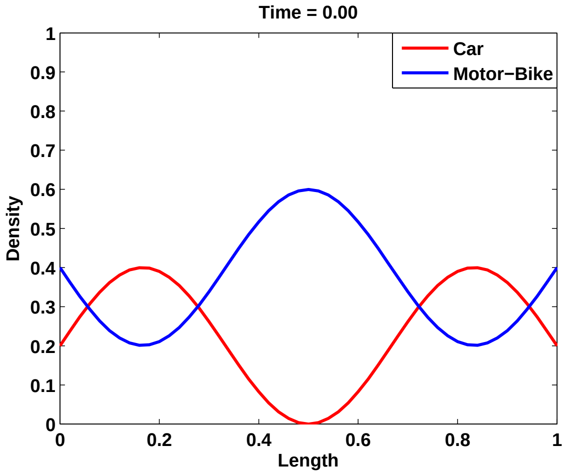
$$Q_{i+\frac{1}{2}} = \frac{1}{2}(Q_{i+1} + Q_i), \quad Q_{i-\frac{1}{2}} = \frac{1}{2}(Q_i + Q_{i-1}).$$

4. Results and discussions

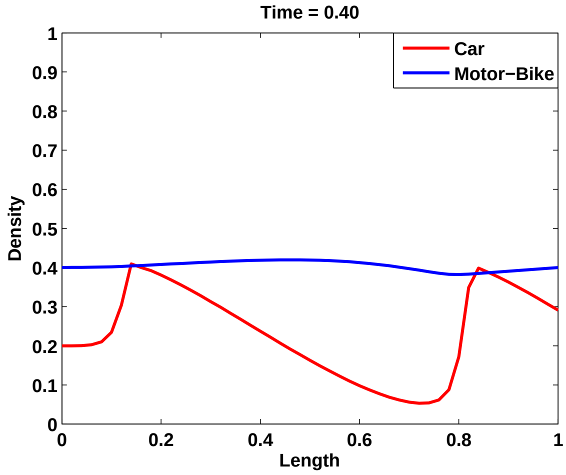
The Péclet number measures the relative dominance of advection versus diffusion phenomenon. A large Péclet number indicates an advectively dominated flow whereas a small number indicates a diffusive flow. We have plotted the density profile of cars and motor-bikes for Péclet number $pe = 0.5$, 1 and 10 to observe the behavior of the traffic flow. The average density profile of cars and motor-bikes for different values of Péclet number is observed for final time $T_f = 10$. We have also carried out the comparison of average density profile of cars and motor-bikes for different values of γ, which is the ratio of initial density of cars and motorbikes keeping the initial density of cars fixed and varying the initial density of motor-bikes, to observe the impact of motor-bikes on cars. In Figs. 5–7, densities of motor-bike and car for different values of Péclet number at different time points have been observed.

Figure 5 represents the snap shots of densities of car and motor-bike at different time points 0.00, 0.40, 0.80, 1.20, 1.60 and 1.99 for Péclet number $pe = 0.5$. We observed that the densities of motor-bikes comes in steady state condition faster where as density of cars changes slowly.

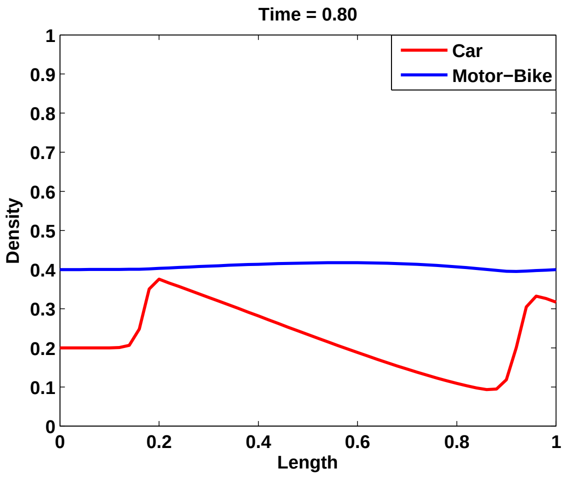
Figure 6 represents the snap shots of densities of car and motor-bike at different time for Péclet number $pe = 1$ at different time points. This illustrates the impact on densities of cars and motor-bikes at different time.



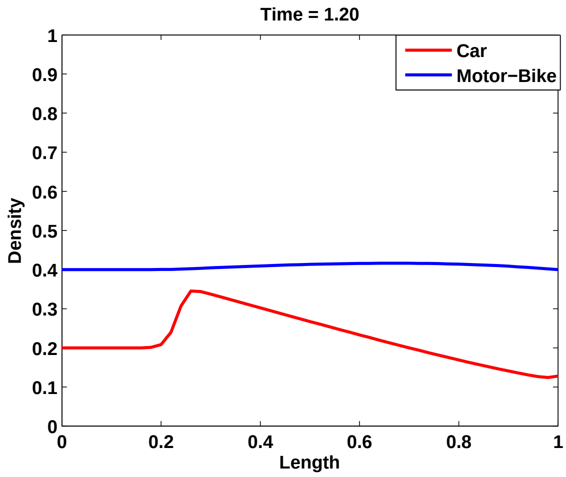
a Initial density profile.



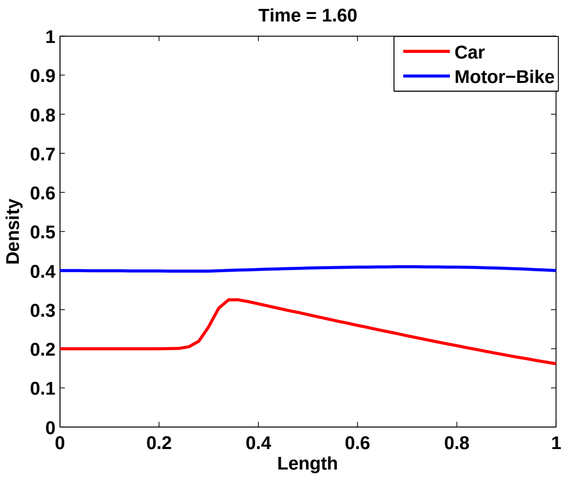
b Density at time 0.40.



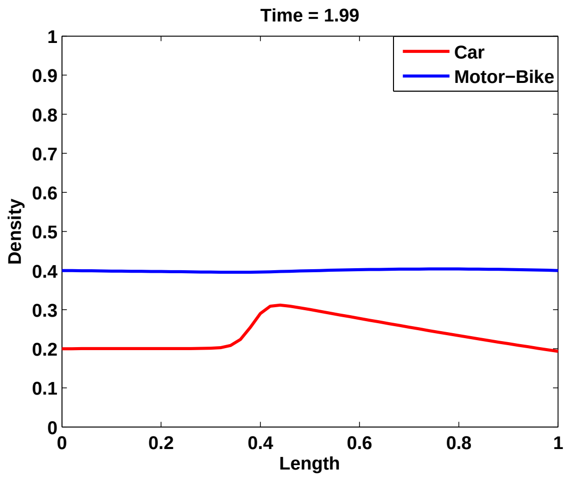
c Density at time 0.80.



d Density at time 1.20.

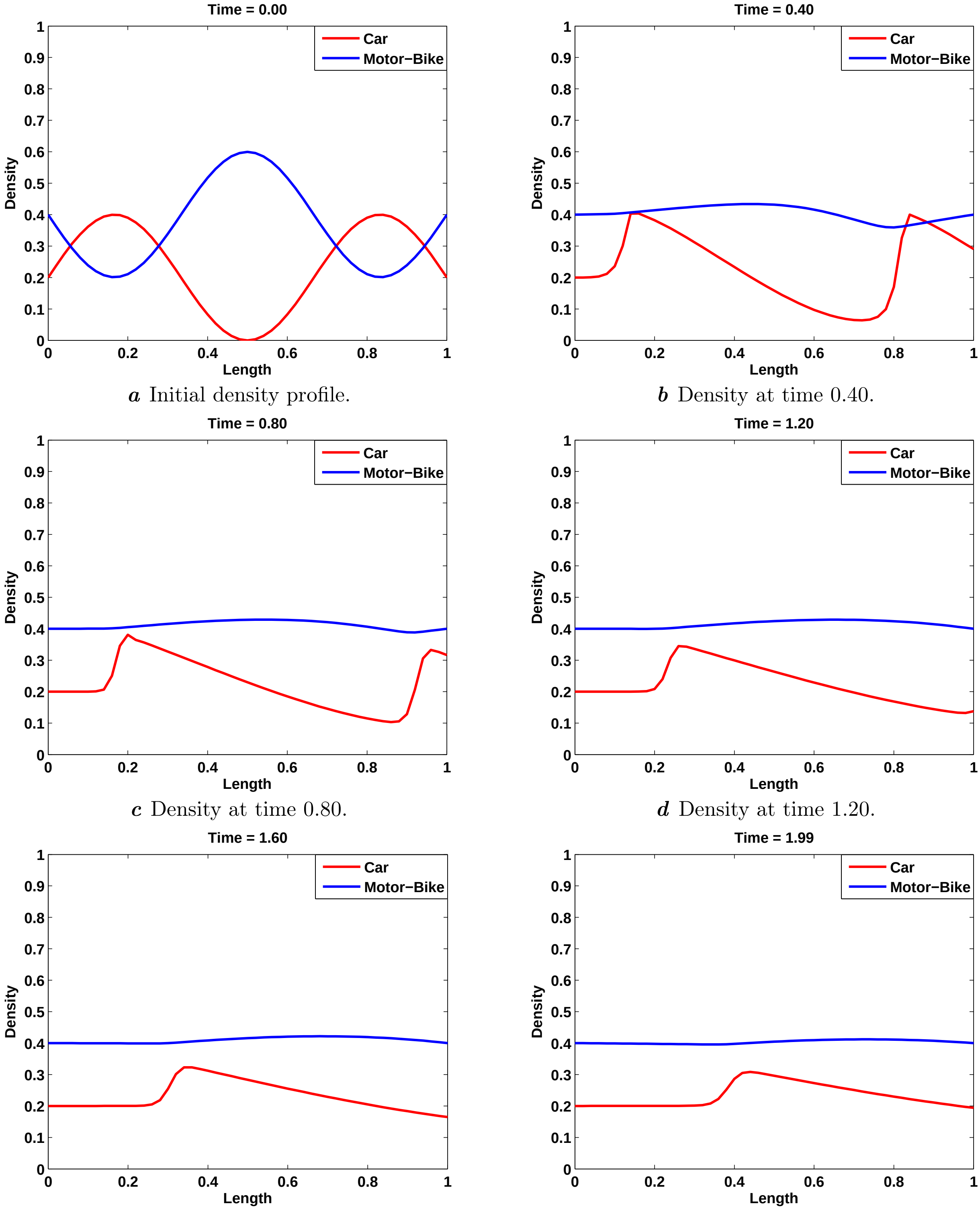


e Density at time 1.60.



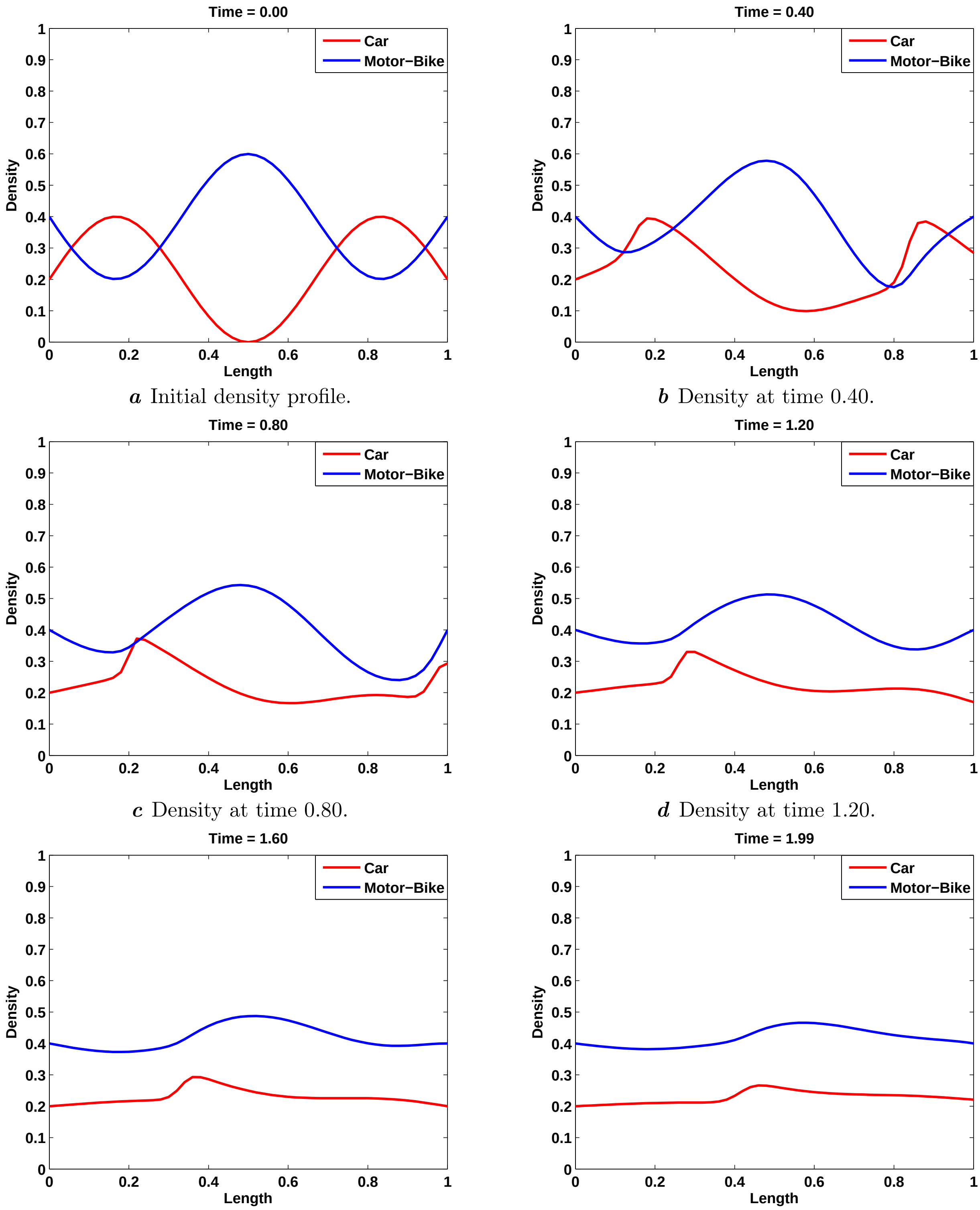
f Density at time 1.99.

Fig. 5. Density of motor-bike and car versus space for Péclet $pe = 0.5$ at different time points in non-dimensional setting: the blue and red solid curves respectively represent the density of motor-bike and car. The results in the plots a, b, c, d, e and f respectively represent the density of motor-bike and car at different times 0.00, 0.40, 0.80, 1.20, 1.60 and 1.99.



a Initial density profile.
b Density at time 0.40.
c Density at time 0.80.
d Density at time 1.20.
e Density at time 1.60.
f Density at time 1.99.

Fig. 6. Density of motor-bike and car versus space for Péclet $pe = 1$ at different time: the blue and red solid curves respectively represent the density of motor-bike and car. The results in the plots a, b, c, d, e and f respectively represent the density of motor-bike and car at different times 0.00, 0.40, 0.80, 1.20, 1.60 and 1.99.



a Initial density profile.

b Density at time 0.40.

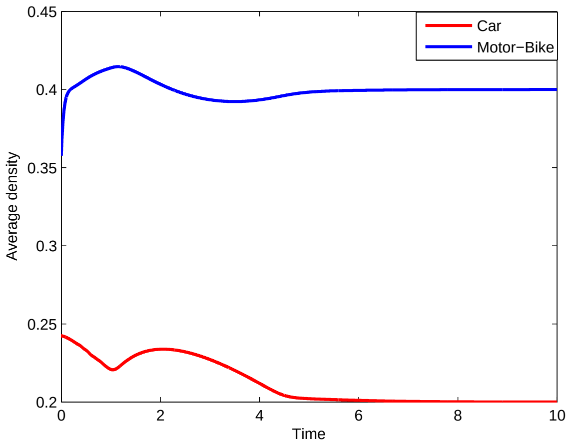
c Density at time 0.80.

d Density at time 1.20.

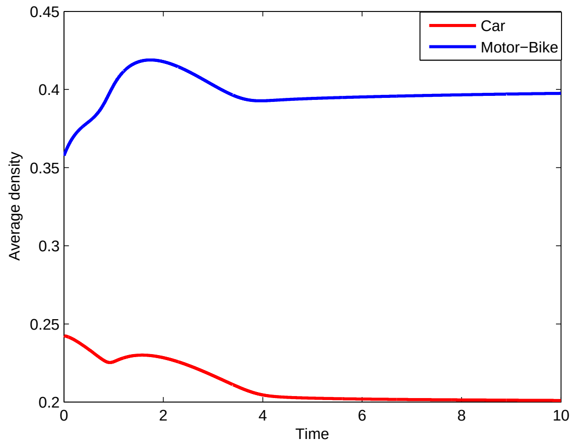
e Density at time 1.60.

f Density at time 1.99.

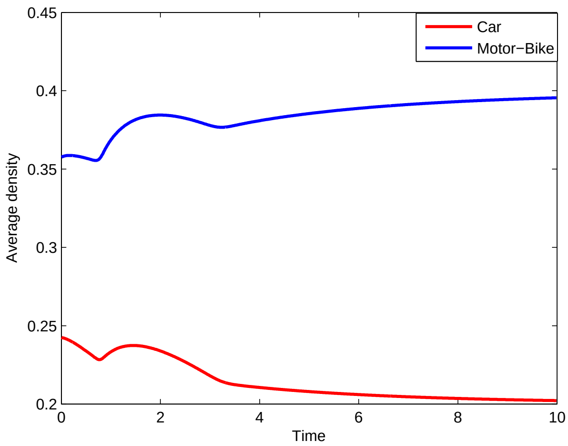
Fig. 7. Density of motor-bike and car versus space for Péclet $pe = 10$ at different time: the blue and red solid curves respectively represent the density of motor-bike and car. The results in the plots a, b, c, d, e and f respectively represent the density of motor-bike and car at different times 0.00, 0.40, 0.80, 1.20, 1.60 and 1.99.



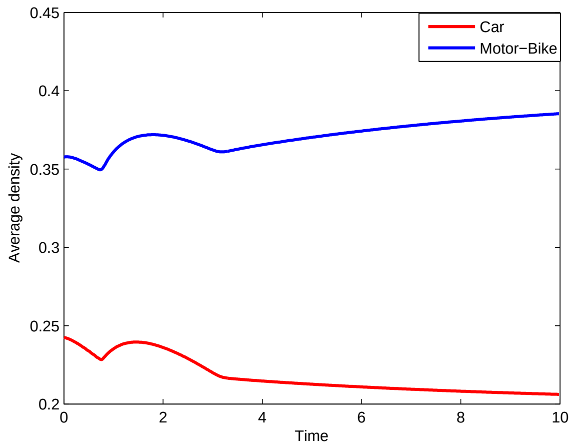
a Péclet number 1.



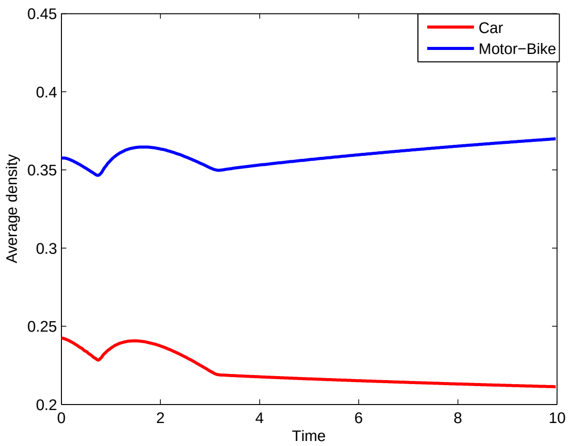
b Péclet number 10.



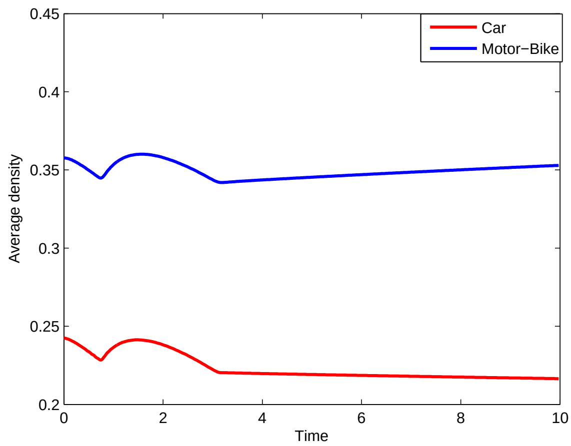
c Péclet number 50.



d Péclet number 100.



e Péclet number 200.



f Péclet number 500.

Fig. 8. Average density of motor-bike and car versus space for different values of Péclet number at final time $T_f = 10$. The blue and red solid curves respectively represent the average density of motor-bike and car. The results in the plots a, b, c, d, e and f respectively represent the average density of motor-bike and car at different values of Péclet number 1, 10, 50, 100, 200 and 500.

Figure 7 represents the snap shots of densities of cars and motor-bikes at different time points for Péclet number $pe = 10$. We observed that the densities of motor-bikes and cars take more time to come in equilibrium state for larger value of Péclet number. Comparing Figs. 5–7, we conclude that when the value of the Péclet number pe increases it takes more time for the density of motor-bikes to come in equilibrium. It is due to the fact that diffusivity causes the system to come in the equilibrium state faster. If the diffusivity is dominated then the value of Péclet number increases and it takes longer time for the densities to come in equilibrium state.

Figure 8 represents the average densities of cars and motor-bikes at final time $T_f = 10$ for different values of Péclet numbers $pe = 1$, 10, 50, 100, 200 and 500. As the value of Péclet number increases the densities of both cars and motor-bikes increase and takes more time to come in equilibrium state. It is observed that the average densities of motor-bike and car approach in equilibrium state in early time for smaller value of Péclet number and take longer time to approach in equilibrium state for the larger value of Péclet number.

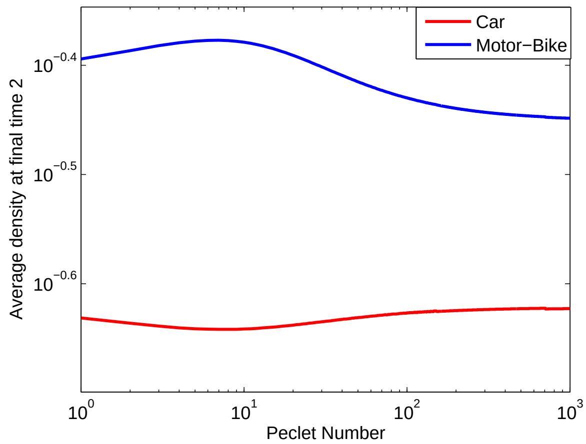
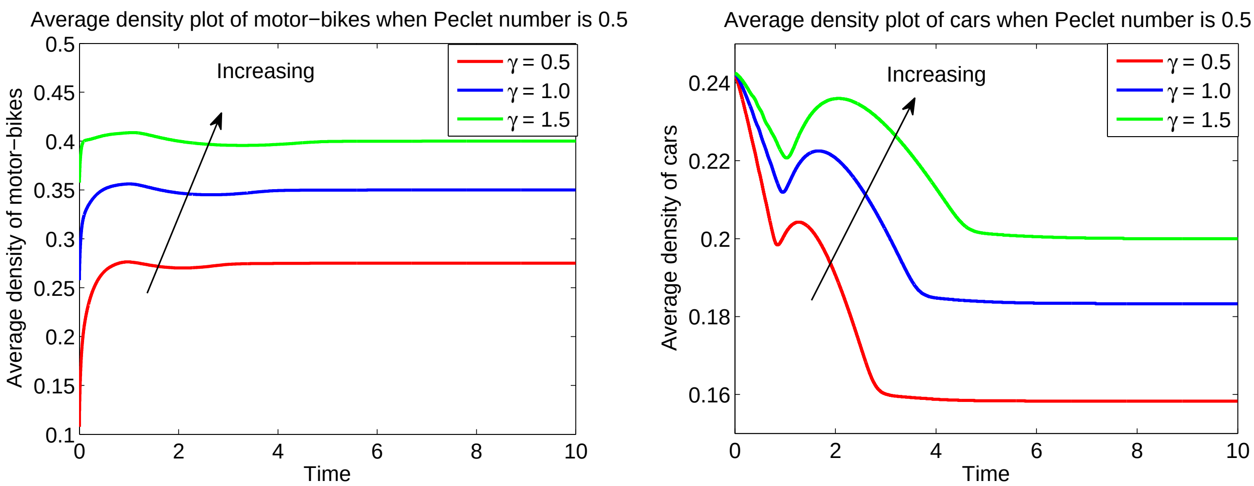


Fig. 9. Density of motor-bike and car versus Péclet number at final time $T_f = 2$.

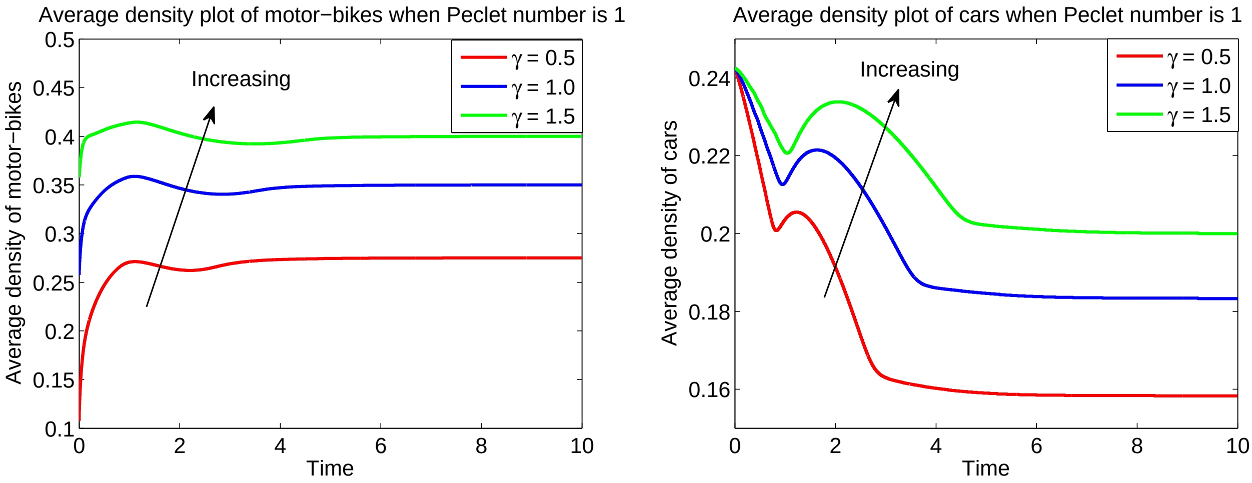
Figure 9 represents the average densities of cars and motor-bikes versus Péclet numbers at final time $T_f = 2$. We observed that when the value of Péclet number increases the average density of cars increases where as the average density profile of motor-bikes has decreased.



a Average density of motor-bikes over time. ***b*** Average density of cars over time.

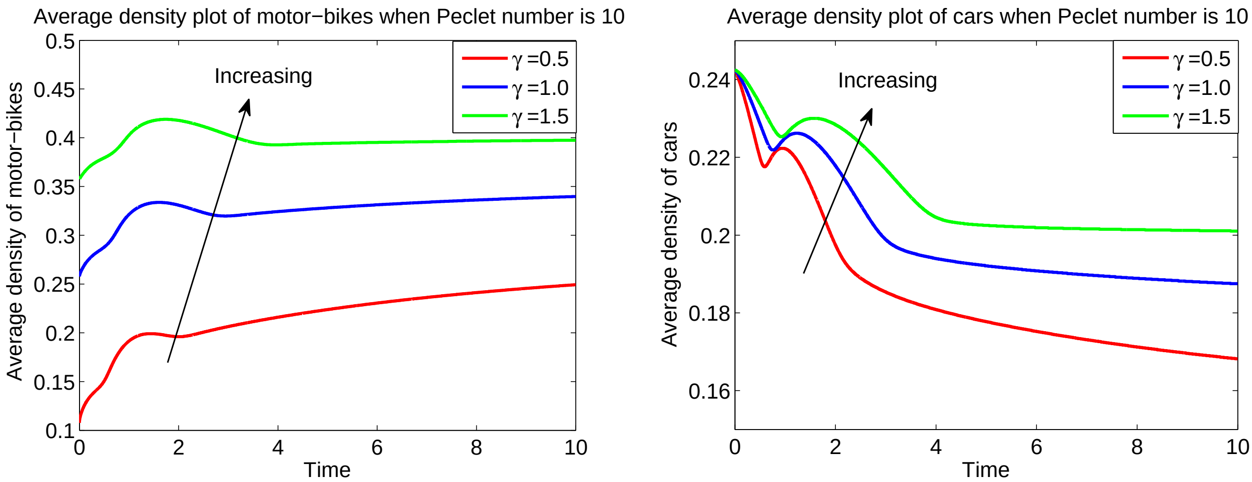
Fig. 10. Average density of motor-bike and car at different ratio γ of initial density when Péclet number $pe = 0.5$. The green, blue and red solid curves respectively represent the average density of motor-bike in the plot ***a*** (left) and average density of car in the plot ***b*** (right) for $\gamma = 0.5$, 1.0, 1.5.

Figures 10–12 show the average density profiles of cars and motor-bikes at final time $T_f = 10$ for different initial density ratio $\gamma = 0.5$, 1.0, 1.5 of cars and motor-bikes for different values of Péclet number $pe = 0.5$, 1.0, 10. The results in the plots ***a*** (left) of Figs. 10–12 represent the average density of motor-bike and the results in the plots ***b*** (right) of Figs. 10–12 represent the average density of car. Since the initial density profile of motor-bike and car is in reciprocal phenomenon, the density profiles of motor-bike and car in the time interval $[0, 2]$ is reversed. When the average density of motor-bikes increases, the average density of cars decreases in the time interval $[0, 2]$ then after they come in equilibrium condition as time increases. It is observed from Figs. 10–12 that the average densities



a Average density of motor-bikes over time. ***b*** Average density of cars over time.

Fig. 11. Average density of motor-bike and car at different ratio γ of initial density when Péclet number $pe = 1$. The green, blue and red solid curves respectively represent the average density of motor-bike in the plot ***a*** (left) and average density of car in the plot ***b*** (right) for $\gamma = 0.5$, 1.0, 1.5.



a Average density of motor-bikes over time. ***b*** Average density of cars over time.

Fig. 12. Average density of motor-bike and car at different ratio γ of initial density when Péclet number is $pe = 10$. The green, blue and red solid curves respectively represent the average density of motor-bike in the plot ***a*** (left) and average density of car in the plot ***b*** (right) for $\gamma = 0.5$, 1.0, 1.5.

of both motor-bike and car approach in equilibrium state in longer time for larger value of Péclet number. In each case, we have observed that as the ratio γ increases, the average density of cars also increases. At the same time, as the average density of cars decreases over the time, the average density of motor-bikes has increased at that time which shows the natural phenomenon of the traffic density in roads. Thus, it is concluded that the initial density profile of motor-bike and car influences both entities to reach in equilibrium condition.

5. Conclusions

In this paper, we extended the LWR traffic model equation in the mixed-traffic flow of cars and motor-bikes without counter flow in a single lane of a road segment. The advection equation has been used to model the flow of cars and the advection-diffusion equation has been used to model the flow behavior of motor-bikes depicting the phenomenon of the motor-bike riders in traffic congestion. Model equations of cars and motor-bikes are coupled based on the total traffic density on the segment of the road and have been non-dimensionalized to introduce a non-dimensional Péclet number. The model equations are solved numerically by using the explicit finite difference method, and the average density of motor-bikes and cars is computed by using $\frac{1}{3}$-Simpson rule in the domain. From the numerically simulated results and graphs, we claim that the proposed mixed-traffic model is able to capture the real mixed-traffic behavior of cars and motor-bikes in the road segment. The mixed-traffic behavior of cars and motor-bikes depends upon the Péclet number. The Péclet number becomes smaller if the traffic is dominated by diffusion term and becomes larger if it is dominated by advection. The density of motor-bikes and cars in mixed-traffic approaches in equilibrium state earlier if the Péclet number is less than 1 whereas it takes more time for greater values of Péclet number. The initial density profile of motor-bikes and cars also influences both entities to reach an equilibrium condition.

[1] Greenshields B. D. A Study of Highway Capacity. Highway Research Board Proceeding. 448–477 (1935).

[2] Wierbos M. J., Knoop V. L., Hänseler F. S., Hoogendoorn S. P. A macroscopic flow model for mixed bicycle–car traffic. Transportmetrica A: Transport Science. **17** (3), 340–355 (2021).

[3] Lighthill M. J., Whitham G. B. On kinematic waves II. A theory of traffic flow on long crowded roads. Proceedings of the Royal Society of London Series A: Mathematical, Physical and Engineering Sciences. **229**, 317–345 (1955).

[4] Richards P. I. Shock Waves on the Highway. Operations Research. **4** (1), 42–51 (1956).

[5] Pandey G., Rao K, R., Mohan D. Modelling vehicular interactions for heterogeneous traffic flow using cellular automata with position preference. Journal of Modern Transportation. **25** (3), 163–177 (2017).

[6] Lenorzer A., Casas J., Dinesh R., Zubair M., Sharma N., Dixit V., Torday A., Brackstone M. Modelling and simulation of mixed traffic. Australasian Transport Research Forum (ATRF), Sydney, New South Wales, Australia. **37**, 1–12 (2015).

[7] Shiomi Y., Hanamori T., Eng M., Nobuhiro U., Shimamoto H. Modeling traffic flow dominated by motorcycles based on discrete choice approach. Proceedings of 1st LATSIS Conference (2012).

[8] Chen G., Fu G., Deng M., Li L. A cell automation traffic flow model for mixed traffic. Procedia – Social and Behavioral Sciences. **96**, 1412–1419 (2013).

[9] Chanut S., Buisson C. Macroscopic model and its numerical solution for two-flow mixed traffic with different speeds and lengths. Transportation Research Record: Journal of the Transportation Research Board. **1852** (1), 209–219 (2003).

[10] Logghe S., Immers L. H. Multi-class kinematic wave theory of traffic flow. Transportation Research Part B: Methodological. **42** (6), 523–541 (2008).

[11] Wong G. C. K., Wong S. C. A multi-class traffic flow model — an extension of LWR model with heterogeneous drivers. Transportation Research Part A: Policy and Practice. **36** (9), 827–841 (2002).

[12] Zhang P., Zhang R. X., Wong S., Dai S. Q. Hyperbolicity and kinematic waves of a class of multi-population partial differential equations. European Journal of Applied Mathematics. **17** (2), 171–200 (2006).

[13] Ngoduy D. Multiclass first-order modelling of traffic networks using discontinuous flowdensity relationships. Transportmetrica. **6** (2), 121–141 (2010).

[14] Lebacque J., Lesort J., Giorgi F. Introducing Buses Into First-Order Macroscopic Traffic Flow Models. Transportation Research Record. **1644** (1), 70–79 (1998).

[15] Daganzo C. F., Knoop V. L. Traffic Flow on Pedestrianized Streets. Transportation Research Part B: Methodological. **86**, 211–222 (2016).

[16] Fan S., Work D. B. Heterogeneous Multiclass Traffic Flow Model With Creeping. SIAM Journal on Applied Mathematics. **72** (2), 813–835 (2015).

[17] Gashaw S., Goatin P., Härri J. Modeling and Analysis of Mixed Flow of Cars and Powered Two Wheelers. Transportation Research Part C: Emerging Technologies. **89**, 148–167 (2018).

[18] Van Wageningen-Kesselsa F., van't Hof B., Hoogendoorna S. P., van Lint H., Vuik K. Anisotropy in generic multi-class traffic flow models. Transportmetrica A: Transport Science. 451–472 (2013).

Математичне моделювання змішаного руху в містах

Прадхан Р. К.[1,2], Шрестха С.[2], Гурунг Д. Б.[2]

[1] *Кафедра математики, Коледж Хвопа, Бхактапур, Непал*
[2] *Кафедра математики, Школа науки,*
Університет Катманду, Непал

Транспорт — це засіб пересування. У зв'язку із зростанням населення у містах спостерігається зростання дорожнього потоку, затримки руху транспортних засобів та транспортний хаос. Перевантаження руху викликає багато соціальних та економічних проблем. Завдяки зручності та швидкості, мотоцикли поступово стають основним засобом пересування містом. У цій статті ми розширили рівняння моделі транспортного потоку Лайтхілла–Уізема–Річардса (LWR) на випадок змішаного потоку двох об'єктів: автомобіля та мотоцикла на однонаправленому сегменті дороги з однією смугою руху. Потік автомобілів моделюється за допомогою рівняння адвекції, а потік мотоциклів — адвекції–дифузії. Рівняння моделі для автомобілів та мотоциклів пов'язані змінною загальної густини руху на ділянці дороги, і вони безрозмірні, щоб ввести безрозмірне число, відоме як число Пекле. Явні скінченно-різницеві схеми, що задовольняють умови CFL, використовуються для чисельного розв'язування модельних рівнянь для розрахунку густин автомобілів та мотоциклів. Досліджено та представлено графічно моделювання густин у різні моменти часу. Також розрахована середня густина автомобілів та мотоциклів на ділянці дороги для різних значень числа Пекле та обговорено її поведінку при змішаному русі. Помічено, що поведінка автомобілів та мотоциклів у змішаному русі залежить від числа Пекле. Густини мотоциклів і автомобілів у змішаному транспортному потоці наближаються з часом до рівноважного стану раніше для менших значень числа Пекле, тоді як для великих значень числа Пекле ця густина наближається до рівноваги за довший час.

Ключові слова: *змішаний дорожній рух, модель дорожнього руху LWR, число Пекле, умова CFL.*