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Call warrants pricing formula under mixed-fractional Brownian motion with Merton jump-diffusion

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Mixed fractional Brownian motion (MFBM) is a linear combination of a Brownian motion and an independent fractional Brownian motion which may overcome the problem of arbitrage, while a jump process in time series is another problem to be address in modeling stock prices. This study models call warrants with MFBM and includes the jump process in its dynamics. The pricing formula for a warrant with mixed-fractional Brownian motion and jump, is obtained via quasi-conditional expectation and risk-neutral valuation.

Keywords: *mixed-fractional Brownian motion, Merton jump diffusion, call warrant.*
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1. Introduction

Warrant is a derivative that gives the holder the right, not an obligation, to purchase a given quantity of an underlying share at a predetermined exercise price, on or before the expiry date. Hence, the holder can either exercise the right to buy the shares at the exercise price or sell the warrant. Warrants can be categorized into call warrants or put warrants, where the Bursa Malaysia allows the former type to be traded. Warrants may be modelled using the Black–Scholes model [1] because of similar properties to options.

The payoff of a warrant allows the holder the right to buy k number of shares of stock at the price of G, which is

$$\frac{1}{N+Mk}\max\left(kS_T-NG,0\right), \tag{1}$$

where N is the number of shares for common stocks, M is the number of shares for outstanding warrants, and S is the value of the underlying asset at maturity T.

Brownian motion, which enables arbitrage-free market has been employed as the stochastic model to simulate logarithmic returns, of which research by [2–4] demonstrated that the logarithmic return distribution has characteristics such as fat tails, volatility smile and long-range dependence. This encourages studies, such as the ones by [5–8] that modelled long-range dependency by characterising the distribution of the logarithmic returns with fractional Brownian motion (FBM). However, [9, 10] proposed the mixed-fractional Brownian motion (MFBM) which combines both the Brownian motion and the FBM with Hurst parameter $H \in (0,1)$ to solve the issue of allowing arbitrage in the FBM [11, 12]. According to [13], selecting $H \in \left(\frac{3}{4},1\right)$ reduces the MFBM to the standard Brownian motion, preventing arbitrage and allowing complete market. Furthermore, with reference to [14], the MFBM can capture the local variability of the process model, providing a more accurate representation of the financial data.

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The MFBM has drawn a lot of interest for its potential in option pricing [15–17]. Additionally, [18, 19] added jumps to the MFBM to value currency options, compound options, and extendible options. In [20], an actuarial technique was used to price Asian options under MFBM with jumps. On that account, and the similarities between options and warrants, [21] has investigated the pricing of warrants with the MFBM.

In order to derive the closed-form pricing formula for call warrants, this work aims to model call warrants with the MFBM and includes a jump process. The paper is organized as follows. Section 2 which explains the model formulation for call warrants in the MFBM environment with jumps, briefly provides some description of the model. The analytical solution is derived in Section 3, and the study is concluded in Section 4.

2. Preliminaries

This section briefly presents some description of the mixed-fractional Brownian motion (MFBM) as described in [13] and the jump-diffusion dynamics as described in [22]. This model can be used in arbitrage-free, equilibrium, and complete markets, as well as those that are arbitrage-filled, out of equilibrium, and incomplete markets.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_{t,t\geqslant 0}\}, \mathbb{P})$ be a filtered probability space where $\mathbb{P}$ is the physical probability measure. Then an MFBM, $B_t^H(\alpha,\beta)$ is defined as

$$B_t^H(\alpha,\beta) = \alpha W_t + \beta W_T^H,$$

where W_t is a Brownian motion, W_t^H is a fractional Brownian motion with Hurst parameter $H \in \left(\frac{3}{4}, 1\right)$, $\{\mathcal{F}_{t,t\geqslant 0}\}$ is the $\mathbb{P}$-augmentation of the filtration generated by (W_T, W_T^H) for $\tau \leqslant t$, and α and β are some real non-zero constants. Additional properties of the MFBM can be found in [15, 16].

In a risk-neutral framework, assuming a market with no friction, the underlying asset price moves according to the following dynamics:

$$dS_t = (r - \lambda\mu_{J_t})\, S_t\, dt + \sigma S_t\, dW_t + \sigma_H S_t\, dW_t^H + (y_t - 1) S_t\, dN_t, \tag{2}$$

where r is the risk-free rate, σ and σ_H are constant volatilities of the logarithmic returns when jumps do not occur, N_t denotes a Poisson process with intensity λ, while y_t is the size of absolute price jump, and $(y_t - 1)$ is the size of the relative price jump (see [22]). Furthermore, according to [22], the logarithmic asset price jump sizes are normally distributed, such that $\ln y_t \sim \mathcal{N}\left(\mu_{J_t}, \delta_t^2\right)$, where μ_J is an i.i.d. logarithmic return jump size, and δ is an i.i.d. volatility of the logarithmic return jump. Hence,

$$(y_t - 1) \sim \ln\left(e^{\mu_J + \frac{\delta^2}{2}} - 1, e(2\mu_J + \delta^2\left[e^{\delta^2} - 1\right]\right).$$

Suppose that $\mathbb{Q}$ is a risk-neutral probability, the quasi-conditional expectation of the stock price is given as

$$\mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t^H\right] = S_t\, e^{r(T-t)}.$$

Therefore, following [23], under $\mathbb{Q}$, the solution to Equation (2) is obtained via Itô formula as follows

$$S_T = S_t \prod_{i=1}^{N_{T-t}} e^{J_{t_i}} e^{\left(r-\lambda\mu_{J_t}-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\sigma_H^2\left(T^{2H}-t^{2H}\right)+\sigma(W_T-W_t)+\sigma_H\left(W_T^H-W_t^H\right)},$$

or

$$S_T = S_t e^{\left(r-\lambda\mu_{J_t}-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\sigma_H^2\left(T^{2H}-t^{2H}\right)+\sigma(W_T-W_t)+\sigma_H\left(W_T^H-W_t^H\right)+\sum_{i=1}^{N_{T-t}}\ln J_i}. \tag{3}$$

The derivation of the call warrants pricing formula under the MFBM-MJD market is under the assumption of no transaction costs or taxes, and no riskless arbitrage opportunities exist.

3. Results and discussion

This section provides the derivation of the closed-form pricing formula for call warrants under the mixed-fractional Brownian motion with Merton jump-diffusion model (MFBM-MJD).

Suppose that the underlying asset price follows the process in Equation (2), and the payoff is presented as a function of $P(S_T)$. At a risk-free interest rate, r, the price of the call warrant is the discounted risk-neutral conditional expectation of its payoff function given by

$$w_t(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[P(S_T)\middle|\mathcal{F}_t^H\right],$$

where $\mathbb{E}$ is a quasi-conditional expectation with respect to the risk-neutral measure $\mathbb{Q}$.

From Equation (3) and following [24], it can be observed that the Wiener processes are Gaussian, such that $(W_T - W_t) \sim \mathcal{N}(0, T-t)$ and $(W_T^H - W_t^H) \sim \mathcal{N}(0, T^{2H} - t^{2H})$. It follows that $(W_T - W_t) = \sqrt{T-t}Z_1$ and $(W_T^H - W_t^H) = \sqrt{T^{2H} - t^{2H}}Z_2$, where $Z_1 \sim \mathcal{N}(0,1)$ and $Z_2 \sim \mathcal{N}(0,1)$. Additionally, $J_{t_1}, \dots, J_{t_n}$ is a sequence of i.i.d. random variables, where $\ln J_{t_i} \sim \mathcal{N}(\mu_{J_t}, \delta_t^2)$, which implies $\sum_{i=1}^n \ln J_{t_i} \sim \mathcal{N}(n\mu_{J_t}, n\delta_t^2)$. Consequently, $\sum_{i=1}^n \ln J_{t_i} = n\mu_{J_t} + \sqrt{n}\delta_t Z_3$, where $Z_3 \sim \mathcal{N}(0,1)$. Therefore, we have

$$\sigma_n\sqrt{T-t}Z_n = \sigma\sqrt{T-t}Z_1 + \sigma_H\sqrt{T^{2H} - t^{2H}}Z_2 + \sqrt{n}\,\delta_t Z_3, \tag{4}$$

where $Z_n \sim \mathcal{N}(0,1)$ and $\sigma_n^2 = \sigma^2 + \frac{\sigma_H^2(T^{2H}-t^{2H})}{T-t} + \frac{n\delta_t^2}{T-t}$. Hence, by Equations (3) and (4), and let $N_{T-t} = n$,

$$S_T^n = S_t e^{\left(r - \lambda\mu_{J_t} - \frac{1}{2}\sigma^2\right)(T-t) - \frac{1}{2}\sigma_H^2\left(T^{2H}-t^{2H}\right) + \sigma(W_T - W_t) + \sigma_H\left(W_T^H - W_t^H\right) + n\mu_{J_t} + \sigma_n\sqrt{T-t}Z_n}. \tag{5}$$

We further let $r_n = r - \lambda\mu_{J_t} + \frac{\mu_{J_t} + \frac{\delta_t^2}{2}n}{T-t}$. So, Equation (5) can be written as follows

$$S_T^n = S_t e^{\left(r_n - \frac{\sigma_n^2}{2}\right)(T-t) + \sigma_n\sqrt{T-t}Z_n}.$$

Consider the payoff function that is defined by Equation (1). Reference [25] states that warrant holders should exercise their warrants only when $kS_T \geqslant NG$. Then, under a risk-neutral world, the price of a call warrant at $t \in [0, T]$ is defined by

$$\begin{aligned} w_t(t, S_t) &= \frac{e^{-r(T-t)}}{N + Mk}\mathbb{E}^{\mathbb{Q}}\left[\max(kS_T - NG, 0)\middle|\mathcal{F}_t^H\right] \\ &= \frac{e^{-r(T-t)}}{N + Mk}k\,\underbrace{\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbb{I}_{\left\{S_T > \frac{NG}{k}\right\}}\middle|\mathcal{F}_t^H\right]}_{A} - \frac{e^{-r(T-t)}}{N + Mk}NG\,\underbrace{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\left\{S_T > \frac{NG}{k}\right\}}\middle|\mathcal{F}_t^H\right]}_{B}. \end{aligned}$$

Now, let

$$d_2 = \frac{\ln\frac{NG}{kS_t} - (r_n - \frac{\sigma_n^2}{2})(T-t)}{\sigma_n\sqrt{T-t}}.$$

From Equations (3) and (5), and using the independence of $N(T-t)$ and J_{t_i}, also the Poisson distribution with intensity $\lambda(T-t)$, we solved A as follows

$$\begin{aligned} A &= \sum_{n=0}^{\infty}\mathbb{E}^{\mathbb{Q}}\left[S_T^n\mathbb{I}_{\left\{S_T > \frac{NG}{k}\right\}}\middle|\mathcal{F}_t^H\right]P(N_{T-t} = n) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}\mathbb{E}^{\mathbb{Q}}\left[\left(S_t e^{\left(r_n - \frac{\sigma_n^2}{2}\right)(T-t) + \sigma_n\sqrt{T-t}Z_n}\right)\mathbb{I}_{\left\{S_T > \frac{NG}{k}\right\}}\middle|\mathcal{F}_t^H\right] \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}S_t\,e^{r_n(T-t)}\int_{d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x - \sigma_n\sqrt{T-t})^2}{2}}dx \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}S_t\,e^{r_n(T-t)}\int_{d_2 - \sigma_n\sqrt{T-t}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}S_t\,e^{r_n(T-t)}\left[1 - \phi(d_2 - \sigma_n\sqrt{T-t})\right] \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}}{n!} S_t \, e^{r_n(T-t)} \phi(-d_2 + \sigma_n \sqrt{T-t}).$$

Then, B can be solved as follows

$$\begin{aligned} B &= \sum_{n=0}^{\infty} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{S_T > \frac{NG}{k}\}} \middle| \mathcal{F}_t^H \right] P(N_{T-t} = n) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}}{n!} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{S_T > \frac{NG}{k}\}} \middle| \mathcal{F}_t^H \right] \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}}{n!} \int_{d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}}{n!} \left[1 - \phi(d_2) \right] \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}}{n!} \phi(-d_2). \end{aligned}$$

Hence, the following proposition presents the closed-form pricing formula of a call warrant under MFBM-MJD.

Proposition 1. The price of a call warrant under a mixed-fractional Brownian motion with jumps with expiry date T and k number of shares of stocks for payment J, is given by

$$w_t = \frac{e^{-r(T-t)}}{N+Mk} \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)}}{n!} \left[k S_t e^{r_n(T-t)} \phi(d_{1J}) - NG \phi(d_{2J}) \right],$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{kS_t}{NG} + (r_n + \frac{\sigma_n^2}{2})(T-t)}{\sigma_n \sqrt{T-t}}, \\ d_2 &= \frac{\ln \frac{kS_t}{NG} + (r_n - \frac{\sigma_n^2}{2})(T-t)}{\sigma_n \sqrt{T-t}}, \\ r_n &= r - \lambda \mu_{J_t} + \frac{\mu_{J_t} + \frac{\delta_t^2}{2} n}{T-t}, \\ \sigma_n &= \sqrt{\sigma^2 + \frac{\sigma_H^2 \left(T^{2H} - t^{2H} \right)}{T-t} + \frac{n \delta_t^2}{T-t}}. \end{aligned}$$

Hence we have obtained the closed–form formula for call warrants under MFBM-MJD.

4. Conclusion

In this study, the closed-form pricing formula for call warrants under mixed-fractional Brownian motion with jump-diffusion (MFBM-MJD) is presented. The derivation applies quasi-conditional expectation and uses the risk-neutral valuation approach. This model captures long-memory phenomenon and discontinuous behavior in the logarithmic returns. Future work may include utilizing the path integral method [26, 27] to model the pricing of warrants.

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Формула ціноутворення колл-варантів за змішано-дробового броунівського руху зі стрибкоподібною дифузією Мертона

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Змішаний дробовий броунівський рух (ЗДБР) — це лінійна комбінація броунівського руху та незалежного дробового броунівського руху, яка може вирішити проблему арбітражу, тоді як стрибковий процес у часових рядах є ще однією проблемою, яку слід вирішити під час моделювання цін на акції. Це дослідження спрямоване на моделювання варантів за допомогою ЗДБР і включення стрибкоподібного процесу в його динаміку. Формула ціноутворення для варанта зі змішаним дробовим броунівським рухом і стрибком отримана за допомогою квазіумовного очікування та нейтральної до ризику оцінки.

Ключові слова: *змішано-дробовий броунівський рух, стрибкоподібна дифузія Мертона, колл–варант.*