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New development of homotopy analysis method for non-linear integro-differential equations with initial value problems

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Homotopy analysis method (HAM) was proposed by Liao in 1992 in his PhD thesis for non-linear problems and was applied in many different problems of mathematical physics and engineering. In this note, a new development of homotopy analysis method (ND-HAM) is demonstrated for non-linear integro-differential equation (NIDEs) with initial conditions. Practical investigations revealed that ND-HAM leads an easy way how to find initial guess and it approaches the exact solution faster than the standard HAM, modified HAM (MHAM), new modified of HAM (mHAM) and more general method of HAM (q-HAM). Uniqueness solution of the problem and convergence of ND-HAM are proved in the Banach space. Finally, two examples are illustrated to show the accuracy and validity of the proposed method. Five methods are compared in each example.

Keywords: *homotopy analysis method, new development of homotopy analysis method (ND-HAM), non-linear integro-differential equation, comparisons, convergence.*

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1. Introduction

It is known that non-linear phenomena appear in many applications in scientific fields, such as fluid dynamics, solid-state physics, plasma physics, mathematical biology, chemical kinetics, aerodynamics, kinetic theory of gases, quantum mechanics, mathematical economics, and elasticity theory. Many problems in the fields of physics, engineering and science are modeled by non-linear integro-differential equations (IDEs) and the analytic solution of the non-linear problems is not known or can be found in rare cases. The obtained linear and non-linear integro-differential equations can be solved through many numerical methods. The existence of solution and approximation of the problems have been studied by many authors via: homotopy analysis method (HAM) [1–4], modified HAM [5–9], optimal HAM [10–12], q-HAM [13–17], for variety of problems. A number of series numerical methods were derived for integro-differential equations, such as variational iteration method [18], Taylor-successive approximation method [19], the Adomian decomposition method (ADM) [20–22], differential transform method [23], Wavelet–Galerkin method [24], the Tau method [25], modified HPM [26], Laplace transform ADM [27], finite element method [28] for partial differential equations (PDEs).

In this note, the following IDEs are considered: Non-linear Volterra–Fredholm integro-differential equations (VFIDEs) of order p

$$u^{(p)}(t)+\sum_{j=1}^{p-1}a_j(t)u^{(j)}(t)=f(t)+\lambda_1\int_a^t K_1(t,s)F_1(u(s))\,ds+\lambda_2\int_a^b K_2(t,s)F_2(u(s))\,ds, \tag{1}$$

with initial conditions

$$u^{(k)}(a)=\alpha_k,\quad k=0,\dots,p-1,\quad p\in\mathbb{N},\quad (p\geqslant 2), \tag{2}$$

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where $t \in \Omega = [a,b]$ and $K_1, K_2 \colon \Omega \times \Omega \longrightarrow \mathbb{R}$, $f \colon \Omega \longrightarrow \mathbb{R}$ and $a_j \colon \Omega \longrightarrow \mathbb{R}$, $j = 1, \dots, p-1$ are known functions, λ_1, λ_2 are parameters and $F_1, F_2 \colon C(\Omega, \mathbb{R}) \longrightarrow \mathbb{R}$ are non-linear function as well as $u(t)$ is unknown function to be determined.

For the non-linear IDEs (1)–(2), application of ND-HAM (new development of HAM), standard HAM proposed by Liao [1], modified HAM named (MHAM) developed by Bataineh et al. [5], new development of HAM (mHAM) initiated by Ayati et al. [6], and more general method of homotopy analysis method called (q-HAM) proposed by El-Tawil and Huseen [13] are described in details. Norm convergence of ND-HAM is proven in the Banach space.

In Example 1 (Table 1 and 2), advantages of ND-HAM over standard HAM [1], modified MHAM [5], mHAM [6] and q-HAM [13], are shown practically. Matlab codes are developed for five methods (ND-HAM, HAM, MHAM, mHAM, q-HAM) and shown that by suitable choice of $x_i(s)$ of $f(s) = x_0(s) + x_1(s) + \dots + x_n(s)$, ND-HAM can give better results than the standard HAM and all modified HAM.

Organization of the paper is as follows. In section 2, basic concept of HAM is recalled, few modified HAM (MHAM, mHAM, q-HAM) are described and the detailed description of the proposed method is presented. Section 3 describes detailed application of the standard HAM and ND-HAM for the Problems 1–2. The existence and uniqueness solution of Eqs. (1)–(2) and convergence of the ND-HAM are proven in Section 4. In Section 5, three examples are solved to illustrate the performance of proposed method and comparisons with other methods are demonstrated. Finally, the paper ends with a conclusion and acknowledgement in Section 5.

2. Basic idea of standard HAM, modified HAM (MHAM, mHAM, q-HAM) and ND-HAM

Basic idea of HAM is as follows. Let non-linear equation be given by

$$N[u(t)] = 0. \tag{3}$$

Liao [1] has constructed the zero-order deformation equation in the form

$$(1-q)\pounds[\phi(t;q) - u_0(t)] = q\hbar H(t)\big[N[\phi(t;q)]\big], \tag{4}$$

where $\pounds$ is the linear operator, $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(t)$ is auxilary function, $u_0(t)$ is an initial guess of the solution $u(t)$ satisfying initial or boundary conditions and $\phi(t;q)$ is an unknown function to be determined depending on the variables t and q and satisfies the following equation

$$\phi^{(i)}(t;0) = u_0^{(i)}(t), \quad i = 0, 1, 2, \dots.$$

When the parameter q increases from 0 to 1, the homotopy solution $\phi(t;q)$ varies from $u_0(t)$ to solution $u(t)$ of the original equation (3). Using the parameter q as dummy variable, the function $\phi(t;q)$ can be expanded in Taylor series

$$\phi(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m, \tag{5}$$

where $u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}\Big|_{q=0}$.

Assuming that the auxiliary parameter $\hbar$ in (4) is properly selected so that the series (5) is convergent when $q = 1$

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t).$$

Approximate solution of Eq. (3) can be written as

$$u(t) = u_0(t) + \sum_{m=1}^{n} u_m(t),$$

where $u_m(t)$ is defined from deformation equation (6).

High-order deformation equation. We define the following vector $\boldsymbol{u}_n(t) = (u_0(t), \ldots, u_n(t))$ and differentiating the zero-order deformation equation (4) m times with respect to the embedding parameter q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called m-th order deformation equation

$$\pounds[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)), \tag{6}$$

where

$$\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) = \frac{1}{(m-1)!}\,\frac{\partial^{m-1}\big[N[\phi(t;q)]\big]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{7}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1; \\ 1, & m > 1. \end{cases} \tag{8}$$

Defined by (6)–(8) series solution is called standard HAM. There are few modifications of HAM, for instance:

• In 2009, Bataineh et al. [5] had presented a new modification of the homotopy analysis method (MHAM) for solving non-linear systems of the second order boundary-value problems (BVPs). In this modification, they consider the non-linear systems of the second order differential equation in the form

$$N_i[u_i(t)] = f_i(t), \quad i = \{1, 2\},$$

subject to the boundary conditions

$$u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0.$$

Representing the right side function

$$f_i(t) = x_{i,0}(t) + x_{i,1}(t) + \ldots + x_{i,n}(t), \quad i = \{1, 2\},$$

and establishing $\varphi_i(t, q)$ in powers of the embedding parameter q

$$\varphi_i(t, q) = x_{i,0}(t) + q x_{i,1}(t) + \ldots + q^n x_{i,n}(t), \quad i = \{1, 2\},$$

then constructing the mth-order deformation equation in the form

$$\pounds[u_{i,m}(t) - \chi_m u_{i,m-1}(t)] = \hbar H(t)\,\mathrm{Re}_{i,m}(\boldsymbol{u}_{i,m-1}(t)), \tag{9}$$

where χ_m is defined by (8) and

$$\mathrm{Re}_{i,m}(\boldsymbol{u}_{i,m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\big[N_i[\phi_i(t;q)] - \varphi_i(t,q)\big]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{10}$$

is called a new modified version of HAM named MHAM.

• In 2012, El-Tawil and Huseen [13] considered the following differential equation

$$N(u(x,t)) - f(x,t) = 0, \tag{11}$$

where N is a non-linear operator, (x, t) denotes independent variables, $f(x, t)$ is a known function and $u(x, t)$ is an unknown function.

They constructed the so-called zero-order deformation equation in the form

$$(1 - qn)\pounds[\phi(x,t,q) - u_0(x,t)] = q\hbar H(x,t)\big[N[\phi(x,t,q) - f(x,t)]\big], \tag{12}$$

where $n \geqslant 1$, $q \in \left[0, \frac{1}{n}\right]$, denotes the so-called embedded parameter, L is an auxiliary linear operator with the property $L[f] = 0$ when $f = 0$, and $\hbar \neq 0$ is an auxiliary parameter, $H(x, t)$ denotes a non-zero auxiliary function.

It is obvious, that when $q = 0$ and $q = \frac{1}{n}$ equation (12) becomes

$$\phi(x,t,0) = u_0(x,t), \quad \phi\left(x, t, \frac{1}{n}\right) = u(x,t),$$

respectively. Thus, as increases q from 0 to $\frac{1}{n}$, the solution $u(x, t)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$.

Expanding $\phi(x,t,q)$ in Taylor series with respect to q,

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m. \tag{13}$$

where $u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}$.

Assume that $\hbar$, $H(x,t)$, $u_0(x,t)$, L are so properly chosen such that the series (13) converges at $q = \frac{1}{n}$ and the solution of (11) can be written as

$$u(x,t) = \phi\left(x,t,\frac{1}{n}\right) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{14}$$

Differentiating equation (12) m times with respect to q and setting $q = 0$ and finally dividing them by $(m-1)!$, we have the so-called m-th order deformation equation

$$\pounds[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)\,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(x,t)), \tag{15}$$

where $\boldsymbol{u}_r(x,t) = (u_0(x,t), u_1(x,t), \ldots u_r(x,t))$ is a vector function and

$$\mathrm{Re}_m(\boldsymbol{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\big[N[\phi(x,t,q)] - f(x,t)\big]}{\partial q^{m-1}}\Bigg|_{q=0}, \tag{16}$$

with

$$\chi_m = \begin{cases} 0, & m \leqslant 1; \\ n, & m > 1. \end{cases} \tag{17}$$

The iterative solution $u_m(x,t)$ which are defined by (15)–(17) is called q-HAM. It is practically shown in El-Tawil and Huseen [13] that series solution (14) where component functions $u_m(x,t)$ are governed by the equation (15) converges to the exact solution $u(x,t)$, faster than the standard HAM. Due to the existence of factor $(1/n)^m$, more chances of convergence may occur. It should be noted that the case of $n = 1$, standard HAM can be reached. In [14] convergence of q-HAM is proved and development of q-HAM and application of q-HAM are shown in [15,16].

• In 2014, Ayati et al. [6] developed a new modified version of HAM named (mHAM). They have considered the following non-linear differential equation

$$N[u(t)] = f(t). \tag{18}$$

The modified form of HAM, can be established based on assumption that the function $f(r)$ can be divided into several parts, namely,

$$f(t) = \sum_{k=0}^{n} x_k(t). \tag{19}$$

Then, they have constructed the modified m-th order deformation equation in the form

$$\begin{aligned} &\pounds[u_0(t)] = x_0(t), \\ &\pounds[u_1(t) - u_0(t)] = \hbar\big[\mathrm{Re}_1(\boldsymbol{u}_0(t)) - x_1(t)\big], \\ &\pounds[u_m(t) - u_{m-1}(t)] = \hbar\big[\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) - x_m(t)\big], \quad 2 \leqslant m \leqslant n, \\ &\pounds[u_m(t) - u_{m-1}(t)] = \hbar\,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)),\ m > n. \end{aligned} \tag{20}$$

$$\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) = \frac{1}{(m-1)!}\,\frac{\partial^{m-1}\big[N[\phi(t;q)]\big]}{\partial q^{m-1}}\Bigg|_{q=0}, \tag{21}$$

where non-linear term $N[\phi(t;q)]$ is the left side of the equation (18).

The series solution u_m defined by (19)–(21) is called new modified HAM in short mHAM.

- In 2015, Yin et al. [7] proposed a modified HAM to obtain quick and accurate solution of wave-like fractional physical models. This modified semi-analytical approach is the combination of Laplace transform algorithm and homotopy analysis method named homotopy analysis transform method (HATM) and was applied it for fractional partial differential equations. The HATM utilizes a simple and powerful method to adjust and control the convergence region of the infinite series solution using an auxiliary parameter. The numerical solutions obtained by this proposed method indicate that the approach is easy to implement, highly accurate, and computationally very attractive. A good agreement between the obtained solutions and some well-known results which have been obtained by other methods exist.
- In 2017, Ziane and Cherif [8] proposed a new modification of HAM by combining the natural transformation with homotopy analysis method to solve non-linear fractional partial differential equations. This method is called the fractional homotopy analysis natural transform method (FHANTM). The FHANTM can easily be applied to many problems and is capable of reducing the size of computational work. The fractional derivative is described in the Caputo sense. The results show that the FHANTM is an appropriate method for solving non-linear fractional partial differential equation.

To derive the ND-HAM, we rewrite Eq. (3) in the form

$$N[u(t)] = f(t), \tag{22}$$

and assume that the function $f(t)$ is split into n terms, namely

$$f(t) = x_0(t) + x_1(t) + \ldots + x_n(t). \tag{23}$$

Expanding $g(t, q)$ into powers of the embedding parameter q, we obtain

$$g(t; q) = x_0(t) + x_1(t) + x_2(t)(qh) + \ldots + x_n(t)(qh)^{n-1}. \tag{24}$$

For ND-HAM we rewrite Eq. (6)–(8), in the form

$$\pounds[u_0(t)] = x_0(t), \tag{25}$$

$$\pounds[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)), \tag{26}$$

where χ_m is defined by (8) and

$$\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\big[N[\phi(t;q)] - g(t;q)\big]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{27}$$

In ND-HAM, there are the following advantages:

1. Free choice of $x_0(t)$ depending on given function $f(t)$ and solving Eq. (25), we can get exact solution $u(t)$ of the Eq. (22) from the first iteration. In this case, the next iteration obtained by Eq. (26) gives exactly zero solution $u_i(t) = 0$, $i = 1, 2, \ldots$.
2. If Eq. (25) does not give exact solution, then it will serve as a choice of initial guess $u_0(t)$ satisfying initial or boundary conditions.
3. Due to (24), residual term $\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t))$ computed by (27) gives us more economic computations for any value of $\hbar$.
4. Practical investigations reveal that ND-HAM might be dominated by the suitable choice of decomposed functions $x_i(t)$ of $f(t)$ over standard HAM [1], MHAM [5], mHAM [6] and for small values of n of Oq-HAM [13] (see Example 1).

3. Main results. Application of HAM and ND-HAM

To solve non-linear VFIDEs (1)–(2) using HAM and ND-HAM, we introduce a non-linear operator

$$N[\phi(t;q)] = \left(D_a^p + \sum_{j=1}^{p-1} a_j(t) D_a^j\right)\phi(t;q)$$
$$- \lambda_1 \int_a^t K_1(t,s)F_1(\phi(s;q))ds - \lambda_2 \int_a^b K_2(t,s)F_2(\phi(s;q))\,ds, \tag{28}$$

where D_a^j is a differential operator of order j and $\phi(t;q)$ is unknown function to be determined and write Eqs. (1)–(2) in the operator equation form

$$N[\phi(t;q)] = f(t), \tag{29}$$
$$\phi^{(k)}(a;1) = u^{(k)}(a) = \alpha_k, \quad k = 0,1,\dots,p-1. \tag{30}$$

The m^{th}-order deformation equation (standard HAM) (6)–(8) with $H(t) = 1$, for the non-linear equations (29)–(30) are given by

$$\mathcal{L}[u_1(t)] = \hbar \,\mathrm{Re}_1(\boldsymbol{u}_0(t)) = \hbar \left.\{N[\phi(t;q)] - f(t)\}\right|_{q=0},$$
$$\mathcal{L}[u_m(t) - u_{m-1}(t)] = \hbar \,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)), \quad m = 2,3,\dots,$$
$$u_0^{(k)}(a) = \alpha_k, \quad k = 0,\dots,p-1, \quad u_m^{(k)}(a) = 0, \quad m = 1,2,\dots, \quad k = 0,1,\dots. \tag{31}$$

where residual term $\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t))$ is as follows

$$\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}[N(\phi(t;q)) - f(t)]}{\partial q^{m-1}}\right]_{q=0}.$$

For ND-HAM, we split right side function $f(t)$ of Eq. (29) in the form (23)–(24), then we obtain m-th order deformation equation for non-linear operator equation (29)–(30) as follows

$$\mathcal{L}[u_0(t)] = x_0(t),$$
$$\mathcal{L}[u_1(t)] = \hbar \,\mathrm{Re}_1(\boldsymbol{u}_0(t)) = \left.\{N[\phi(t;q)] - g(t,q)\}\right|_{q=0},$$
$$\mathcal{L}[u_m(t) - u_{m-1}(t)] = \hbar \,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)), \quad m = 2,3,\dots,$$
$$u_0^{(k)}(a) = \alpha_k, \quad k = 0,\dots,p-1, \quad u_m^{(k)}(a) = 0, \quad m = 1,2,\dots, \quad k = 0,1,\dots, \tag{32}$$

where $g(t,q)$ is defined by (24) and residual term $\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t))$ is of the form

$$\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}[N[\phi(t;q)] - g(t,q)]}{\partial q^{m-1}}\right|_{q=0}. \tag{33}$$

Before applying standard HAM and ND-HAM for operator equation (29), we need the following important formulas Kanwal [29, p. 285].

Lemma 1. *Let $f \in L^1[a,b]$ then n tuple integrals $J_a^n(f)$ is computed in the form*

$$J_a^n(f(s)) = \int_a^s \int_a^{s_n} \dots \int_a^{s_2} f(s_1)\,ds_1\,ds_2 \cdots ds_n$$
$$= \frac{1}{(n-1)!}\int_a^s (s-t)^{n-1} f(t)\,dt. \tag{34}$$

Lemma 2. *Let $n, m \in N$ and $f \in L^1[a,b]$, then the the following properties hold*

$$\begin{aligned}
&J_a^n J_a^m f(t) = J_a^m J_a^n f(t) = J_a^{n+m} f(t),\\
&D_a^n[J_a^n f(t)] = f(t),\\
&J_a^n[D_a^n f(t)] = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k,\\
&J_a^m (t-a)^k = \frac{k!}{(m+k)!}(t-a)^{m+k},\\
&D_a^n (t-a)^k = 0, \quad k = 0,1,\ldots,n-1.
\end{aligned} \tag{35}$$

Let us apply ND-HAM (32)–(33) to the Eq. (1) with initial conditions (2). Choosing $L = D^p_{a+}$ and acting integral operator J^p_{a+} on both sides of first equation of (32) taking into account (30), we have

$$J_a^p\big[D_a^p u_0(t)\big] = J_a^p(x_0(t)) = \frac{1}{(p-1)!}\int_a^t (t-s)^{p-1} x_0(s)\,ds,$$

which leads to

$$u_0(t) = \sum_{k=0}^{p-1} \frac{\alpha_k}{k!}(t-a)^k + \frac{1}{(p-1)!}\int_a^t (t-s)^{p-1} x_0(s)\,ds.$$

Since $u_m^{(k)}(a) = 0$, $m = 1,2,\ldots$, $k = 0,1,\ldots$ and applying integral operator J_a^p on both sides of the second equation (32) and taking $u_0(t)$ as an initial guess, we obtain

$$\begin{aligned}
u_1(t) &= \hbar J_a^p\big[R_1(\boldsymbol{u_0}(t))\big] = \hbar J_a^p\big\{N[\phi(t;q)] - g(t,q)\big\}\big|_{q=0}\\
&= \hbar J_a^p\left[\left(D_a^p + \sum_{j=1}^{p-1} a_j(t) D_a^j\right)u_0(t) - \big(\lambda_1 G_1^0(t,q) + \lambda_2 G_2^0(t,q)\big) - \big(x_0(t) + x_1(t)\big)\right]\\
&= \hbar\Bigg[u_0(t) - \sum_{k=0}^{p-1}\frac{\alpha_k}{k!}(t-a)^k + \sum_{j=1}^{p-1} J_a^p\big(a_j(t) D_a^j u_0(t)\big)\\
&\quad - J_a^p\big(\lambda_1 G_1^0(t,q) + \lambda_2 G_2^0(t,q)\big) - J_a^p\big(x_0(t) + x_1(t)\big)\Bigg],
\end{aligned}$$

where

$$G_i^0(t,q) = \int_a^t K_i(t,s)\big[F_i(\phi(s;q))\big]_{q=0} ds, \quad i \in \{1,2\}.$$

For $2 \leqslant m \leqslant n$, due to (28) and (32)–(33),

$$\begin{aligned}
\pounds[u_m(t) - u_{m-1}(t)] &= \hbar\,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t))\\
&= \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}[N[\phi(t;q)] - g(t,q)]}{\partial q^{m-1}}\right]_{q=0}\\
&= \left[\left(D^p + \sum_{j=1}^{p-1} a_j(t) D^j\right)u_{m-1}(t) - \big(\lambda_1 G_1^{m-1}(t,q) + \lambda_2 G_2^{m-1}(t,q)\big) - x_m(t)\hbar^{m-1}\right],
\end{aligned} \tag{36}$$

where

$$G_i^{m-1}(t,q)|_{q=0} = \int_a^t K_i(t,s)\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}} F_i(\phi(s;q))\right]_{q=0} ds, \quad i \in \{1,2\}. \tag{37}$$

Similarly, for $m \geqslant n+1$

$$\pounds[u_m(t)-u_{m-1}(t)] = \hbar\,\mathrm{Re}_m(\boldsymbol{u}_{m-1}(t)) = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}[N[\phi(t;q)]]}{\partial q^{m-1}}\right]_{q=0}$$
$$= \left[\left(D^p + \sum_{j=1}^{p-1} a_j(t) D^j\right) u_{m-1}(t) - \left(\lambda_1 G_1^{m-1}(t,q)|_{q=0} + \lambda_2 G_2^{m-1}(t,q)|_{q=0}\right)\right], \quad (38)$$

where $G_i^{m-1}(t,q)|_{q=0}$, $i \in \{1,2\}$ are defined by (37).

In view of $u_m^{(k)}(a)=0$, $m=1,2,\ldots$, $k=0,1,\ldots$ and acting integral operator J_a^p on both sides of the equation (36) and taking into account the properties of integral operator J_a^p (34)–(35), we obtain

$$u_m(t) = (1+\hbar)u_{m-1}(t) + \sum_{j=1}^{p-1} J_a^p\big(a_j(t) D_a^j u_{m-1}(t)\big)$$
$$- J_a^p\big(\lambda_1 G_1^{m-1}(t,q) + \lambda_2 G_2^{m-1}(t,q)\big) - J_a^p\big(x_m(t)\big)\hbar^{m-1}, \quad 2 \leqslant m \leqslant n.$$

For $m \geqslant n+1$ and acting J_a^p to both sides of Eq. (38) together with the properties of integral operators (34)–(35), one can get

$$u_m(t) = (1+\hbar)u_{m-1}(t) + \sum_{j=1}^{p-1} J_a^p\big(a_j(t) D_a^j u_{m-1}(t)\big) - J_a^p\big(\lambda_1 G_1^{m-1}(t,q) + \lambda_2 G_2^{m-1}(t,q)\big),$$

where $J_a^p f(t)$ is computed by (34) and the derivative of the product of two functions $f(t)$ and $g(t)$ is computed by Leibniz rule,

$$D_a^N[f(t)g(t)] = \sum_{n=0}^{N} \binom{N}{n} D_a^{N-n} f(t) D_a^n g(t).$$

4. Uniqueness and convergence

In this section, we prove existence and uniqueness solution of the Eq. (1)–(2). In addition the convergence of ND-HAM is also proved in Banach space.

4.1. Uniqueness solution

In this section, we will give uniqueness solutions of Eq. (1), with the initial condition (2) and prove theorems. Before starting and proving the main results, we introduce the following hypotheses:

— (H1): There exist two constants $L_1, L_2 > 0$ such that for any $u_1, u_2 \in C(J,R)$, $J=[a,b]$

$$|F_1(u_1(t)) - F_1(u_1(t))| \leqslant L_1|u_1-u_2|,$$
$$|F_2(u_1(t)) - F_2(u_2(t))| \leqslant L_2|u_1-u_2|,$$
$$|D^j(u_1(t)) - D^j(u_1(t))| \leqslant \gamma_j|u_1-u_2|, \quad j=\{0,1,\ldots,p-1\}.$$

where D^j is a derivative operator and γ_j are the Lipschitz constants.

— (H2): There exist two functions $K_1^*, K_2^* \in C(D,R^+)$ the set of all positive functions continuous on $D=\{(t,s)\in R\times R\colon a \leqslant s \leqslant t \leqslant b\}$ such that

$$K_1^* = \sup_{t\in[a,b]} \int_a^t |K_1(t,s)ds| < \infty, \quad K_2^* = \sup_{t\in[a,b]} \int_a^b |K_2(t,s)ds| < \infty.$$

— (H3): The functions $a_j(t)$, $j=\{1,2,\ldots,p-1\}$ and $f(t)$ mapping $J\to R$ are continuous functions.

Theorem 1 (Banach's Fixed Point Theorem). *Let (X,d) be a complete metric space and let $T\colon X\to X$ be a contraction on X. Then T has a unique fixed point $x\in X$, such that $T(x)=x$.*

Before the prove of the uniqueness solution of the non-linear IDEs (1)–(2), we prove the following lemma.

Lemma 3. *Let $\varphi(t)\in C(J,R^+)$ then $u(t)\in C(J,R^+)$ is a solution of the problem (1)–(2) iff u is satisfying*

$$u(t)=\varphi(t)-\sum_{j=1}^{p-1}\frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}\Big(a_j(s)D^ju(s)\Big)ds + \frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}\left[\int_a^s K_1(s,r)F_1(u(r))dr\right]ds + \frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}\left[\int_a^s K_2(s,r)F_2(u(r))dr\right]ds, \quad (39)$$

for $t\in J=[a,b]$ and

$$\varphi(t)=\sum_{k=0}^{p-1}\frac{\alpha_k}{k!}(t-a)^k+\frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}f(s)\,ds. \quad (40)$$

Proof. Acting integral operator J_a^p on both sides of (1) taking into account initial conditions (2), we have

$$u(t)-\sum_{k=0}^{p-1}\frac{\alpha_k}{k!}(t-a)^k+\sum_{j=1}^{p-1}J_a^p\big(a_j(s)D^ju(s)\big) = J_a^p\left[\lambda_1\int_a^s K_1(s,r)F_1(u(r))\,dr+\lambda_2\int_a^s K_2(s,r)F_2(u(r))\,dr\right].$$

Using properties of integral operator (Lemma 3), we obtain equivalent non-linear IEs to nonlinear IDEs (1)–(2).

Thus, first uniqueness result can be obtained based on the Banach contraction principle. ∎

Theorem 2. *Assume that the hypothesises (H1), (H2) and (H3) hold. If*

$$\delta^*=\left(\frac{\gamma^*a^*p}{p!}+\frac{\lambda_1K_1^*L_1}{p!}+\frac{\lambda_1K_2^*L_2}{p!}\right)(b-a)^p<1, \quad (41)$$

where $\gamma^*=\max\limits_{1\leqslant j\leqslant p-1}\gamma_j$ and $a^*=\max\limits_{1\leqslant j\leqslant p-1}|a_j(t)|$, then there exists a unique solution $u(x)\in C(J)$ to (1)–(2).

Proof. Let the operator $T\colon C(J,R)\to C(J,R)$ be defined by

$$(Tu)(t)=\varphi(t)-\sum_{j=1}^{p-1}\frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}\big(a_j(s)D^ju(s)\big)\,ds + \frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}\left[\lambda_1\int_a^s K_1(s,r)F_1(u(r))dr\right]ds + \frac{1}{(p-1)!}\int_a^t(t-s)^{p-1}\left[\lambda_2\int_a^s K_2(s,r)F_2(u(r))dr\right]ds,$$

where $\varphi(t)$ is defined by (40).

It is known by Lemma 5, that a function u is a solution to (1)–(2) iff u satisfies Eq. (39). Now we prove that T has a fixed point u in $C(J,R)$ under condition (41). To do this end, let $u_1, u_2 \in C(J,R)$ then for any $t \in [a,b]$

$$\begin{aligned}|(Tu_1)(t)-(Tu_2)(t)| &\leqslant \sum_{j=1}^{p-1}\frac{1}{(p-1)!}\int_a^t (t-s)^{p-1}|a_j(s)|\big|D^j u_1(s)-D^j u_2(s)\big|ds\\ &+\frac{1}{(p-1)!}\int_a^t (t-s)^{p-1}\left[\lambda_1\int_a^s |K_1(s,r)||F_1(u_1(r))-F_1(u_2(r))|dr\right]ds\\ &+\frac{1}{(p-1)!}\int_a^t (t-s)^{p-1}\left[\lambda_2\int_a^s |K_2(s,r)||F_2(u_1(r))-F_2(u_2(r))|dr\right]ds\\ &\leqslant \left[\frac{a^*\gamma^* p(b-a)^p}{p!}+\frac{\lambda_1 K_1^* L_1(b-a)^p}{p!}+\frac{\lambda_2 K_2^* L_2(b-a)^p}{p!}\right]||u_1-u_2||\\ &= \delta\|u_1-u_2\|, \quad \delta<1.\end{aligned}$$

Thus, operator T is the contraction map. By the Banach contraction principle (Theorem 4), we can conclude that T has a unique fixed point u in $C(J,R)$. ∎

Now, we will give conditions of the existence solution by means of Schauder's fixed point theorem.

4.2. Convergence of the ND-HAM

Before prove the convergence of the series $u(t)=\sum_{m=0}^{\infty}u_m(t)$ solution of (3), we need some preliminaries.

— Let nonlinear operator N in Eq. (3) be contraction operator (i.e. $\|N\|<1$).
— Assume that nonlinear equation (3) has a unique solution (Theorem 6).

Let $\phi(t;\lambda)=\sum_{m=0}^{\infty}\lambda^m u_m(t)$ and $N\big(\sum_{m=0}^{\infty}\lambda^m u_m(t)\big)=\sum_{m=0}^{\infty}\lambda^m A_m$, where λ is a parameter and A_n Adomian polynomials defined by

$$A_n=\frac{1}{n}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{m=0}^{\infty}\lambda^m u_m(t)\right)\right]_{\lambda=0}. \tag{42}$$

The parameter λ is introduced for convenience and it allows for the determination of A_n by the formula (42). Generally it is possible to obtain A_n exactly as a function of $(u_0,\ldots,u_n)$ from the non-linear function N.

Convergence Results. For every convergent series $u=\sum_{m=0}^{\infty}u_m(t)$, we define $N(u)$ by

$$N(u)=\sum_{m=0}^{\infty}A_m(u_0,u_1,\ldots,u_m) \tag{43}$$

where the A_m of (43) are obtained from relations (42). Convergence of series $\sum_{m=0}^{\infty}A_m$ is proven in the following theorem.

Theorem 3 (Cherruault [30]). *Let N be a contraction operator. If $\|N_n-N\|=\varepsilon_n \to_{n\to\infty}=0$ then the sequence S_n given by*

$$S_{n+1}=N(u_0+S_n), \quad S_0=0,$$

converges towards the S solution of

$$N(u_0+S)=S,$$

where $N_n(U_n)=N_n\big(\sum_{m=0}^{n}u_m\big)=\sum_{m=0}^{n}A_m(u_0,u_1,\ldots,u_m)$, $S_n=u_1+u_2+\ldots+u_n$ and $S=\lim_{n\to\infty}S_n$.

Let us first prove the convergence of the standard HAM to the solution of (1).

Theorem 4 (Convergence theorem). *Suppose that the series $\sum_{m=0}^{\infty} u_m(t)$ converges to a function $u(t)$, where the functions $u_m \in C(J,\mathbb{R})$ are governed by the high-order deformation equation (25)–(27) of ND-HAM. Then, $u(t)$ is the exact solution of problem (1)–(2).*

Proof. Let us define

$$H_m(t) = \frac{1}{m!}\left[\frac{\partial^m F[\phi(t;q)]}{\partial q^m}\right]_{q=0}.$$

From Theorem 8 (Cherruault [30]), we can conclude that, if the series $\sum_{m=0}^{\infty} u_m(t)$ approaches $u(t)$, then the series $\sum_{k=0}^{\infty} H_k(t)$ must converge to $F[u(t)]$. Suppose every $u_m(t)$ satisfies (32)–(33). Then, from the convergence of $\sum_{m=0}^{\infty} u_m(t)$, it follows that

$$\lim_{m\to\infty} u_m(t) = 0 \tag{44}$$

for $t \in [0,T]$. Summing on the left side of (31) without acting operator $\pounds$ and taking into account (44), we have

$$\sum_{m=1}^{\infty}[u_m(t) - \chi_m u_{m-1}(t)] = \lim_{n\to+\infty}\sum_{m=1}^{n}[u_m(t) - \chi_m u_{m-1}(t)] = \lim_{n\to+\infty} u_n(t) = 0.$$

By the linearity of the differential operator $\pounds = D^p$, (31) implies

$$\begin{aligned}\hbar\sum_{m=1}^{+\infty}\mathrm{Re}_m(\overline{u}_{m-1}(t)) &= \sum_{m=1}^{+\infty} D^p[u_m(t) - \chi_m u_{m-1}(t)]\\ &= \lim_{n\to\infty}\sum_{m=1}^{n} D^p[u_m(t) - \chi_m u_{m-1}(t)] = \lim_{n\to\infty} D^j[u_n(t)]\\ &= D^j\left[\lim_{n\to\infty} u_n(t)\right] = D^p(0) = 0.\end{aligned}$$

Since $\hbar \neq 0$, we must have

$$\sum_{m-1}^{+\infty}\mathrm{Re}_m(\overline{u}_{m-1}(t)) = 0.$$

On the other hand,

$$\begin{aligned}\mathrm{Re}_m(\overline{u}_{m-1}(t)) = &\left(D^p + \sum_{j=1}^{p-1}\xi_j D^j\right)[u_{m-1}(t)] - (1-\chi_m)f(t)\\ &- \frac{\lambda_1}{(m-1)!}\int_a^t K_1(t,s)\left[\frac{\partial^{m-1}F_1[\phi(s;q)]}{\partial q^{m-1}}\right]_{q=0} ds\\ &- \frac{\lambda_2}{(m-1)!}\int_a^b K_2(t,s)\left[\frac{\partial^{m-1}F_2[\phi(s;q)]}{\partial q^{m-1}}\right]_{q=0} ds.\end{aligned}$$

Therefore,

$$\begin{aligned}0 = \sum_{m=1}^{+\infty}\mathrm{Re}_m(\overline{u}_{m-1}(t)) = &\sum_{m=1}^{+\infty}\left\{\left(D^p + \sum_{j=1}^{p-1}a_j(t)D^j\right)\left[u_{m-1}(t)\right] - (1-\chi_m(t))f(t)\right.\\ &\left.- \frac{1}{(m-1)!}\int_a^t K_1(t,s)\left[\frac{\partial^{m-1}F_1[\phi(s;q)]}{\partial q^{m-1}}\right]_{q=0} ds - \frac{\lambda_2}{(m-1)!}\int_a^b K_2(t,s)\left[\frac{\partial^{m-1}F_2[\phi(s;q)]}{\partial q^{m-1}}\right]_{q=0} ds\right\}\\ = &\left(D^p + \sum_{j=1}^{p-1}a_j(t)D^j\right)\sum_{m=1}^{+\infty}[u_{m-1}(t)] - f(t)\\ &- \lambda_1\int_a^t\left(K_1(t,s)\sum_{m=1}^{\infty}H_{1,m-1}(s)\right)ds - \lambda_2\int_a^b\left(K_2(t,s)\sum_{m=1}^{\infty}H_{2,m-1}(s)\right)ds\end{aligned}$$

$$= \left(D^p + \sum_{j=1}^{p-1} a_j(t)D^j\right)u(t) - f(t) - \lambda_1 \int_a^t K_1(t,s)F_1(u(s))\,ds - \lambda_2 \int_a^b K_2(t,s)F_2(u(s))\,ds.$$

which implies

$$\left(D^p + \sum_{j=1}^{p-1} a_j(t)D^j\right)u(t) = f(t) + \lambda_1 \int_a^t K_1(t,s)F_1(u(s))\,ds + \lambda_2 \int_a^b K_2(t,s)F_2(u(s))\,ds.$$

■

In the next section, we demonstrate numerical advantages of the proposed method.

5. Numerical Experiments

In this section, we demonstrated five methods: Standard HAM [1], MHAM [5], mHAM [6] and q-HAM [15] in each of the example.

Example 1 (Waleed Al-Hayani [27]). Let us solve the following second-order integro-differential equation

$$\begin{cases} y''(s) = e^s - s + \displaystyle\int_0^1 s\,t\,y(t)\,dt, \\ y(0) = 1, \quad y'(0) = 1. \end{cases} \tag{45}$$

Exact solution is $y(s) = e^s$.

Solution. Let us find the exact and approximate solution of problem (45) using ND-HAM. Let us rewrite Eq. (45) in the operator form

$$\begin{cases} N(\phi(s,q)) = f(s), \\ \phi(0,1) = y(0) = 1, \quad \phi'(0,1) = y'(0) = 1, \end{cases} \tag{46}$$

where

$$\begin{cases} N(\phi(s,q)) = \dfrac{\partial^2}{\partial s^2}\phi(s,q) - \displaystyle\int_0^1 s\,t\,\phi(t,q)\,dt, \\ f(s) = e^s - s. \end{cases}$$

We attempt to find exact solution of IDEs (45) by four methods.

1. **ND-HAM**. Let us expand right side function of Eq. (46) in the form

$$f(s) = e^s - s = x_0(s) + x_1(s) = [g(s;q)]_{q=0},$$

where $x_0(t) = e^s$, $x_1(s) = -s$. Choosing $L = \frac{d^2}{ds^2}$, and from the first equation of (32), we have

$$L[y_0(s)] = x_0(s) \rightarrowtail y_0(s) = e^s.$$

Let $m = 1$, then from second equation of (32),

$$\begin{aligned} \pounds[y_1(s)] &= \hbar R_1(y_0(s)) = h[N(\phi(s;q)) - g(s;q)]\,|_{q=0} \\ &= \hbar\left[\frac{d^2}{ds^2}[y_0(s)] - s\int_0^1 t y_0(t)dt - (x_0(s) + x_1(s))\right] \\ &= \hbar\big[e^s - s - (e^s - s)\big] = 0, \end{aligned}$$

which implies $y_1(s) = 0$. By continuing this procedure, one can obtain $y_2(s) = y_3(s) = y_4(s) = \ldots = 0$. So that the solution of equation (46) is

$$y(s) = \phi(s,1) = \sum_{m=0}^{\infty} y_m(s) = y_0(t) = e^s,$$

which is identical with the exact solution of Eq. (45).

2. **Standard HAM**. Assume that initial guess is $y_0(s) = e^s$, then from Eqs. (6)–(7) it follows that

$$\begin{aligned}\pounds[y_1(s)] &= \hbar R_1(y_0(s)) = \hbar[N(\phi(s;q)) - f(s)]\,|_{q=0}\\ &= \hbar\left[\frac{d^2}{ds^2}[y_0(s)] - s\int_0^1 t\,y_0(t)\,dt - f(s)\right]\\ &= \hbar\big[e^s - s - (e^s - s)\big] = 0.\end{aligned} \tag{47}$$

Since $y_1(0) = y_1'(0) = 0$ by integrating twice (47), we get $y_1(s) = 0$. Continuing in this manner, one can obtain $y_2(s) = y_3(s) = y_4(s) = \ldots = 0$. Thus, the solution of the equation (46) is

$$y(s) = \phi(s,1) = \sum_{m=0}^{\infty} y_m(s) = y_0(t) = e^s,$$

which is identical with the exact solution of Eq. (45).
Since first iteration of q-HAM developed by El-Tawil and Huseen [13] coincides with first iteration of HAM, therefore it gives exact solution when initial guess is chosen as exact solution.

3. **mHAM** is developed by Ayati et al. [6] in 2014. In this case function $f(s)$ is divided into two parts

$$f(s) = e^s - s = x_0(s) + x_1(s),$$

where $x_0(s) = e^s$, $x_1(s) = -s$.
In view of the first equation of (20), we get

$$L[y_0(s)] = x_0(s) \rightarrowtail y_0(s) = e^s.$$

From the second equation of (20), it follows that

$$\begin{aligned}\pounds[y_1(s) - y_0(s)] &= \hbar[R_1(y_0(s)) - x_1(s)]\\ &= \hbar\left[\frac{d^2}{ds^2}[y_0(s)] - s\int_0^1 t\,y_0(t)\,dt - x_1(s)\right] = \hbar\, e^s,\end{aligned}$$

which leads to

$$y_1(s) = (1+\hbar)(e^s - 1 - s).$$

From the third equation of (20), we obtain the second iteration

$$\begin{aligned}\pounds[y_2(s) - y_1(s)] &= \hbar R_2(y_1(s)) = \hbar\frac{\partial}{\partial q}\big[N(\phi(s;q))\big]_{q=0}\\ &= \hbar\left[\frac{d^2}{ds^2}[y_1(s)] - s\int_0^1 t\,y_1(t)\,dt\right] = \hbar(1+\hbar)\left[e^s - \frac{s}{6}\right],\end{aligned} \tag{48}$$

Acting operator J_0^2 on both sides of Eq. (48) leads to

$$\begin{aligned}y_2(s) &= y_1(s) + \hbar(1+\hbar)\int_0^s\int_0^{s_1}\left[e^t - \frac{t}{6}\right]dt ds_1\\ &= (1+\hbar)(e^s - 1 - s) + \hbar(1+\hbar)\left(e^s - 1 - s - \frac{s^3}{36}\right)\\ &= (1+\hbar)^2(e^s - 1 - s) - \hbar(1+\hbar)\frac{s^3}{36}.\end{aligned}$$

Three term iterations at $\hbar = -1$ yields

$$Y_{2\mathrm{mHAM}}(s) = y_0(s) + y_1(s) + y_2(s) = e^s,$$

which is identical with the exact solution of Eq. (45).

4. **In the modified HAM (MHAM)** developed by Bataineh [5], function $\varphi(t)$ is constructed as

$$\varphi(s,q) = x_0(s) + q x_1(s), \tag{49}$$

where $x_0(s) = e^s - \frac{4s}{5}$, $x_1(s) = -\frac{s}{5}$.

Choosing $y_0(s) = \exp(s)$ and from (9)–(10) and (49) it follows that

$$\begin{aligned} \pounds[y_1(s)] &= \hbar R_1(u_0(s)) = \hbar[N(\phi(s;q)) - \varphi(s;q)]\big|_{q=0} \\ &= \hbar\left[\frac{d^2}{ds^2}[y_0(s)] - s\int_0^1 t\, y_0(t)\,dt - x_0(s)\right] = -\frac{4}{5}\hbar\, s, \end{aligned}$$

which leads to

$$y_1(s) = -\frac{4}{5}\hbar\int_0^s\int_0^{s_1} t\,dt\,ds_1 = -\hbar\frac{s^3}{30}.$$

Next iteration is

$$\begin{aligned} \pounds[y_2(s) - y_1(s)] &= \hbar R_2(y_1(s)) = \hbar\frac{\partial}{\partial q}\big[N(\phi(s;q)) - \varphi(s;q)\big]_{q=0} \\ &= \hbar\left[\frac{d^2}{ds^2}[y_1(s)] - s\int_0^1 t\, y_1(t)\,dt - x_1(s)\right] \\ &= \hbar\, s\left[\frac{1}{5} - \hbar\frac{29}{150}\right], \end{aligned} \tag{50}$$

Acting operator J_0^2 on both sides of Eq. (50) leads to

$$y_2(s) = y_1(s) + \hbar\left[\frac{1}{5} - \hbar\frac{29}{150}\right]\int_0^s\int_0^{s_1} t\,dt\,ds_1 = -\hbar^2\frac{29}{900}s^3.$$

Three term iterations at $\hbar = -1$ yields

$$Y_{2\mathrm{MHAM}}(s) = y_0(s) + y_1(s) + y_2(s) = e^s + \frac{1}{900}s^3,$$

which gives highly accurate solution but does not coincide with the exact solution.

Let us apply ND-HAM for the different initial guess. To find approximate solution let us split right side function of Eq. (46) in the form

$$f(s) = 0 + \left(e^s - \frac{3s}{4}\right) - \frac{s}{4} = x_0(s) + x_1(s) + x_2(s),$$

where $x_0(s) = 0$, $x_1(s) = e^s - 3s/4$, $x_2(s) = -s/4$ and construct $g(s,t)$ function as follows

$$g(s,q) = x_0(s) + x_1(s) + (q\hbar)x_2(s).$$

Knowing $L = \frac{d^2}{ds^2}$, and solving the first equation of (32), we obtain

$$y_0(s) = 1 + s. \tag{51}$$

In Table 1, we have summarized all five methods with the same initial guess (51) and with five iterations ($m = \{3, 5, 10\}$).

Here

— For MHAM function $\varphi(s,q) = x_0(s) + q x_1(s) = \left(e^s - \frac{s}{2}\right) + q\left(-\frac{s}{2}\right)$.

— For mHAM function $f(s) = x_0(s) + x_1(s) + x_2(s) = 0 + \left(e^s - \frac{s}{2}\right) + \left(-\frac{s}{2}\right)$.

— For q-HAM function $f(s) = e^s - s$ and $n = 2$, $\hbar = -2$.

From the Table 1, we can conclude that ND-HAM dominated over all modified HAM. Control parameter $\hbar = -1$ is taken for ND-HAM, HAM, MHAM, mHAM and $\hbar = -2$ for q-HAM.

Table 1. Numerical results for Example 1, $m=5$.

m	Exact	ND-HAM	HAM = q-HAM	MHAM = m-HAM
3	e^s	$e^s+\frac{s^3}{2160}$	$e^s-\frac{s^3}{1080}$	$e^s+\frac{s^3}{540}$
5	e^s	$e^s+\frac{s^3}{5832}\cdot 10^{-4}$	$e^s-\frac{s^3}{2916}\cdot 10^{-4}$	$e^s+\frac{s^3}{1458}\cdot 10^{-4}$
10	e^s	$e^s+\frac{s^3}{1417176}\cdot 10^{-9}$	$e^s-\frac{s^3}{708588}\cdot 10^{-9}$	$e^s+\frac{s^3}{354294}\cdot 10^{-9}$

Example 2 (Huseen et al. [17]). Let us consider non-linear VIEs

$$\begin{aligned} &u'(s)=-1+\int_0^s u^2(t)\,dt,\\ &u(0)=0. \end{aligned} \tag{52}$$

Solution. Analytical solution of Eq. (52) does not exist, fortunately Wavelet Galerkin method [24] is taken as analytical numerical solution. To solve (52) by the ND-HAM we rewrite it in the operator form

$$N(\phi(s,q))=f(s),$$

where

$$\begin{cases} N(\phi(s,q))=\dfrac{\partial}{\partial s}\phi(s,q)-\displaystyle\int_0^s \phi^2(t,q)\,dt,\\ f(s)=-1. \end{cases}$$

Let us construct $g(s,q)$ function as follows

$$g(s,q)|_{q=0}=f(s)=-1+0=x_0(s)+x_1(s),\quad x_0(s)=-1,\quad x_1(s)=0. \tag{53}$$

In view of Eqs. (25) and (53), it follows that $u_0(s)=-s$. Since

$$\begin{cases} \phi(s,q)=u_0(s)+\displaystyle\sum_{m=1}^{\infty}q^m u_m(s),\\ \dfrac{\partial}{\partial q}\phi(s,q)|_{q=0}=u_1(s),\\ \dfrac{\partial^{m-1}}{\partial q^{m-1}}\phi(s,q)|_{q=0}=(m-1)!u_{m-1}(s),\\ \phi^2(s,q)|_{q=0}=u_0^2(s),\quad \dfrac{\partial}{\partial q}\phi^2(s,q)|_{q=0}=2u_0(s)u_1(s),\\ \dfrac{1}{(m-1)!}\dfrac{\partial^{m-1}}{\partial q^{m-1}}\phi^2(s,q)|_{q=0}=\displaystyle\sum_{i=0}^{m-1}u_i(s)u_{m-1-i}, \end{cases}$$

we have

$$\begin{aligned} L[u_1(s)]&=\hbar\left[N(\phi(s,q))-(x_0(s)+x_1(s))\right]|_{q=0}\\ &=\hbar\left[\frac{\partial}{\partial s}u_0(s)-\int_0^s u_0^2(t)\,dt-(x_0(s)+x_1(s))\right],\\ L[u_2(s)-u_1(s)]&=\hbar\frac{\partial}{\partial q}\left[N(\phi(s,q))\right]|_{q=0}=\hbar\left[\frac{\partial}{\partial s}u_1(s)-2\int_0^s u_0(t)u_1(t)\,dt\right],\\ L[u_m(s)-u_{m-1}(s)]&=\hbar\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[N(\phi(s,q))\right]\Big|_{q=0}\\ &=\hbar\left[\frac{\partial}{\partial s}u_{m-1}(s)-\int_0^s\sum_{i=0}^{m-1}u_i(t)u_{m-1-i}(t)\,dt\right],\quad m\geqslant 3. \end{aligned} \tag{54}$$

From (54), it follows that

$$u_1(s) = -\frac{\hbar}{12} \cdot s^4,$$
$$u_2(s) = u_1(s) - \frac{\hbar^2}{252} s^4 \cdot \left(21 + s^3\right),$$
$$u_3(s) = u_2(s) + \frac{\hbar^2}{6048} s^4 \left(504 \cdot (1+\hbar) + 24(1+2h)s^3 + \hbar s^6\right).$$

The fifth terms approximation of the ND-HAM at $\hbar = -1$ is

$$\begin{aligned} U_{5\text{ND-HAM}}(s) &= u_0(s) + u_1(s) + u_2(s) + u_3(s) + u_4(s) + u_5(s) \\ &= -s + \frac{1}{12}s^4 - \frac{1}{252}s^7 + \frac{1}{6048}s^{10} - \frac{1}{157248}s^{13} + \frac{37}{158505984}s^{16}. \end{aligned} \tag{55}$$

Since $f(s) = -1$ is constant, the fifth terms approximation of the HAM, MHAM and mHAM are the same as ND-HAM defined by (55). The fifth terms approximation of the Oq-HAM, developed by Sh. N. Huseen et al. [17] in 2013, for the same problem of Eq. (52), at $n = 2$ yields

$$\begin{aligned} U_{5\text{Oq-HAM}}(s) &= u_0(s) + \frac{u_1(s)}{2} + \frac{u_2(s)}{4} + \frac{u_3(s)}{8} + \frac{u_4(s)}{16} + \frac{u_5(s)}{32} \\ &= -s + \frac{1}{24}s^4 - \frac{1}{1008}s^7 + \frac{1}{48384}s^{10} - \frac{1}{2515968}s^{13} + \frac{37}{5072191488}s^{16}. \end{aligned}$$

The fifth terms approximation of the Adomian decomposition method (ADM) developed in El-Sayed and Abdel–Aziz [22] is in the form,

$$\begin{aligned} U_{5\text{ADM}}(s) &= u_0(s) + u_1(s) + u_2(s) + u_3(s) + u_4(s) + u_5(s) \\ &= -s + \frac{1}{12}s^4 - \frac{1}{252}s^7 + \frac{1}{6048}s^{10} - \frac{1}{157248}s^{13} + \frac{79}{264176640}s^{16}. \end{aligned}$$

Numerical comparisons of the methods are given in the Table 3. Five iterations are taken for all methods. Parameter $n = 2$ is taken for the Oq-HAM [17].

Table 2. Numerical results for Example 2.

s value	Exact [24]	ND-HAM	Oq-HAM in [17]	ADM in [22]
0.0000	0.0000	0.000000000	0.00000000	0.000000000
0.0938	−0.0937	−0.093793549	−0.09379677	−0.093793549
0.3125	−0.3117	−0.311706425	−0.31210292	−0.311706425
0.5000	−0.4948	−0.494822508	−0.49740330	−0.494822508
0.7188	−0.6969	−0.696941464	−0.70776777	−0.696941463
0.9062	−0.8520	−0.851934173	−0.86852216	−0.851934160
1.0000	−0.9205	−0.920475703	−0.92911909	−0.920475637

For the numerical comparisons of the methods Wavelet Galerkin method [24] is taken as analytical numerical solution. From Table 2, we can note that ND-HAM which is identical with HAM is a bit more accurate than others.

6. Conclusion

In this paper, we explored the superiority of the ND-HAM over the HAM [1], MHAM [5], mHAM [6] and q-HAM [13] in terms of accuracy. It provides a good opportunity to choose initial guess by solving linear operator equation depending on the given function $f(t)$. We also studied the non-linear IDEs (28)–(29) and its scheme of approximate solutions. Five methods were compared with the same examples and the same initial conditions. Table 1 demonstrated that ND-HAM can give better results over other methods because of suitable choice $x_i(t)$ of the expansion $f(t)$. Example 2 is also solved by Oq-HAM [17] and Adomian decomposition method (ADM) [21] until the fifth iteration. Practical applications show that ND-HAM gave fixed point in case when initial guess coincides with the exact solutions. Even if it does not coincide with the exact solution, it brings one closer to the exact solutions with small number of iterations. Standard HAM is well developed and converges to exact solution fast and accurate. Matlab code is used to get the numerical solutions. In addition, theoretical study is also made on uniqueness solution of the problem and convergence of the proposed method.

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Новий розвиток методу гомотопічного аналізу для нелінійних інтегро-диференціальних рівнянь з початковими умовами

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Метод гомотопічного аналізу (МГА) був запропонований Ляо в 1992 році в його докторській дисертації для нелінійних проблем і застосовувався в багатьох різних проблемах математичної фізики та техніки. У цій статті демонструється новий розвиток методу гомотопічного аналізу (НР-МГА) для нелінійного інтегро-диференціального рівняння (НІДР) із початковими умовами. Практичні дослідження показали, що НР-МГА дозволяє легко знайти початкове припущення та наближається до точного розв'язку швидше, ніж стандартний МГА, модифікований МГА (ММГА), новий модифікований МГА (мНАМ) і більш загальний метод МГА (з-МГА). Доведено єдиність розв'язку задачі та збіжність ND-HAM у банаховому просторі. Нарешті, два приклади проілюстровані, щоб показати точність і достовірність запропонованого методу. У кожному прикладі порівнюються п'ять методів.

Ключові слова: *метод гомотопічного аналізу, новий розвиток методу гомотопічного аналізу, нелінійне інтегро-диференціальне рівняння, порівняння, збіжність.*