MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 4, pp. 833–841 (2022)

The solution of an infinite system of ternary differential equations

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(Received 11 August 2022; Revised 9 October 2022; Accepted 10 October 2022)

The present paper is devoted to an infinite system of differential equations. This system consists of ternary differential equations corresponding to 3×3 Jordan blocks. The system is considered in the Hilbert space l_2. A theorem about the existence and uniqueness of solution of the system is proved.

Keywords: *differential equation, infinite system, existence and uniqueness of solution, Hilbert space.*

2010 MSC: 91A23, 91A24 **DOI:** 10.23939/mmc2022.04.833

1. Introduction

Many real-life problems are reduced to the control problems described by partial differential equations (PDE) (see, for example, [1–3]). It is well known that one of the main methods to solve such problems for the PDE is the decomposition one (see, for example, [1, 4–8]). As a result, we obtain a control problem for infinite system of differential equations.

Indeed, let a controlled distributed system be described by the following parabolic equation

$$\frac{\partial y}{\partial t} + Ay = w, \quad y(x,0) = y_0(x), \quad x \in \Delta, \quad y(x,t) = 0, \quad x \in \partial\Delta, \quad 0 < t < T, \tag{1}$$

where $y = y(x,t)$ is the unknown function, $x = (x_1, x_2, \dots, x_n) \in \Delta \subset \mathbb{R}^n$, $n \geqslant 1$, Δ is a bounded domain, and the boundary $\partial\Delta$ of the domain Δ is assumed to be piecewise smooth, $t \in [0,T]$, and T is a given positive number, $w = w(x,t)$ is the control function $w(x,t) \in L_2(C_T)$,

$$C_T = \{(x,t) | x \in \Delta,\ 0 < t < T\},$$

is an open cylinder in $\mathbb{R}^{n+1}$, $y_0(x) \in L_2(\Delta)$,

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right), \quad a_{ij}(x) = a_{ji}(x),$$

$a_{ij}(x)$ is a bounded measurable function. Also, there exists positive number k such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geqslant k\sum_{i=1}^{n}\eta_i^2, \quad \text{for all} \quad (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}^n \quad \text{and} \quad x \in \Delta.$$

This work was partially supported by the National Fundamental Research Grant Scheme FRGS of Malaysia, FRGS/1/2020/STG06/UPM/02/2.

Further, recall that $W_2^1(\Delta)$ is Hilbert space of elements of $L_2(\Delta)$ with first-order generalized derivatives being square integrable on Δ, $\mathring{W}_2^{1,0}(C_T)$ is the subspace of $W_2^1(\Delta)$ where smooth compactly supported functions form a dense subset, $W_2^{1,0}(C_T)$ is Hilbert space of elements of $L_2(C_T)$ with generalized derivatives $\frac{\partial y}{\partial x_i}$, $i = 1, 2, \dots, n$, being square integrable on C_T, and $\mathring{W}_2^{1,0}(C_T)$ is the subspace of $W_2^{1,0}(C_T)$ were smooth functions vanishing near C_T form a dense set. The inner products in $L_2(\Delta)$ and $W_2^1(\Delta)$ are defined by the formulas

$$(u, v)_{L_2} = \int_\Delta u(x)v(x)\,dx, \quad (u, v)_{W_2^1} = \int_\Delta \big(u(x)v(x) + u_x(x)v_x(x)\big)\,dx,$$

respectively, and norms in these spaces are defined by the formulas

$$\|u\|_{L_2} = \sqrt{(u, u)_{L_2}}, \quad \|u\|_{W_2^1} = \sqrt{(u, u)_{W_2^1}},$$

respectively.

If the above-mentioned conditions are satisfied [9], then for any $w(x, t) \in L_2(C_T)$ and $y_0(x) \in L_2(\Delta)$, problem (1) has a unique generalized solution $y = y(x, t)$ in the class $\mathring{W}_2^{1,0}(C_T)$. Moreover, the solution is in the form (see [9, III. 3])

$$y(x, t) = \sum_{i=1}^{\infty} y_i(t)\varphi_i(x), \tag{2}$$

where the functions $y_i(t)$, $0 \leqslant t \leqslant T$, $i = 1, 2, \dots$, form a solution of Cauchy problem for the infinite system of differential equations

$$\dot{y}_i = \lambda_i y_i + w_i(t), \quad y_i(0) = y_{i0}, \quad i = 1, 2, \dots, \tag{3}$$

$\lambda_1, \lambda_2, \dots, \lambda_i, \dots$ are the generalized eigenvalues of the operator A [4], all these eigenvalues are positive, and $\lambda_i \to +\infty$ as $i \to \infty$, the functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_i(x), \dots$ form an orthonormal complete system of generalized eigenfunctions of A in $L_2(\Delta)$, and $w_i(t)$ and y_{i0} are Fourier coefficients of $w(x, t)$ and $y_0(x)$, respectively, in the system $\{\varphi_i(x)\}$, that is

$$w(x, t) = \sum_{i=1}^{\infty} w_i(t)\varphi_i(x), \quad y_0(x, t) = \sum_{i=1}^{\infty} y_{i0}(t)\varphi_i(x).$$

In addition, the series (2) converges uniformly in $L_2(C_T)$ and its sum $y(x, t)$ belongs to the space $\mathring{W}_2^1(\Delta)$ for each $t \in [0, T]$ and is a continuous function of the variable t in the norm of $\mathring{W}_2^1(\Delta)$ [9].

Thus, there is close relationship between the control problems described by partial differential equations (1) and infinite system of differential equations (3). For example, in [6,7,10,11], differential game problems described by the linear partial differential equation of the next form

$$\frac{\partial z}{\partial t} = Az + u - v, \quad Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right),$$

where u and v are the control parameters of pursuer and evader respectively, $z = z(x, t)$ is a scalar function, were reduced to a differential game described by the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = u_k - v_k, \quad k = 1, 2, \dots,$$

where u_k and v_k, $k = 1, 2, \dots$, are control parameters of pursuer and evader respectively, $z_k, u_k, v_k \in \mathbb{R}$, and coefficients λ_k, $k = 1, 2, \dots$, satisfy the condition

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \longrightarrow \infty.$$

The paper [12] also relates to such games.

The papers [7, 10, 11, 13, 14] suggested studying differential game problems described by infinite system of differential equations (4) in one theoretical frame independently of partial differential equations assuming that λ_k, $k = 1, 2, \ldots$, in (4) are any real numbers. Later on various differential game problems described for infinite systems of differential equations were studied in the works [15–19].

So, there is a significant relationship between control problems described by partial differential equations and those described by infinite system of differential equations.

We recall the vector space of all sequences of real numbers

$$l_2 = \Big\{\xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) \mid \sum_{n=1}^{\infty} \xi_n^2 < \infty\Big\},$$

is Hilbert space with the inner product and norm defined by

$$(\xi, \eta) = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \|\xi\| = \sqrt{(\xi, \xi)}.$$

In [20] existence and uniqueness theorem was proved for the infinite system (4) for any positive numbers λ_i, $i = 1, 2, \ldots$, in Hilbert space associated with the operator A. Later on such a theorem was proved for the following infinite system of differential equations [21]

$$\begin{aligned} \dot{x}_i &= -\alpha_i x_i - \beta_i y_i + w_{i1}, \quad x_i(0) = x_{i0}, \\ \dot{y}_i &= \quad \beta_i x_i - \alpha_i y_i + w_{i2}, \quad y_i(0) = y_{i0}, \end{aligned}$$

in Hilbert space l_2, where α_i, β_i are real numbers, $\alpha_i \geqslant 0$, $(x_{10}, x_{20}, \ldots)$, $(y_{10}, y_{20}, \ldots) \in l_2$, the function $w(t) = (w_1(t), w_2(t), \ldots)$, $t \in [0, T]$, and components $w_i(t) = (w_{i1}(t), w_{i2}(t))$ are measurable and satisfy the condition

$$\int_0^T \sum_{i=1}^{\infty} \big(w_{i1}^2(t) + w_{i2}^2(t)\big) dt \leqslant \rho^2, \quad 0 \leqslant t \leqslant T,$$

T is a sufficiently large fixed number.

The general purpose of this paper is to study the following infinite system of differential equations:

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + y_i + w_{i1}(t), & x_i(0) &= x_{i0}, \\ \dot{y}_i &= -\lambda_i y_i + z_i + w_{i2}(t), & y_i(0) &= y_{i0}, \quad i = 1, 2, \ldots, \\ \dot{z}_i &= -\lambda_i z_i + w_{i3}(t), & z_i(0) &= z_{i0}, \end{aligned} \tag{4}$$

in Hilbert space l_2, where λ_i is a given non negative real number,

$$x_0 = (x_{10}, x_{20}, \ldots), \quad y_0 = (y_{10}, y_{20}, \ldots), \quad z_0 = (z_{10}, z_{20}, \ldots) \in l_2.$$

The class of functions $w(t) = (w_1(t), w_2(t), \ldots)$, $w \colon [0, T] \to l_2$, with measurable coordinates $w_i(t) = (w_{i1}(t), w_{i2}(t), w_{i3}(t))$, $0 \leqslant t \leqslant T$, $i = 1, 2, \ldots$, satisfying the condition

$$\sum_{i=1}^{\infty} \int_0^T \big(w_{i1}^2(s) + w_{i2}^2(s) + w_{i3}^2(s)\big)\, ds \leqslant \rho_0^2,$$

we denote by $S(\rho_0)$, where ρ_0 is a positive number.

The problem is to determine does there a unique solution of the system (4) in Hilbert space l_2 exist? Let

$$\begin{aligned} \eta_i(t) &= ((x_i(t), y_i(t), z_i(t)), \quad |\eta_i(t)| = \sqrt{x_i^2(t) + y_i^2(t) + z_i^2(t)}, \\ \eta(t) &= (\eta_1(t), \eta_2(t), \ldots) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t), \ldots), \\ \eta_0 &= (\eta_{10}, \eta_{20}, \ldots) = (x_{10}, y_{10}, z_{10}, x_{20}, y_{20}, z_{20}, \ldots), \\ \|\eta(t)\| &= \left(\sum_{i=1}^{\infty} (x_i^2(t) + y_i^2(t) + z_i^2(t))\right)^{1/2}, \quad \|\eta_0\| = \left(\sum_{i=1}^{\infty} (x_{i0}^2 + y_{i0}^2 + z_{i0}^2)\right)^{1/2}. \end{aligned}$$

2. Notation and preliminary results

In this section, we state some necessary basic definition and properties related to the study.

Definition 1. *Let $w(\cdot) \in S(\rho_0)$. A function $\eta(t) = (\eta_1(t), \eta_2(t), \ldots)$, with continuous coordinates $\eta_i(t)$ satisfying initial conditions $\eta_i(0) = \eta_{i0}$, $i = 1, 2, \ldots$ is said to be solution of the system (4) if $\eta_i(t)$ is differentiable almost everywhere on $[0, T]$ and satisfies almost everywhere on $[0, T]$ the system (4).*

It can be shown that for the matrix

$$A_i = \begin{bmatrix} -\lambda_i & 1 & 0 \\ 0 & -\lambda_i & 1 \\ 0 & 0 & -\lambda_i \end{bmatrix}, \quad i = 1, 2, \ldots,$$

we have

$$e^{A_i t} = e^{-\lambda_i t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad i = 1, 2, \ldots.$$

It is not difficult to verify that the following statement is true.

Property 1. For the matrix $e^{A_i t}$, the following relations hold

(i) $e^{A_i(t+h)} = e^{A_i t} \cdot e^{A_i h}$;

(ii) $\left|e^{A_i t}\eta_i\right| \leqslant e^{-\lambda_i t} a(t)|\eta_i|$; $\left\|e^{A_i t}\right\| \leqslant a(T)$, $a(t) = 1 + t + \frac{1}{2}t^2$.

(iii) $\left\|e^{A_i t} - E_3\right\| \leqslant 1 - e^{-\lambda_i t} + t + \frac{1}{2}t^2$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\|A\| = \max\limits_{|x|=1} |Ax|$.

The second inequality in (iii) can be established as follows. For $x = (x_1, x_2, x_3)$, $x_1^2 + x_2^2 + x_3^2 = 1$,

$$\begin{aligned}
\|e^{A_i t} - I\| &= \max_{|x|=1} \left|\left(e^{A_i t} - I\right) x\right| = \max_{|x|=1} \left| \begin{bmatrix} e^{-\lambda_i t} - 1 & te^{-\lambda_i t} & \frac{1}{2}t^2 e^{-\lambda_i t} \\ 0 & e^{-\lambda_i t} - 1 & te^{-\lambda_i t} \\ 0 & 0 & e^{-\lambda_i t} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right| \\
&= \max_{|x|=1} \left| \begin{bmatrix} \left(e^{-\lambda_i t} - 1\right) x_1 + te^{-\lambda_i t} x_2 + \frac{1}{2}t^2 e^{-\lambda_i t} x_3 \\ \left(e^{-\lambda_i t} - 1\right) x_2 + te^{-\lambda_i t} x_3 \\ \left(e^{-\lambda_i t} - 1\right) x_3 \end{bmatrix} \right| \\
&\leqslant \max_{|x|=1} \left(\left| \begin{bmatrix} \left(e^{-\lambda_i t} - 1\right) x_1 \\ \left(e^{-\lambda_i t} - 1\right) x_2 \\ \left(e^{-\lambda_i t} - 1\right) x_3 \end{bmatrix} \right| + \left| \begin{bmatrix} t\, e^{-\lambda_i t} x_2 \\ t\, e^{-\lambda_i t} x_3 \\ 0 \end{bmatrix} \right| + \left| \begin{bmatrix} \frac{1}{2}t^2 e^{-\lambda_i t} x_3 \\ 0 \\ 0 \end{bmatrix} \right| \right) \\
&= \max_{|x|=1} \left(\left(1 - e^{-\lambda_i t}\right) \sqrt{x_1^2 + x_2^2 + x_3^2} + te^{-\lambda_i t} \sqrt{x_2^2 + x_3^2} + \frac{1}{2} t^2 e^{-\lambda_i t} |x_3| \right) \\
&\leqslant 1 - e^{-\lambda_i t} + te^{-\lambda_i t} + \frac{1}{2} t^2 e^{-\lambda_i t} \leqslant 1 - e^{-\lambda_i t} + t + \frac{1}{2} t^2.
\end{aligned}$$

We need this property in the following section to prove the main result of the present paper.

3. Main result

We denote the space of continuous functions $\eta(t) \in l_2$, $0 \leqslant t \leqslant T$, by $C(0, T; l_2)$. The following statement is the main result of the present paper.

Theorem 1. *If $w(\cdot) \in S(\rho_0)$ and $\lambda_i \geqslant 0$, $i = 1, 2, \ldots$, then there exists a unique solution of the infinite system of differential equations (4) in the space $C(0, T; l_2)$.*

Proof. Clearly, each ternary differential equation of the infinite system (4) has the unique solution $\eta(t)=(\eta_1(t),\eta_2(t),\ldots)$,

$$\eta_i(t)=e^{A_it}\eta_{i0}+\int_0^t e^{A_i(t-s)}w_i(s)\,ds, \tag{5}$$

where $w_i=(w_{i1},w_{i2},w_{i3})$, $\eta_{i0}=(x_{i0},y_{i0},z_{i0})$. Therefore, infinite system (4) can't have more than one solution in the space l_2.

Next, we show that $\eta(\cdot)=(\eta_1(\cdot),\eta_2(\cdot),\ldots)\in C(0,T;l_2)$. To prove this we need to show that $\eta(t)=(\eta_1(t),\eta_2(t),\ldots)\in l_2$ for each t, $0\leqslant t\leqslant T$, and that $\eta(t)$, $0\leqslant t\leqslant T$, is continuous in the norm of the space l_2.

We prove that $\eta(t)=(\eta_1(t),\eta_2(t),\ldots)\in l_2$ *for each* $t\in[0,T]$. One can obtain from (5) that

$$|\eta_i(t)|^2\leqslant 2\left(\left|e^{A_it}\eta_{i0}\right|^2+\left(\int_0^t\left|e^{A_i(t-s)}w_i(s)\right|ds\right)^2\right).$$

By using the relations following from Property 1

$$\begin{aligned}\left|e^{A_i(t-s)}w_i(s)\right|&\leqslant e^{-\lambda_i(t-s)}\left(1+(t-s)+\frac{1}{2}(t-s)^2\right)|w_i(s)|\\&\leqslant a(t)\,|w_i(s)|,\quad t\geqslant s,\end{aligned}$$

and the Cauchy–Schwartz inequality

$$\left(\int_0^t 1\cdot|w_i(s)|\,ds\right)^2\leqslant t\int_0^t|w_i(s)|^2ds,$$

we obtain

$$\begin{aligned}|\eta_i(t)|^2&\leqslant 2\left(e^{-2\lambda_it}a^2(t)|\eta_{i0}|^2+t\,a^2(t)\int_0^t|w_i(s)|^2ds\right)\\&\leqslant 2a^2(T)\left(|\eta_{i0}|^2+T\int_0^T|w_i(s)|^2ds\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{\infty}|\eta_i(t)|^2&\leqslant 2a^2(T)\left(\sum_{i=0}^{\infty}|\eta_{i0}|^2+T\sum_{i=0}^{\infty}\int_0^T|w_i(s)|^2ds\right)\\&\leqslant 2a^2(T)\left(\|\eta_0\|^2+T\rho_0^2\right).\end{aligned}$$

Thus, $\eta(t)\in l_2$ for each $t\in[0,T]$.

Let us prove that the function $\eta(t)$, $0\leqslant t\leqslant T$, is continuous. We show that, for any positive ε, there exists $\delta>0$ such that $\|\eta(t+h)-\eta(t)\|<\varepsilon$ whenever $|h|<\delta$. For $h>0$, using Property 1

$$\begin{aligned}\eta_i(t+h)-\eta_i(t)&=e^{A_i(t+h)}\eta_{i0}+\int_0^{t+h}e^{A_i(t+h-s)}w_i(s)\,ds-e^{A_it}\eta_{i0}-\int_0^t e^{A_i(t-s)}w_i(s)\,ds\\&=\left(e^{A_ih}-E_3\right)e^{A_it}\eta_{i0}+\int_0^t\left(e^{A_ih}-E_3\right)e^{A_i(t-s)}w_i(s)\,ds\\&\quad+\int_t^{t+h}e^{A_i(t+h-s)}w_i(s)\,ds.\end{aligned}$$

Using the inequality $(a+b+c)^2\leqslant 3(a^2+b^2+c^2)$ yields

$$\begin{aligned}|\eta_i(t+h)-\eta_i(t)|^2&\leqslant 3\left|\left(e^{A_ih}-E_3\right)e^{A_it}\eta_{i0}\right|^2+3\left|\int_0^t\left(e^{A_ih}-E_3\right)e^{A_i(t-s)}w_i(s)\,ds\right|^2\\&\quad+3\left|\int_t^{t+h}e^{A_i(t+h-s)}w_i(s)\,ds\right|^2\leqslant 3\left\|e^{A_ih}-E_3\right\|^2e^{-2\lambda_it}a^2(t)|\eta_{i0}|^2\end{aligned}$$

$$+ 3\|e^{A_i h} - E_3\|^2 a^2(t) \left(\int_0^t e^{-\lambda_i (t-s)} |w_i(s)|\, ds \right)^2$$
$$+ 3a^2(t) \left(\int_t^{t+h} e^{-\lambda_i (t+h-s)} |w_i(s)|\, ds \right)^2.$$

Then,

$$\|\eta(t+h) - \eta(t)\|^2 = \sum_{i=0}^{\infty} |\eta_i(t+h) - \eta_i(t)|^2 \leqslant I_1 + I_2 + I_3,$$

where

$$I_1 = 3a^2(T) \sum_{i=1}^{N} \left\|e^{A_i h} - E_3\right\|^2 \left(e^{-2\lambda_i t} |\eta_{i0}|^2 + \left(\int_0^t e^{-\lambda_i (t-s)} |w_i(s)|\, ds \right)^2 \right),$$
$$I_2 = 3a^2(T) \sum_{i=N+1}^{\infty} \left\|e^{A_i h} - E_3\right\|^2 \left(e^{-2\lambda_i t} |\eta_{i0}|^2 + \left(\int_0^t e^{-\lambda_i (t-s)} |w_i(s)|\, ds \right)^2 \right),$$
$$I_3 = 3a^2(T) \sum_{i=1}^{\infty} \left(\int_t^{t+h} |w_i(s)|\, ds \right)^2,$$

where N is a positive integer to be chosen below.

Since $\lambda_i \geqslant 0$ and by Property 1 (iii) $\|e^{A_i h} - E_3\|^2 \leqslant \left(1 + h + \frac{1}{2}h^2\right)^2$, we have

$$I_2 \leqslant 3a^2(T) \sum_{i=N+1}^{\infty} \left\|e^{A_i h} - E_3\right\|^2 \left(|\eta_{i0}|^2 + \left(\int_0^t |w_i(s)|\, ds \right)^2 \right)$$
$$\leqslant 3a^2(T) \left(1 + h + \frac{1}{2}h^2\right)^2 \sum_{i=N+1}^{\infty} \left(|\eta_{i0}|^2 + T \int_0^T |w_i(s)|^2 ds \right).$$

Since the series

$$\sum_{i=1}^{\infty} |\eta_{i0}|^2, \quad \sum_{i=1}^{\infty} \int_0^T |w_i(s)|^2 ds$$

are convergent, then for any positive number ε, we can choose the positive integer N such that $I_2 < \varepsilon/3$.

Next, we estimate I_1. Since $\lambda_i \geqslant 0$,

$$I_1 = 3a^2(T) \sum_{i=1}^{N} \left\|e^{A_i h} - E_3\right\|^2 \left(e^{-2\lambda_i t} |\eta_{i0}|^2 + \left(\int_0^t e^{-\lambda_i (t-s)} |w_i(s)|\, ds \right)^2 \right)$$
$$\leqslant 3a^2(T) \sum_{i=1}^{N} \left\|e^{A_i h} - E_3\right\|^2 \left(|\eta_{i0}|^2 + T \int_0^T |w_i(s)|^2 ds \right). \tag{6}$$

By Property 1 (iii) $\|e^{A_i h} - E_3\|^2 \leqslant \left(1 - e^{-\lambda_i h} + h + \frac{1}{2}h^2\right)^2 \to 0$ as $h \to 0$ for each i. Since the expression on the right hand side of (6) has finite number of terms, we can choose δ_1 such that $I_1 < \varepsilon/3$ whenever $|h| < \delta_1$.

To estimate I_3, we use Cauchy–Schwartz inequality

$$I_3 \leqslant 3a^2(T) \sum_{i=1}^{\infty} h \int_t^{t+h} |w_i(s)|^2 ds$$
$$\leqslant 3ha^2(T) \sum_{i=1}^{\infty} \int_0^T |w_i(s)|^2 ds \leqslant 3ha^2(T)\rho_0^2.$$

Obviously, we can choose δ_2 such that $I_3 < \varepsilon/3$ whenever $|h| < \delta_2$. Hence, $\|\eta(t+h) - \eta(t)\|^2$ can be done less than any given positive number ε by choosing $\delta = \min\{\delta_1, \delta_2\}$.

Now, we consider $\|\eta(t) - \eta(t-h)\|$, $h > 0$. Since

$$\begin{aligned}
\eta_i(t) - \eta_i(t-h) &= e^{A_i(t)}\eta_{i0} + \int_0^t e^{A_i(t-s)} w_i(s)\,ds - e^{A_i(t-h)}\eta_{i0} - \int_0^{t-h} e^{A_i(t-h-s)} w_i(s)\,ds \\
&= \left(e^{A_i h} - E_3\right) e^{A_i(t-h)}\eta_{i0} + \int_0^{t-h} \left(e^{A_i h} - E_3\right) e^{A_i(t-h-s)} w_i(s)\,ds \\
&\quad + \int_{t-h}^{t} e^{A_i(t-h-s)} w_i(s)\,ds.
\end{aligned}$$

Similar to the estimation of $\|\eta(t+h) - \eta(t)\|^2$ we can establish that for any positive number $\varepsilon > 0$, there exists number $\delta > 0$ such that $\|\eta(t) - \eta(t-h)\|^2 < \varepsilon$ whenever $|h| < \delta$. Therefore, $\eta(\cdot) \in C(0,T;l_2)$. This completes the proof of the theorem. ∎

4. Conclusion

In the present paper, we have studied the existence and uniqueness of the solution of an infinite system of ternary differential equations (4). The infinite system can be written as follows

$$\dot{z} = Az + w,$$

where the infinite block diagonal matrix $A = \mathrm{diag}(A_1, A_2, \ldots)$ consisting of matrices

$$A_i = \begin{bmatrix} -\lambda_i & 1 & 0 \\ 0 & -\lambda_i & 1 \\ 0 & 0 & -\lambda_i \end{bmatrix}, \quad i = 1, 2, \ldots,$$

is studied in this paper for the first time. We have proved the existence and uniqueness of the solution of an infinite system of ternary differential equations in the space $C(0,T;l_2)$. Clearly, A_i is of the form of Jordan block.

In the past, the following cases were studied (see, for example, [5, 20])

$$A = \begin{bmatrix} -\lambda_1 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & -\lambda_2 & 0 & \ldots & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & 0 & \ldots & \ldots & -\lambda_n & \ldots \\ \ldots & \ldots & \ldots & \ldots & 0 & \ldots \end{bmatrix}, \quad \lambda_i \geqslant 0, \quad i = 1, 2, \ldots,$$

and (see, for example, [21])

$$A = \mathrm{diag}(A_1, A_2, \ldots), \quad A_i = \begin{bmatrix} -\alpha_i & -\beta_i \\ \beta_i & -\alpha_i \end{bmatrix}, \quad \alpha_i \geqslant 0, \quad i = 1, 2, \ldots.$$

Then for the corresponding infinite system, control and differential game problems were studied. Such problems can be now studied for the infinite ternary differential equations (4) the cases of integral and geometric constraints on controls of players.

In future, for the case

$$A = \mathrm{diag}(A_1, A_2, \ldots), \quad A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \quad i = 1, 2, \ldots.$$

the existence and uniqueness of the solution of infinite systems of differential equations can be studied.

Acknowledgement

The present research was partially supported by the National Fundamental Research Grant Scheme FRGS of Malaysia, FRGS/1/2020/STG06/UPM/02/2.

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Розв'язок нескінченної системи потрійних диференціальних рівнянь

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Стаття присвячена нескінченній системі диференціальних рівнянь. Ця система складається з потрійних диференціальних рівнянь, що відповідають 3×3 жордановим блокам. Система розглядається в гільбертовому просторі l_2. Доведено теорему про існування та єдиність розв'язку системи.

Ключові слова: *диференціальне рівняння, нескінченна система, існування та єдиність розв'язку, гільбертовий простір.*