Properties of fundamental solutions, correct solvability of the Cauchy problem and integral representations of solutions for ultraparabolic Kolmogorov–type equations with three groups of spatial variables and with degeneration on the initial hyperplane

Voznyak O. G.[1], Dron V. S.[2], Medynskyi I. P.[3]

[1] *West Ukrainian National University,*
11 Lvivska Str., 46009, Ternopil, Ukraine
[2] *Pidstryhach Institute for Applied Problems of Mechanics and Mathematics*
of the National Academy of Sciences of Ukraine,
3-B Naukova Str., 79060, Lviv, Ukraine
[3] *Lviv Polytechnic National University,*
12 S. Bandera Str., 79013, Lviv, Ukraine

(Received 1 April 2022; Revised 20 September 2022; Accepted 21 September 2022)

Some properties of the fundamental solution of the Cauchy problem for homogeneous ultraparabolic Kolmogorov–type equation with three groups of spatial variables including two groups of degeneration and with degeneration on the initial hyperplane are established. For different type of degeneration on the initial hyperplane the theorems on integral representations of solutions and correct solvability of the Cauchy problem are presented. These results for such type of equations are obtained in appropriate classes of weight functions.

Keywords: *ultraparabolic equations of the Kolmogorov type, fundamental solution of the Cauchy problem, weight spaces, integral representations of solutions, correct solvability of the Cauchy problem.*

2010 MSC: 35K45, 35K65, 35K70 **DOI:** 10.23939/mmc2022.03.779

1. Introduction

The paper aims to study the class of equations which is a natural generalization of the classical equation of diffusion with inertia of A. M. Kolmogorov [1]. This equation and its various generalizations have been studied by many authors. Linear and nonlinear ultraparabolic equations arise in some problems of probability theory, in the mathematical modeling of options, in Brownian motion theory, in convective diffusion theory, in binary electrolytes theory, during the modeling of diffusion with inertia and electron scattering in age approximation of the slowed-down electrons theory, in the biology, economics and other fields of science [2–11]. Levi method, including its modifications, is the main information source and it is a method to construct the Cauchy problem fundamental solution (FSCP) [11, 12]. However, the application of this method for degenerate equations of Kolmogorov type is significantly complicated in case when coefficients are dependent on all variables. Besides traditional difficulties, there are serious ones associated with the equation degeneracy. The Levi stepwise method was used for ultraparabolic equations of Kolmogorov type with the different numbers of groups of spatial variables, both without degeneration [13–17], and with degeneration on the initial hyperplane [18]. In particular, FSCP Z for equations with coefficients that are independent on the variables of degeneration was constructed in [13,18], in [14,15] it was constructed for equations with one group of spatial variables of degeneration, and in [16,17] it was done for equations with two groups of spatial variables of degeneration. Estimates of the function Z and its derivatives, and estimates of the increments of principal derivatives of Z with respect to spatial variables are also found in these papers.

This work is a continuation of the launched research of ultraparabolic equations of the Kolmogorov type with two groups of degenerate variables that also have degeneration on the initial hyperplane.

The main purpose is to obtain additional properties of FSCP and their application for establishing of the theorems about integral representations of solutions and about correct solvability of the Cauchy problem for the corresponding homogeneous and inhomogeneous equation. These results are similar to the results for the same equations in the case without degeneration on the initial hyperplane.

2. Notation, assumptions and supporting information

Let n, n_1, n_2 and n_3 be given positive integers such that $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$ and $n = n_1 + n_2 + n_3$; $\mathbb{N}_j := \{1, \ldots, j\}$, $j \in \mathbb{N}$, $\mathbb{Z}_j := \mathbb{N}_j \cup \{0\}$, $m_j = j - 1/2$, $j \in \mathbb{N}_3$. We suppose that spatial variable $x \in \mathbb{R}^n$ consists of three groups of variables $x := (x_1, x_2, x_3)$, where components $x_j := (x_{j1}, \ldots, x_{jn_j}) \in \mathbb{R}^{n_j}$, $j \in \mathbb{N}_3$. Accordingly, the multi-index $k \in \mathbb{Z}_+^{n_j}$ we will write in the form $k := (k_1, k_2, k_3)$, where $k_j := (k_{j1}, \ldots, k_{jn_j}) \in \mathbb{Z}_+^{n_j}$, $|k_j| := |k_{j1}| + \ldots + |k_{jn_j}|$, $j \in \mathbb{N}_3$; $M := m_1 n_1 + m_2 n_2 + m_3 n_3$; $M_k := m_1|k_1| + m_2|k_2| + m_3|k_3|$; $\Pi_H := \{(t,x) | t \in H, x \in \mathbb{R}^n\}$, if $H \subset \mathbb{R}$.

In addition, we will use the following notations:

$$z^{(0)} := x, \quad z^{(1)} := (z_1, x_2, x_3), \quad z^{(2)} := (x_1, z_2, x_3), \quad z^{(3)} := (x_1, x_2, z_3),$$

$$x^{(1)} := (x_1, z_2, z_3), \quad x^{(2)} := (x_1, x_2, z_3),$$

$$\Delta_x^z f(\cdot, x, \cdot) := f(\cdot, x, \cdot) - f(\cdot, z, \cdot), \quad \Delta_{x_s}^{z_s} f(\cdot, x, \cdot) := \Delta_x^{z^{(s)}} f(\cdot, x, \cdot), \quad s \in \mathbb{N}_3,$$

$$\hat{x}_1 := (x_{11}, \ldots, x_{1n_2}), \quad x_1' := (x_{11}, \ldots, x_{1n_3}), \quad x_2' := (x_{21}, \ldots, x_{2n_3}),$$

$$X_1(t) := x_1, \quad X_2(t) := x_2 + t\hat{x}_1, \quad X_3(t) := x_3 + t x_2' + 2^{-1} t^2 x_1', \quad t \in \mathbb{R},$$

$$X(t) := (X_1(t), X_2(t), X_3(t)), \quad \Xi^{(1)}(t,\tau) := (\xi_1, X_2(t,\tau), X_3(t,\tau)), \quad \Xi^{(2)} := (\xi_1, \xi_2, X_3(t,\tau)),$$

$$\rho(t; x, \xi) := \sum_{j=1}^{3} t^{1-2j} |X_j(t) - \xi_j|^2, \quad t \in \mathbb{R}, \quad \{x, \xi\} \subset \mathbb{R}^n,$$

$$A(t,\tau) := \int_\tau^t \frac{d\theta}{\alpha(\theta)}, \quad B(t,\tau) := \int_\tau^t \frac{\beta(\theta) d\theta}{\alpha(\theta)}, \text{ where } \alpha, \beta \text{ are some functions on } \mathbb{R},$$

$$E_c(t, x; \tau, \xi) := \exp\{-c\rho(B(t,\tau); x, \xi)\}, \quad t > \tau, \ \{x, \xi\} \subset \mathbb{R}^n,$$

$$E^d(t,\tau) := \exp\{dA(t,\tau)\}, \quad E_c^d(t, x; \tau, \xi) := E_c(t, x; \tau, \xi) E^d(t,\tau), \quad t > \tau, \quad \{x, \xi\} \subset \mathbb{R}^n, \quad d \in \mathbb{R}.$$

Consider the equation

$$Lu(t,x) := (S - A(t, x, \partial_{x_1})) u(t,x) = f(t,x), \quad (t,x) \in \Pi_{(0,T]}, \tag{1}$$

with

$$S := \alpha(t)\partial_t - \beta(t)\left(\sum_{j=1}^{n_2} x_{1j}\partial_{x_{2j}} + \sum_{j=1}^{n_3} x_{2j}\partial_{x_{3j}}\right),$$

$$A(t, x, \partial_{x_1}) := \beta(t) \sum_{j,l=1}^{n_1} a_{jl}(t,x) \partial_{x_{1j}} \partial_{x_{1l}} + \beta(t) \sum_{j=1}^{n_1} a_j(t,x) \partial_{x_{1j}} + a_0(t,x),$$

where f is a given function, and u is an unknown function; α and β are continuous on the interval $[0,T]$ functions and $\alpha(t) > 0$, $\beta(t) > 0$ for $t \in (0,T]$, $\alpha(0)\beta(0) = 0$ and β is a monotonically increasing function. The degenerations at $t = 0$ in the equation (1) are generated by the functions α and β included in the equation.

We will classify the degeneration with the help of values $A(T,0)$ and $B(T,0)$. Thus, in the case of $A(T,0) < \infty$ we will say that the equation (1) is weakly degenerate. In the case $A(T,0) = \infty$ it is called strongly degenerate, and very strongly degenerate, if $A(T,0) = \infty$ and $B(T,0) = \infty$. For the equation (1) the Cauchy problem with the initial condition as $t = 0$ isn't always considered in the common formulation. But we can say about FSCP as about the following function: $Z(t, x; \tau, \xi)$, $0 < \tau < t \leqslant T$, which has with respect to variables t and x all derivatives from the equation (1) and for any $\tau \in (0,T)$ and for arbitrary continuous and bounded function $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ the formula

$$u(t,x) := \int_{\mathbb{R}^n} Z(t,x;\tau,\xi)\varphi(\xi)d\xi, \quad (t,x) \in \Pi_{(\tau,T]}, \tag{2}$$

determines in the layer $\Pi_{(\tau,T]}$ the solution of the Cauchy problem for the homogeneous equation (1) with the initial condition

$$u(t,x)|_{t=\tau} = \varphi(x), \quad x \in \mathbb{R}^n. \tag{3}$$

In the case of weak degeneration, it can be considered the Cauchy problem for the equation (1) with the initial condition

$$u(t,x)|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n; \tag{4}$$

in the case of strong degeneration, the initial condition is

$$(u(t,x)E^d(T,t))|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n, \tag{5}$$

where d is the constant from the estimates of FSCP. In the case of very strong degeneration, the equation is considered without any initial condition. To combine the cases of weak and strong degeneration let introduce the following notation

$$Z_0(t,x;\xi) := \begin{cases} Z(t,x;0,\xi), \quad (t,x) \in \Pi_{(0,T]}, & \text{if } A(T,0) < \infty; \\ \lim\limits_{\tau \to 0} \left(Z(t,x;\tau,\xi)E^{-d}(T,\tau)\right), \quad (t,x) \in \Pi_{(0,T]}, & \text{if } A(T,0) = \infty. \end{cases}$$

Assume that the coefficients a_{jl}, a_j and a_0 of the equation (1) are complex-valued functions on $\Pi_{[0,T]}$ satisfying the following conditions:

(i) a_{jl}, a_j, a_0 are bounded and continuous with respect to t and there exists such constant $\delta > 0$ that for any $(t,x) \in \Pi_{[0,T]}$ and $\sigma_1 := (\sigma_{11}, \ldots, \sigma_{1n_1}) \in \mathbb{R}^{n_1}$ the inequality $\mathrm{Re}\sum_{j,l=1}^{n_1} a_{jl}(t,x)\sigma_{1j}\sigma_{1l} \geqslant \delta|\sigma_1|^2$ is valid;

(ii) a_{jl}, a_j, a_0 are Hölder continuous in spatial variables function in the following sense:

$$\exists\, H_1 > 0\ \exists\, \gamma_1 \in (0,1]\ \forall \{(t,x),(t,z^{(1)})\} \subset \Pi_{[0,T]}\colon |\Delta_{x_1}^{z_1} a(t,x)| \leqslant H_1|x_1 - z_1|^{\gamma_1}, \tag{6}$$

$$\exists\, H_2 > 0\ \exists\, \gamma_2 \in (1/3,2/3]\ \forall \{(t,x),(t,z^{(2)})\} \subset \Pi_{[0,T]},\ \forall h \in [\tau,T]\colon$$
$$|\Delta_{x_2}^{z_2} a(t,x)| \leqslant H_2\big((B(h,\tau))^{m_2\gamma_2} + |X_2(B(h,\tau)) - z_2|^{\gamma_2}\big), \tag{7}$$

$$\exists\, H_3 > 0\ \exists\, \gamma_3 \in (3/5,4/5]\ \forall \{(t,x),(t,z^{(3)})\} \subset \Pi_{[0,T]},\ \forall h \in [\tau,T]\colon$$
$$|\Delta_{x_3}^{z_3} a(t,x)| \leqslant H_3\big((B(h,\tau))^{m_3\gamma_3} + |X_3(B(h,\tau)) - z_3|^{\gamma_3}\big), \tag{8}$$

$$\exists\, H_4 > 0\ \forall \{(t,x),(t,\xi^{(1)}),(t,z^{(2)})\} \subset \Pi_{[0,T]},\ \forall h \in [\tau,T]\colon$$
$$\left|\Delta_{x_1}^{\xi_1}\Delta_{x_s}^{z_s} a(t,x)\right| \leqslant H_4|x_1 - \xi_1|^{\gamma_1}\big((B(h,\tau))^{m_s\gamma_s} + |X_s(B(h,\tau)) - z_s|^{\gamma_s}\big), \quad s \in \{2,3\}, \tag{9}$$

where a defines every coefficients a_{jl}, a_j and a_0. In the condition (9) the constants γ_1, γ_2 and γ_3 are the same as in the appropriated conditions (6)–(8);

(iii) the coefficients a_{jl}, a_j, a_0 of the expression $A(t,x,\partial_{x_1})$ have bounded derivatives of the same forms as theirs multiplier. The derivatives of these coefficients in the layer $\Pi_{[0,T]}$ satisfy the condition **(ii)**.

Note that for $h = \tau$ from the conditions (7), (8) the common Hölder continuous in variables x_2 and x_3 conditions follow. The sufficient conditions for (7), (8) are given in the works [2] and [4] respectively. As it was proved in [18], under the conditions **(i)**–**(ii)** for the equation (1), there exists FSCP Z for which the following estimates hold:

$$\left|\partial_x^k Z(t,x;\tau,\xi)\right| \leqslant C(B(t,\tau))^{-M-M_k} E_c^d(t,x;\tau,\xi), \tag{10}$$

$$|SZ(t,x;\tau,\xi)| \leqslant C(B(t,\tau))^{-M-1} E_c^d(t,x;\tau,\xi), \tag{11}$$

$$\left|\int_{\mathbb{R}^n} \partial_{x_s}^{k_s} Z(t,x;\tau,\xi)d\xi\right| \leqslant C(B(t,\tau))^{-m_s(1-\alpha_s)} E^d(t,\tau), \quad k_s \in \mathbb{Z}_+^{n_s}\backslash\{0\}, \quad s \in \mathbb{N}_3, \tag{12}$$

$$\left|\Delta_{x_l}^{\xi_l}\int_{\mathbb{R}^n}\partial_{x_s}^{k_s}Z(t,x;\tau,\xi)d\xi\right| \leqslant C|x_l-\xi_l|^{\alpha_l}(B(t,\tau))^{-m_s(1-\alpha_s)-m_l\alpha_l}E^d(t,\tau),\ k_s\in\mathbb{Z}_+^{n_s}\backslash\{0\},\ \{l,s\}\in\mathbb{N}_3, \tag{13}$$

$$\left|\int_{\mathbb{R}^{n_2+n_3}}\partial_{x_s}^{k_s}Z(t,x;\tau,\xi)d\xi\right| \leqslant C(B(t,\tau))^{-m_1n_1-m_s(1-\alpha_s)} \times E_c^1(t,\tau,x_1-\xi_1)E^d(t,\tau), \quad k_s\in\mathbb{Z}_+^{n_s}\backslash\{0\}, \quad s\in\{2,3\}, \tag{14}$$

$$\left|\Delta_{x_l}^{\xi_l}\int_{\mathbb{R}^{n_2+n_3}}\partial_{x_s}^{k_s}Z(t,x;\tau,\xi)d\xi\right| \leqslant C|x_l-\xi_l|^{\alpha_l}(B(t,\tau))^{-m_s(1-\alpha_s)-m_l\alpha_l} \times E_c^1(t,\tau,x_1-\xi_1)E^d(t,\tau), \quad k_s\in\mathbb{Z}_+^{n_s}\backslash\{0\}, \quad \{l,s\}\in\{2,3\}, \tag{15}$$

$$\left|\int_{\mathbb{R}^{n_3}}\partial_{x_3}^{k_3}Z(t,x;\tau,\xi)d\xi\right| \leqslant C(B(t,\tau))^{-m_1n_1-m_2n_2-m_3(1-\alpha_3)} \times E_c^1(t,\tau,x_1-\xi_1)E_c^2(t,\tau,X_2(t,\tau)-\xi_2)E^d(t,\tau), \quad k_3\in\mathbb{Z}_+^{n_3}\backslash\{0\}, \quad |k_3|=1, \tag{16}$$

$$\left|\Delta_{x_l}^{\xi_l}\int_{\mathbb{R}^{n_2+n_3}}\partial_{x_s}^{k_s}Z(t,x;\tau,\xi)d\xi\right| \leqslant C|x_l-\xi_l|^{\alpha_l}(B(t,\tau))^{-m_s(1-\alpha_s)-m_ln_l} \times E_c^1(t,\tau,x_1-\xi_1)E_c^2(t,\tau,X_2(t,\tau)-\xi_2)E^d(t,\tau), \quad k_3\in\mathbb{Z}_+^{n_3}\backslash\{0\}, \quad |k_3|=1, \tag{17}$$

where $0<\tau<t\leqslant T$, $\{x,\xi\}\subset\mathbb{R}^n$, $k:=(k_1,k_2,k_3)\in\mathbb{Z}_+^n$, $m_1|k_1|+|k_2|+|k_3|\leqslant 1$, C is a positive constant.

Note that the conditions (6)–(8) occur for the Hölder indexes, if we choose $\alpha_3=(3+\alpha_1)/5$, $\alpha_2=(1+\alpha_1)/3$, $\alpha_1=\alpha$, $\alpha\in(0,1]$. In this case with the help of special modifying method from the work [3] for the case of two groups of spatial variables of degeneration we also obtain the following estimations:

$$\left|\Delta_x^\xi\partial_x^kZ(t,x;\tau,\xi)\right| \leqslant (d(x;\xi))^\alpha(B(t,\tau))^{-M-M_k-m_1\alpha}E_c^d(t,x;\tau,\xi), \tag{18}$$

$$\left|\Delta_x^\xi SZ(t,x;\tau,\xi)\right| \leqslant (d(x;\xi))^\alpha(B(t,\tau))^{-M-1-m_1\alpha}E_c^d(t,x;\tau,\xi), \tag{19}$$

where $0<\tau<t\leqslant T$, $k:=(k_1,k_2,k_3)\in\mathbb{Z}_+^n$, $m_1|k_1|+|k_2|+|k_3|\leqslant 1$, C is a positive constant, and $d(x;\xi):=\sum_{l=1}^3|x_l-\xi_l|^{1/(2l-1)}$ is the parabolic distance between the points x and ξ, $\{x,\xi\}\subset\mathbb{R}^n$.

3. Definitions of norms and spaces

We introduce function spaces to research correct solvability of the problems with the initial conditions and the problems without the initial conditions depending on type of the degeneration of the equation. Since the function Z tends to 0 exponentially as $|x|\to\infty$ then the density of potentials where FSCP is a kernel can appropriately increase. Really these potentials and, thus, the solutions can exponentially increase as $|x|\to\infty$. The increasing orders are determined by the orders of equations, and the types of growth are described by special functions dependent on t.

Consider sets of the functions $\boldsymbol{k}(t,\boldsymbol{a})$ and $\boldsymbol{s}(t)$, $t\in[0,T]$, which are defined by the following way:

$$\boldsymbol{k}(t,\boldsymbol{a}):=(k_1(t,a_1),k_2(t,a_2),k_3(t,a_3)), \quad \boldsymbol{s}(t):=(s_1(t),s_2(t),s_3(t)),$$

where

$$k_j(t,a_j):=\begin{cases}c_0a_j(c_0-a_j(T-B(T,t))^{2(j-1)+1})^{-1} & \text{for } t\in(0,T],\\ c_0a_j(c_0-a_j(T-B(T,0))^{2(j-1)+1})^{-1}, & \text{if } t=0,\ B(T,0)<\infty,\\ 0, & \text{if } t=0,\ B(T,0)=\infty,\end{cases} \quad j\in\mathbb{N}_3;$$

$$s_1(t):=k_1(t,a_1)+2(B(t,0))^2k_2(t,a_2)+2^{-2}3(B(t,0))^4k_3(t,a_3),$$

$$s_2(t):=2k_2(t,a_2)+3(B(t,0))^2k_3(t,a_3), \quad s_3(t):=3k_3(t,a_3); \tag{20}$$

$c_0 \in (0, c)$, c is the constant from the estimates (10) and (11); $\boldsymbol{a} := (a_1, a_2, a_3)$ is a set of such numbers that $0 \leqslant a_j < c_0 T^{1-2j}$, $j \in \mathbb{N}_3$.

Let's introduce also the following notation:

$$[\boldsymbol{k}(t, \boldsymbol{a}), \xi] := \sum_{j=1}^{3} k_j(t, a_j)|\xi_j|^2, \quad t > 0, \quad \xi_j \in \mathbb{R}^{n_j}, \quad j \in \mathbb{N}_3.$$

The functions $\boldsymbol{k}(t, a)$ and $\boldsymbol{s}(t)$ have the properties:

$$k_j(t, a_j) \geqslant k_j(0, a_j), \quad j \in \mathbb{N}_3, \quad s_1(t) \geqslant k_1(0, a_1), \quad s_j(t) > k_j(0, a_j), \quad j \in \{2, 3\}. \tag{21}$$

Note that functions $k_j(t, a_j)$, $j \in \mathbb{N}_3$, and the functions $k_j^0(t, a_j)$, $j \in \mathbb{N}_3$, in the next form

$$k_j^0(t, a_j) := c_0 a_j \big(c_0 - a_j t^{2(j-1)+1}\big)^{-1}, \quad 0 \leqslant t < (c_0/a_j)^{1/(2j-1)}, \quad j \in \mathbb{N}_3,$$

are conducted by relationships

$$k_1(t, a_1) = k_1^0\big(B(t, \tau), k_1(\tau, a_1)\big), \quad k_j^0\big(B(t, \tau), k_j(\tau, a_j)\big) \leqslant k_j(t, a_j), \quad j \in \{2, 3\}.$$

And since $k_j(t, a_j) := k_j^0\big(T - B(T, t), a_j\big)$, $t \in [0, T]$, $j \in \mathbb{N}_3$, then the inequality

$$-c_0 \rho(t, x; 0, \xi) + [\boldsymbol{k}(0, \boldsymbol{a}), \xi] \leqslant [\boldsymbol{k}(t, \boldsymbol{a}), X(t, 0)], \quad t \in (0, T], \quad \{x, \xi\} \in \mathbb{R}^n \tag{22}$$

holds.

Indeed, using the properties of the functions k_j^0, $j \in \mathbb{N}_3$,

$$\begin{aligned} - c_0 \frac{|X_j(t, 0) - \xi_j|^2}{[B(t, 0)]^{2j-1}} + k_j(0, a_j)|\xi_j|^2 = -c_0 \frac{|X_j(t, 0) - \xi_j|^2}{[T - B(T, t) - (T - B(T, 0))]^{2j-1}} \\ + k_j^0(T - B(T, 0), a_j)|\xi_j|^2 \leqslant k_j^0(T - B(T, t) - (T - B(T, 0)), k_j^0(T - B(T, 0), a_j))|X_j(t, 0)|^2 \\ \leqslant k_j^0(T - B(T, t), a_j)|X_j(t, 0)|^2 = k_j(t, a_j)|X_j(t, 0)|^2. \end{aligned}$$

(22) follows from these inequalities.

With the help of functions $\boldsymbol{k}(t, \boldsymbol{a})$ and $\boldsymbol{s}(t)$, $t \in [0, T]$, define the necessary norms and appropriate spaces. Let $p \in [1, \infty]$ and $u(t, x)$, $(t, x) \in \Pi_{[0,T]}$, be given complex-valued measurable for any $t \in [0, T]$ function. For any $t \in [0, T]$ define the norms:

$$\|u(t, \cdot)\|_p^{\boldsymbol{k}(t, \boldsymbol{a})} := \|u(t, x) \exp\{-[\boldsymbol{k}(t, \boldsymbol{a}), X(t, 0)]\}\|_{L_p(\mathbb{R}^n)},$$
$$\|u(t, \cdot)\|_p^{\boldsymbol{s}(t)} := \|u(t, x) \exp\{-[\boldsymbol{s}(t), x]\}\|_{L_p(\mathbb{R}^n)}.$$

We will use the following spaces:

- $L_p^{\boldsymbol{k}(t, \boldsymbol{a})}$, $t \in [0, T]$, $p \in [1, \infty]$, are spaces of measurable functions $\varphi\colon \mathbb{R}^n \to \mathbb{C}$, for which the norms $\|\varphi\|_p^{\boldsymbol{k}(t, \boldsymbol{a})}$ are finite;
- $M^{\boldsymbol{k}(0, \boldsymbol{a})}$ is space countable-additive functions $\mu\colon \mathfrak{B} \to \mathbb{C}$ (generalized Borel measures in $\mathbb{R}^n$) which satisfy the condition

$$\|\mu\|^{\boldsymbol{k}(0, \boldsymbol{a})} := \int_{\mathbb{R}^n} \exp\{-[\boldsymbol{k}(0, \boldsymbol{a}), x]\} d|\mu|(x) < \infty,$$

where $\mathfrak{B}$ is a σ-algebra of Borel sets in the space $\mathbb{R}^n$, and $|\mu|$ is the total variation of μ;

- $L_1^{-\boldsymbol{s}(T)}$ is a space of measurable functions $\psi\colon \mathbb{R}^n \to \mathbb{C}$ with finite norm

$$\|\psi\|_1^{-\boldsymbol{s}(T)} := \big\|\psi(x) \exp\{[\boldsymbol{s}(T), x]\}\big\|_{L_1(\mathbb{R}^n)};$$

- $C_0^{-\boldsymbol{s}(T)}$ is a space of continuous function $\psi\colon \mathbb{R}^n \to \mathbb{C}$ such that for $|x| \to \infty$ we have $|\psi(x)| \exp\{[\boldsymbol{s}(T), x]\} \to 0$. A norm in $C_0^{-\boldsymbol{s}(T)}$ is defined as

$$\|\psi\|_\infty^{-\boldsymbol{s}(T)} := \sup_{x \in \mathbb{R}^n} \big(|\psi(x)| \exp\{[\boldsymbol{s}(T), x]\}\big).$$

Since, on the base of (20),

$$\|u(t, \cdot)\|_p^{\boldsymbol{s}(t)} \leqslant ||u(t, \cdot)||_p^{\boldsymbol{k}(t, \boldsymbol{a})}, \quad t \in [0, T], \quad p \in [1, \infty] \tag{23}$$

and $\boldsymbol{s}(t) \geqslant \boldsymbol{k}(0, \boldsymbol{a})$, $t \in [0, T]$. Then for $\varphi \in L_p^{\boldsymbol{k}(0,\boldsymbol{a})}$ it follows that

$$\|\varphi\|_p^{\boldsymbol{s}(t)} \leqslant \|\varphi\|_p^{\boldsymbol{k}(0,\boldsymbol{a})}, \quad t \in [0, T], \quad p \in [1, \infty]. \tag{24}$$

Let there exists the expression L^* which is Lagrange adjoint one to the expression L. Then adjoint homogeneous equation for (1) is in the form

$$L^* v(\tau, \xi) := \Big(- \alpha(\tau)\partial_\tau + \beta(\tau)\Big(\sum_{j=1}^{n_2} \xi_{1j}\partial_{\xi_{2j}} + \sum_{j=1}^{n_3} \xi_{2j}\partial_{\xi_{3j}} \Big)\Big) v(\tau, \xi) - \beta(\tau) \sum_{j,l=1}^{n_1} \partial_{\xi_{1j}} \partial_{\xi_{1l}} (\overline{a_{jl}(\tau, \xi)} v(\tau, \xi))$$
$$+ \beta(\tau) \sum_{j=1}^{n_1} \partial_{\xi_{1j}} (\overline{a_j(\tau, \xi)} v(\tau, \xi)) - \overline{a_j(\tau, \xi)} v(\tau, \xi) = 0, \quad (\tau, \xi) \in \Pi_{[0,T)}. \tag{25}$$

Here and further, the bar over the expression means the transition in it to a complex-valued conjugation.

4. Formulation of the main results

The main results of the work are contained in the following theorems.

Theorem 1. *Let the coefficients a_{jl}, a_j, a_0 of the equation (1) fulfill the conditions **(i)**–**(iii)**. Then the following assertions hold:*

1) (normality of the solution) there exists FSCP Z^* for the adjoint equation (25) which is related with Z by the equality

$$Z^*(\tau, \xi; t, x) = \overline{Z(t, x; \tau, \xi)}, \quad 0 < \tau < t \leqslant T, \quad \{x, \xi\} \subset \mathbb{R}^n. \tag{26}$$

FSCP Z for which this equality holds is called normal one;

2) (convolution formula) FSCP Z is a solution of the functional equation

$$Z(t, x; \tau, \xi) = \int_{\mathbb{R}^n} Z(t, x; \theta, \lambda) Z(\theta, \lambda; \tau, \xi) d\lambda, \quad 0 < \tau < \theta < t \leqslant T, \quad \{x, \xi\} \subset \mathbb{R}^n; \tag{27}$$

3) there exists only one normal FSCP Z for which the estimates (10) and (11) hold.

Theorem 2. *Let the coefficients of the equation (1) fulfill the conditions **(i)**–**(iii)** and $p \in [1, \infty]$. Then the following assertions hold:*

1) for arbitrary functions $\varphi \in L_p^{\boldsymbol{k}(0,\boldsymbol{a})}$, $p \in (1, \infty]$, and generalized measure $\mu \in M^{\boldsymbol{k}(0,\boldsymbol{a})}$ the formulas

$$u_1^d(t, x) := \begin{cases} \int_{\mathbb{R}^n} Z_0(t, x; \xi)\varphi(\xi) d\xi, & (t, x) \in \Pi_{(0,T]}, \quad \text{if } A(T, 0) < \infty, \\ E^d(T, t) \int_{\mathbb{R}^n} Z_0(t, x; \xi)\varphi(\xi) d\xi, & (t, x) \in \Pi_{(0,T]}, \quad \text{if } A(T, 0) = \infty; \end{cases} \tag{28}$$

$$u_0^d(t, x) := \begin{cases} \int_{\mathbb{R}^n} Z_0(t, x; \xi) d\mu(\xi), & (t, x) \in \Pi_{(0,T]}, \quad \text{if } A(T, 0) < \infty, \\ E^d(T, t) \int_{\mathbb{R}^n} Z_0(t, x; \xi) d\mu(\xi), & (t, x) \in \Pi_{(0,T]}, \quad \text{if } A(T, 0) = \infty \end{cases} \tag{29}$$

determine the unique in the layer $\Pi_{(0,T]}$ solutions of the homogeneous equation (1);

2) there exists constant $C > 0$ which doesn't depend on $\varphi \in L_p^{\boldsymbol{k}(0,\boldsymbol{a})}$, $p \in (1, \infty]$, and $\mu \in M^{\boldsymbol{k}(0,\boldsymbol{a})}$ such that for any $t \in (0, T]$ the estimates

$$\|u_1^d(t, \cdot)\|_p^{\boldsymbol{k}(t,\boldsymbol{a})} \leqslant C\|\varphi\|_p^{\boldsymbol{k}(0,\boldsymbol{a})}, \quad \|u_0^d(t, \cdot)\|_1^{\boldsymbol{k}(t,\boldsymbol{a})} \leqslant C\|\mu\|^{\boldsymbol{k}(0,\boldsymbol{a})}$$

hold;

3) for $p \in (1, \infty)$ the equality $\lim\limits_{t\to\infty} \|u_1^d(t, \cdot) - \varphi(\cdot)\|_p^{\boldsymbol{s}(t)} = 0$ holds, and for $p = 1$ or $p = \infty$ the limit $u_0^d(t, \cdot) \underset{t\to 0}{\longrightarrow} \mu$ or $u_1^d(t, \cdot) \underset{t\to 0}{\longrightarrow} \varphi$ hold respectively in the weak sense, namely for arbitrary functions $\psi \colon \mathbb{R}^n \to \mathbb{C}$ from the spaces $C_0^{-\boldsymbol{s}(T)}$ or $L_1^{-\boldsymbol{s}(T)}$ the relationships

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \psi(x) u_0^d(t, x) dx = \int_{\mathbb{R}^n} \psi(x) d\mu(x) \quad \text{or} \quad \lim_{t\to 0} \int_{\mathbb{R}^n} \psi(x) u_1^d(t, x) dx = \int_{\mathbb{R}^n} \psi(x)\varphi(x) dx,$$

are valid respectively.

The next theorem is in some sense the inverse one of the Theorem 2.

Theorem 3. *Let the conditions **(i)**–**(iii)** hold and u is the solution in $\Pi_{(0,T]}$ of the homogeneous equation (1) which satisfy the condition*

$$\|u(t,\cdot)\|_p^{\boldsymbol{k}(t,\boldsymbol{a})} \leqslant CE^{-d}(T,t), \quad t\in(0,T] \tag{30}$$

with some constant $C>0$ and $p\in[1,\infty]$. Then for $p\in(1,\infty]$ there exists an unique function $\varphi\in L_p^{\boldsymbol{k}(0,a)}$, and for $p=1$ there exists an unique generalized measure $\mu\in M^{\boldsymbol{k}(0,\boldsymbol{a})}$ such that the solution u is presented in the form (28) or (29) respectively.

Let U_p, $p\in[1,\infty]$, be sets of all solutions of the homogeneous equation (1) that for any $t\in(0,T]$ belong to the spaces $L_p^{\boldsymbol{k}(t,\boldsymbol{a})}$ as function of x and the condition (30) holds. From the Theorems 2 and 3 one can formulate the following important corollaries.

Corollary 1. *The spaces $L_p^{\boldsymbol{k}(0,\boldsymbol{a})}$, $p\in(1,\infty]$, and $M^{\boldsymbol{k}(0,\boldsymbol{a})}$ and only they are the sets of initial values of the solutions from the classes U_p, $p\in(1,\infty]$, and U_1 correspondently.*

Corollary 2. *The classes U_p, $p\in(1,\infty]$, and U_1 are the sets of values of Poisson operators determined by the formulas (28) and (29) on the spaces $L_p^{\boldsymbol{k}(0,\boldsymbol{a})}$, $p\in(1,\infty]$, and $M^{\boldsymbol{k}(0,\boldsymbol{a})}$ correspondently, and these operators are isomorphisms.*

Let us present the theorem about the correct solvability of the problem without the initial condition for the homogeneous equation (1) in the case of very strong degeneration.

Theorem 4. *Let the conditions **(i)–(iii)** be valid and u be the solution in $\Pi_{(0,T]}$ of homogeneous equation (1) for which the inequality*

$$\|u(t,\cdot)\|^{\boldsymbol{k}(t,\boldsymbol{a})} := \sup_{x\in\mathbb{R}^n}\left(|u(t,x)|\exp\{-[\boldsymbol{k}(t,\boldsymbol{a}),X(t,0)]\}\right) \leqslant CE^{-d}(T,t)\varepsilon(t), \quad t\in(0,T], \tag{31}$$

holds, where the function $\varepsilon\colon(0,T]\to(0,\infty)$ such that $\varepsilon(t)\to 0$ as $t\to 0$. Then the function u is equal to zero identically.

Note that in a wider class of functions than the class defined by the inequality (31), nontrivial solutions for the homogeneous equation (1)can exist. For instance, we consider the equation

$$\left(\alpha(t)(\partial_t - x_1\partial_{x_2}) - \beta(t)\partial_{x_1}^2 + 1\right)u(t,x) = 0, \quad t\in(0,T], \quad x=(x_1,x_2)\in\mathbb{R}^2,$$

under assumption that $A(T,0)=\infty$, $B(T,0)=\infty$.

FSCP for this equation has a form

$$Z(t,x;\tau,\xi) = (2\pi)^{-1}3^{1/2}E^{-1}(t,\tau)(B(t,\tau))^{-2}$$

$$\times\exp\left\{-(4B(t,\tau))^{-1}|x_1-\xi_1|^2 - 3(B(t,\tau))^{-3}|x_2+2^{-1}B(t,\tau)(x_1+\xi_1)-\xi_2|^2\right\}, 0<\tau<t\leqslant T, \{x,\xi\}\subset\mathbb{R}^2,$$

and the functions $u_1(t,x) := E^1(T,t)$ and $u_2(t,x) := E^{1/2}(T,t)\exp\{2^{-1/2}x_1\}$, $(t,x)\in\Pi_{(0,T]}$, if $\beta(t)=1$, $t\in[0,T]$, are solutions. The functions u_1 and u_2 don't belong to the class determined by the inequality (31).

5. Proof of the results

We start from proof of the theorem 1 assertions. Note that the equation (25) can be a type of homogeneous equation (1) if instead of τ substitute a new variable $\tau'=-\tau$. Thus, under the conditions **(i)–(iii)** for the equation (25) there exists FSCP $Z^*(\tau,\xi;t,x)$, $0<\tau<t\leqslant T$, $\{\xi,x\}\in\mathbb{R}^n$.

We will use below the following Green–Ostrogradskii formula:

$$\int_{t_1}^{t_2}\frac{d\theta}{\alpha(\theta)}\int_{B_R}\left(\overline{v}Lu - u\overline{L^*v}\right)(\theta,y)dy = \int_{B_R}(\overline{v}u)(\theta,y)\Big|_{\theta=t_1}^{\theta=t_2}dy$$

$$-\int_{t_1}^{t_2}d\theta\int_{\Gamma_R}\left(\sum_{j=1}^{n_2}y_{1j}\mu_{2j}+\sum_{j=1}^{n_3}y_{2j}\mu_{3j}\right)(\overline{v}u)(\theta,y)dS_y - \int_{t_1}^{t_2}d\theta\int_{\Gamma_R}\sum_{j=1}^{n_1}B^j[v,u](\theta,y)\mu_{1j}dS_y, \tag{32}$$

where $0<t_1<t_2\leqslant T$, L and L^* are the differential expressions from (1) and (25), B_R is a ball in $\mathbb{R}^n$ with radius R and with center in the origin of coordinates, Γ_R is its boundary, $(\mu_{11},\ldots,\mu_{1n_1},\mu_{21},\ldots,\mu_{2n_2},\mu_{31},\ldots,\mu_{3n_3})$ is a unit vector of the outer normal to the boundary Γ_R,

$$B^j[v,u] := -\sum_{l=1}^{n_1}\left(a_{jl}\partial_{y_{1l}}u\overline{v} - u\partial_{y_{1l}}(a_{jl}\overline{v})\right) + a_ju\overline{v}, \quad j\in\mathbb{N}_{n_1},$$

u and v are enough smooth functions. Transition in the Green–Ostrogradskii formula to limit for suitable functions u and v gives us the formula

$$\int_{t_1}^{t_2} \frac{d\theta}{\alpha(\theta)} \int_{\mathbb{R}^n} \left(\overline{v} Lu - u\overline{L^* v}\right)(\theta, y) dy = \int_{\mathbb{R}^n} (\overline{v} u)(\theta, y)\big|_{\theta=t_1}^{t_2} dy \tag{33}$$

Due to the estimations (10) and the similar estimations for Z^* formula (33) holds, where $u(\theta, y) = Z(\theta, y; \tau, \xi)$, $v(\theta, y) = Z^*(\theta, y; t, x)$, $t_1 = \tau + \varepsilon$ and $t_2 = t - \varepsilon$, ε is an enough small positive number. Thus, one can get the formula

$$\int_{\mathbb{R}^n} \overline{Z^*(\tau+\varepsilon, y; t, x)} Z(\tau+\varepsilon, y; \tau, \xi) dy = \int_{\mathbb{R}^n} \overline{Z^*(t-\varepsilon, y; t, x)} Z(t-\varepsilon, y; \tau, \xi) dy,$$

after passing to the limit as $\varepsilon \to 0$, the inequality (26) follows.

In a similar way we obtain the equality

$$\int_{\mathbb{R}^n} \overline{Z^*(\beta, y; t, x)} Z(\beta, y; \tau, \xi) dy = \int_{\mathbb{R}^n} \overline{Z^*(t-\varepsilon, y; t, x)} Z(t-\varepsilon, y; \tau, \xi) dy. \tag{34}$$

The equality (27) can be written if in (34) to pass to the limit, as $\varepsilon \to 0$, and to use the (26) and the property (3) of the FSCP Z.

To prove the assertion 3 of the Theorem 1 let consider Z_1 and Z_2 be two normal FSCP for the equation (1). Formula (33)is used where $v(\theta, y) = Z_1(\theta, y; \tau, \xi)$, $u(\theta, y) = Z_2(t, x; \theta, y)$. As result we get the equality

$$\int_{\mathbb{R}^n} \overline{Z_1(t_2, y; \tau, \xi)} Z_2(t, x; t_2, y) dy = \int_{\mathbb{R}^n} \overline{Z_1(t_1, y; \tau, \xi)} Z_2(t, x; t_1, y) dy.$$

Since t_1 and t_2 are arbitrary points in the interval (τ, t), the last equality implies that the function

$$\int_{\mathbb{R}^n} \overline{Z_1(\theta, y; \tau, \xi)} Z_2(t, x; \theta, y) dy, \quad \theta \in (\tau, t), \quad \{x, \xi\} \subset \mathbb{R}^n,$$

is independent on θ. Let us denote this function $\Phi(t, x; \tau, \xi)$. Thus,

$$\Phi(t, x; \tau, \xi) = \int_{\mathbb{R}^n} \overline{Z_1(\theta, y; \tau, \xi)} Z_2(t, x; \theta, y) dy, \quad \theta \in (\tau, t), \quad \{x, \xi\} \subset \mathbb{R}^n. \tag{35}$$

Tending the variable θ in the equality (35) first to τ, and then to t, we find that

$$\Phi(t, x; \tau, \xi) = Z_2(t, x; \tau, \xi) = Z_1(t, x; \tau, \xi), \quad 0 < \tau < t \leqslant T, \quad \{x, \xi\} \subset \mathbb{R}^n.$$

The proofs of Theorem 2 and Theorem 3 are quite cumbersome. In the proofs the weight functions (20), their properties (21) and the inequalities (22)–(24) are essential. To prove the Theorem 4 we will use the following lemma.

Lemma 1. *Let t_0 be arbitrary number from the interval $(0, T)$ in the case of strong degeneration and from the semi-interval $[0, T)$, if the degeneration is weak; let a function $u(t, x)\colon \Pi_{[t_0, T]} \to \mathbb{C}$ be continued and it satisfies the condition*

$$\exists M > 0 \; \forall \in (t_0, T]\colon \|u(t, \cdot)\|^{\boldsymbol{k}(t, \boldsymbol{a})} \leqslant M$$

*and it is on $\Pi_{[t_0, T]}$ a solution of the homogeneous equation (1). If the coefficients of this equation fulfill the conditions **(i)**–**(iii)**, then the formula*

$$u(t, x) = \int_{\mathbb{R}^n} Z(t, x; t_0, \xi) u(t_0, \xi) d\xi, \quad (t, x) \in \Pi_{(t_0, T]} \tag{36}$$

holds.

The proof of Lemma 1 will be preceded by proof of the following inequality.

$$\exists\, \delta_1 \in (0, 1) \; \exists\, C_1 > 0 \; \forall\, R > 0 \; \forall\, x \in B_R \; \forall\, \xi \in \mathbb{R} \setminus B_{2R} \; \forall h \in (0, \delta_1)\colon \rho(h, x, \xi) \geqslant C_1 R^2/h. \tag{37}$$

For $h \in (0, 1)$

$$\sum_{j=1}^{3} h^{1-2j}|X_j(h)-\xi_j|^2 \geqslant h^{-1}\left(\sum_{j=1}^{3}|X_j(h)-\xi_j|^2\right) \geqslant |X(h)-\xi|^2 \geqslant \left||\xi|-|X(h)|\right|^2, \tag{38}$$

$$|X(h)| \leqslant |x| + |(0, h\hat{x}_1, hx_1' + 2^{-1}h^2x_2')| \leqslant R + R/2 = 3/2R. \tag{39}$$

The inequality (37) with $C_1 = 9/4$ follows from (38), (39) and from the fact that $\xi \in \mathbb{R} \setminus B_{2R}$. Now, let us turn to the proof of Lemma 1.

Proof. Let $G_R := (t_0, T] \times B_R$; η be a function from the space $C^\infty([0,\infty))$ such that $\eta(r) = 1$ for $r \in (0, 1/2]$, $\eta(r) = 0$ for $r \in [3/4, \infty)$ and $\eta' \leqslant 0$; (t, x) be an arbitrary fixed from $G_{\overline{R}}$, where $\overline{R} > 0$ is a fixed number. Let put in the formula (32) instead of t_1, t_2, $u(\theta, y)$ and $v(\theta, y)$ accordingly $t_0 + h$, $t - \varepsilon$, $u(\theta, y)$ and $v(\theta, y) := \eta(\frac{|y|}{R})Z^*(\theta, y; t, x)$, where $R \geqslant \overline{R}$, $0 < h < 2^{-1}(t - t_0)$, $0 < \varepsilon < 2^{-1}(t - t_0)$, and u is a function satisfying the conditions of the Proposition. Using properties of the function η, properties of the function Z of the Theorem 1 and a fact that $Lu = 0$ we obtain

$$\int_{\mathbb{R}^n} Z(t,x;t-\varepsilon,y)\eta\left(\frac{|y|}{R}\right)u(t-\varepsilon,y)dy = \int_{\mathbb{R}^n} Z(t,x;t_0+h,y)\eta\left(\frac{|y|}{R}\right)u(t_0+h,y)dy$$
$$- \int_{t_0+h}^{t-\varepsilon}\frac{d\theta}{\alpha(\theta)}\int_{B_{3R/4}\setminus B_{R/2}} \overline{\left[L^*(Z^*(t_0+h,y;t,x)\eta\left(\frac{|y|}{R}\right)\right]}u(\theta,y)dy,$$

and after passing to the limit, as $\varepsilon \to 0$, we get the equation

$$u(t,x) = \int_{\mathbb{R}^n} Z(t,x;t_0+h,y)\eta\left(\frac{|y|}{R}\right)u(t_0+h,y)dy$$
$$- \int_{t_0+h}^{t}\frac{d\theta}{\alpha(\theta)}\int_{B_{3R/4}\setminus B_{R/2}} \overline{\left[L^*(Z^*(\theta,y;t,x)\eta\left(\frac{|y|}{R}\right)\right]}u(\theta,y)dy =: I_1^{(R)} - I_2^{(R)}.$$

Let us pass to the limit as $R \to \infty$. The integral $I_1^{(R)}$ tends to

$$I_1 := \int_{\mathbb{R}^n} Z(t,x;t_0+h,y)u(t_0+h,y)dy.$$

Indeed, with the help of (10), (22), (37) and definition of the norm from (31),

$$\left|I_1 - I_1^{(R)}\right| = \left|\int_{\mathbb{R}^n} Z(t,x;t_0+h,y)\left(1-\eta\left(\frac{|y|}{R}\right)\right)u(t_0+h,y)dy\right|$$
$$\leqslant C(B(t,t_0+h))^{-M}\int_{\mathbb{R}^n\setminus B_{R/2}} E^d_{c-c_0}(t,x;t_0+h,y)E_{c_0}(t,x;t_0+h,y)\exp\{[\boldsymbol{k}(t_0+h,\boldsymbol{a}),y]\}$$
$$\times u(t_0+h,y)\exp\{-[\boldsymbol{k}(t_0+h,\boldsymbol{a}),y]\}dy \leqslant C\|u(t_0+h,\cdot)\|^{\boldsymbol{k}(t_0+h,\boldsymbol{a})}E^d(t,t_0+h)$$
$$\times \exp\{[\boldsymbol{k}(t_0+h,\boldsymbol{a}),X(B(t,0))]\}(B(t,t_0+h))^{-M}\int_{\mathbb{R}^n} E_{(c-c_0)/2}(t,x;t_0+h,y)dy$$
$$\times \exp\{-(c-c_0)C_1R^2/(2/B(t,t_0+h))\} \leqslant CC_2\|u(t_0+h,\cdot)\|^{\boldsymbol{k}(t_0+h,\boldsymbol{a})}E^d(t,t_0+h)$$
$$\times \exp\{[\boldsymbol{k}(t_0+h,\boldsymbol{a}),X(B(t,0))]\}\exp\left\{-(c-c_0)C_1R^2/(2B(t,t_0+h))\right\} \to 0,$$

as $R \to \infty$. C_1 is the constant from the inequality (37), and C_2 is a constant from the following inequality:

$$(B(t,t_0+h))^{-M}\int_{\mathbb{R}^n} E_{(c-c_0)/2}(t,x;t_0+h,y)dy \leqslant C_2. \tag{40}$$

This inequality is proved in a similar way as the corresponding inequalities for the equations without degenerations on the initial hyperplane.

Let's prove that $\lim\limits_{R\to\infty} I_2^{(R)} = 0$. Since $L^*Z^* = 0$, then the expression $L^*(Z^*(\theta,y;t,x)\eta\left(\frac{|y|}{R}\right)$ is a sum of products of expressions with constant coefficients including derivatives of the functions $\eta\left(\frac{|y|}{R}\right)$ and function $Z^*(\theta,y;t,x)$ with respect to the variable y. Using the property of normality of FSCP and the estimations (10), we get the estimation

$$\left|L^*(Z^*(\theta,y;t,x)\eta\left(\frac{|y|}{R}\right)\right| \leqslant C(B(t,\theta))^{-M}E_c^d(t,x;\theta,y).$$

With the help of this inequality and (22) similarly it is proved that $\lim\limits_{R\to\infty} I_2^{(R)} = 0$.

Let us turn to the proof of the Theorem 4. Let (t,x) be an arbitrary foxed point from $\Pi_{(0,T]}$ and t_0 be a fixed number from the interval $(0,t/2)$. The solution under consideration u, obviously, satisfies the condition

$$u(t,x)|_{t=t_0} = u(t_0,x), \quad x\in\mathbb{R}^n,$$

and for it the form (36) takes place. Since this form holds for any $t_0 \in (0,t/2)$, we can pass to the limit as $t_0 \to 0$ in it. Limit of the right part (36) is equal to zero. Really, with the help of the inequalities (10), (22), (31) and (40).

$$\begin{aligned}\left|\int_{\mathbb{R}^n} Z(t,x;t_0,\xi)u(t_0,\xi)d\xi\right| &\leqslant C\varepsilon(t_0)\int_{\mathbb{R}^n} E^d_{c-c_0}(t,x;t_0,\xi)\big(E_{c_0}(t,x;t_0,\xi)\\
&\qquad\times\exp\{[\boldsymbol{k}(t_0,\boldsymbol{a}),\xi]\}\big)E^{-d}(T,t_0)(B(t,t_0))^{-M}d\xi\\
&\leqslant C\varepsilon(t_0)\exp\{[\boldsymbol{k}(t,\boldsymbol{a}),X(B(t,0))]\}E^{-d}(T,t)\int_{\mathbb{R}^n}E^d_{c-c_0}(t,x;t_0,\xi)(B(t,t_0))^{-M}d\xi\\
&\leqslant CC_2\varepsilon(t_0)\exp\{[\boldsymbol{k}(t,\boldsymbol{a}),X(B(t,0))]\}E^{-d}(T,t)\to 0, \quad \text{if} \quad t_0\to 0.\end{aligned}$$

Thus, after passing in (36) to the limit as $t_0 \to 0$, we receive that $u(t,x) = 0$. A necessary fact follows because (t,x) is an arbitrary point from $\Pi_{(0,T]}$. The theorem is proved. ■

6. Conclusions

In the paper for homogeneous ultraparabolic equation of the Kolmogorov type with three groups of spatial variables including two groups of degeneration and with degeneration on the initial hyperplane, there are established conditions for the coefficients of the equation under which there exists FSCP for the adjoint equation. The following properties of FSCP for the adjoint equation are proved: normality of the FSCP, the convolution formula, uniqueness of the normal FSCP. Also we presented the theorems about integral representations of the solutions for all types of degeneration and the theorems about correct solvability of the Cauchy problem in the case of weak and strong degenerations, and for the problems without initial condition, if the degeneration of the equation is very strong. For such class of equations these results are new.

[1] Kolmogoroff A. N. Zufällige Bevegungen (Zur Theorie der Brownshen Bewegung). Annals of Mathematics. **35** (1), 116–117 (1934).

[2] Protsakh N. P., Ptashnyk B. Yo. Nonlinear ultraparabolic equations and variational inequalities. Naukova Dumka, Kyiv (2017), (in Ukrainian).

[3] Citti G., Pascucci A., Polidoro S. On the regularity of solutions to a nonlinear ultraparabolic equations arising in mathematical finance. Differential and Integral Equations. **14** (6), 701–738 (2001).

[4] Di Franchesco M., Pascucci A. On a class of degenerate parabolic equations of Kolmogorov type. Applied Mathematics Research eXpress. **2005** (6), 77–116 (2005).

[5] Di Franchesco M., Pascucci A. A continuous dependence result for ultraparabolic equations in option pricing. Journal of Mathematical Analysis and Applications. **336** (2), 1026–1041 (2007).

[6] Foschi P., Pascucci A. Kolmogorov equations arising in finance: direct and inverse problem. Lecture notes of Seminario Interdisciplinare di Matematica. Universita delgi studi della Basilicata. **1**, 145–156 (2007).

[7] Ivashyshen S. D., Medynsky I. P. The Fokker–Planck–Kolmogorov equations for some degenerate diffusion processes. Theory of Stochastic Processes. **16(32)** (1), 57–66 (2010).

[8] Lanconelli E., Pascucci A. On a class of hypoelliptic evolution equations. Rend. Sem. Mat. Univ. Politec. Torino. **52** (1), 29–63 (1994).

[9] Pascucci A. Kolmogorov equations in physics and finance. Progress in Nonlinear Differential Equations and Their Applications. **63**, 313–324 (2005).

[10] Polidoro S. A Global Lower Bound for the Fundamental Solution of Kolmogorov–Fokker–Planck Equations. Archive for Rational Mechanics and Analysis. **137**, 321–340 (1997).

[11] Eidelman S. D., Kochubei A. N., Ivasyshen S. D. Analytic methods in the theory of differential and pseudo-differential equations of parabolic type. Operator Theory: Advances and Applications. **152**. Birkhauser, Basel (2004).

[12] Ivashyshen S. D., Medynsky I. P. On applications of the Levi method in the theory of parabolic equations. Matematychni Studii. **47** (1), 33–46 (2017).

[13] Ivashyshen S. D., Medynsky I. P. The classical fundamental solution of a degenerate Kolmogorov's equation with coefficients independent on variables of degeneration. Bukovinian Mathematical Journal. **2** (2–3), 94–106 (2014), (in Ukrainian).

[14] Ivasyshen S. D., Medyns'kyi I. P. Classical fundamental solutions of the Cauchy problem for ultraparabolic equations of Kolmogorov type with two groups of spatial variables. Diff. equations and related topics. Proceedings of Institute of Mathematics NAS of Ukraine. **13** (1), 108–155 (2016), (in Ukrainian).

[15] Ivasyshen S. D., Medyns'kyi I. P. On the classical fundamental solutions of the Cauchy problem for ultraparabolic Kolmogorov–type equations with two groups of spatial variables. Journal of Mathematical Sciences. **231** (4), 507–526 (2018).

[16] Ivasyshen S. D., Medyns'kyi I. P. Classical fundamental solution of the Cauchy problem for ultraparabolic Kolmogorov–type equations with two qroups of spatial variables of degeneration. I. Journal of Mathematical Sciences. **246** (2), 121–151 (2020).

[17] Ivasyshen S. D., Medyns'kyi I. P. Classical fundamental solution of the Cauchy problem for ultraparabolic Kolmogorov–type equations with two qroups of spatial variables of degeneration. II. Journal of Mathematical Sciences. **247** (1), 1–23 (2020).

[18] Voznyak O. G., Ivashyshen S. D., Medynsky I. P. On fundamental solution of the Cauchy problem for ultraparabolic Kolmogorov's equation with degeneration on the initial hyperplane. Bukovinian Mathematical Journal. **3** (3–4), 41–51 (2015), (in Ukrainian).

Властивості фундаментальних розв'язків, коректна розв'язність задачі Коші та інтегральні зображення розв'язків для ультрапараболічних рівнянь типу Колмогорова з трьома групами просторових змінних та виродженням на початковій гіперплощині

Возняк О. Г.[1], Дронь В. С.[2], Мединський І. П.[3]

[1] *Західноукраїнський національний університет, вул. Львівська, 11, 46009, Тернопіль, Україна*
[2] *Інститут прикладних проблем механіки і метематики ім. Я. С. Підстригача НАН України, вул. Наукова, 3-Б, 79060, Львів, Україна*
[3] *Національний університет "Львівська політехніка", вул. С. Бандери, 12, 79013, Львів, Україна*

Для однорідного ультрапараболічного рівняння типу Колмогорова з трьома групами просторових змінних (в т.ч. двома групами просторових змінних виродження) і виродженням на початковій гіперплощині встановлено деякі властивості фундаментального розв'язку задачі Коші. Для всіх випадків виродження на початковій гіперплощині доведено теореми про інтегральні зображення розв'язків і коректну розв'язність задачі Коші в класах вагових функцій. Для рівнянь з указаного класу ці результати є новими.

Ключові слова: *ультрапараболічне рівняння типу Колмогорова, виродження на початковій гіперплощині, фундаментальний розв'язок задачі Коші, вагові простори, інтегральні зображення розв'язків, коректна розв'язність задачі Коші.*