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Stability analysis of a fractional model for the transmission of the cochineal

El Baz O.[1], Ait Ichou M.[2], Laarabi H.[1], Rachik M.[1]

[1] *Laboratory of Analysis Modeling and Simulation, 20670, Casablanca, Morocco*
[2] *Laboratory of Mathematics and Applications, ENS, Casablanca, Morocco*

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Scale insects are parasitic insects that attack many indoor and outdoor plants, including cacti and succulents. These insects are among the frequent causes of diseases in cacti: for the reason that they are tough, multiply in record time and could be destructive to these plants, although they are considered resistant. Mealybugs feed on the sap of plants, drying them out and discoloring them. In this research, we propose and investigate a fractional model for the transmission of the Cochineal. In the first place, we prove the positivity and boundedness of solutions in order to ensure the well-posedness of the proposed model. The local stability of the disease-free equilibrium and the chronic infection equilibrium is established. Numerical simulations are presented in order to validate our theoretical results.

Keywords: *fractional differential equations; cochineal; Caputo fractional derivative; SIRC epidemic model; local stability; numerical simulations.*

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1. Introduction

Scale insects are parasitic insects that attack many indoor and outdoor plants, including cacti and succulents. These beasts are among the frequent causes of disease in cacti: they are tough, multiply in record time and can therefore be destructive on these plants, which are nevertheless considered resistant. Mealybugs feed on the sap of plants, drying them out and discolouring them [1]!

The cochineal is a parasitic insect that does not exceed half a centimeter in length. It has a kind of rostrum to suck the sap of the plants it colonizes. The female scale insect lays thousands of eggs which appear as fluffy, whitish heaps. These white clusters are often encountered, signs of the presence of the mealybug. Other mealybug eggs take the form of round spots of shiny wax [2].

This pest can be seen on the stems or on the leaves (along the veins or on the underside). We can identify the insects themselves, or spot the infestation by observing cottony, mealy or waxy clumps attached to the organs of the plant. The leaves can also become covered with a sticky honeydew, on which sooty mold (soot-like black deposit) develops. Affected twigs weaken from lack of sap. While mealybugs rarely kill their host, they can still cause significant damage.

Mathematically, it has been observed in mathematical modeling that fractional order models can provide a better fit between measured and simulated data than classical integer order models [3–9]. A first attempt at a fractional model to simulate the Dengue epidemic was reported by Pooseh et al who suggested replacing the first derivative (of integer order) with a fractional derivative in the Riemman–Liouville sense of order $\alpha \in [0,1]$. In his work Diethelm (2012) [10] criticizes Pooseh's approach, explaining, on the one hand, that the choice of fractional derivatives in the R-L sense has the disadvantage that the derivative of order α of a constant is not zero and on the other side, that the differential equations of R-L cannot be combined with the classical initial conditions of the considered form. Then, Diethelm found that a simple dimensional analysis of the model proposed by Poosehet shows that the left-hand sides of the system equations have the dimension $(temps)^{-\alpha}$ so that, the right sides have the dimension $(temps)^{-1}$. To remedy these inconveniences, Diethelm proposes a fractional model to simulate the epidemic of Dengue fever, he considers the fractional derivatives in the sense of Caputo and adjusted the dimensions [11].

The aim of this work is to propose a representative mathematical model to simulate the mealybug epidemic by taking into account the memory of the phenomenon by the fractional derivatives in the sense of Caputo.

2. Preliminaries

In this section we recall some fundamental concepts of fractional differential calculus where the derivative is in the Caputo sense.

Definition 1. *The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^n(\mathbb{R}^+;\mathbb{R})$ is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is a positive integer such that $\alpha \in (n-1,n)$. Also, the corresponding fractional integral of order α with $\mathrm{Re}(\alpha) > 0$ is given by

$$I^\alpha_{[0,t]\}} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

where $\Gamma(\cdot)$ is the Gamma function [12].

Lemma 1. *Let $f, g\colon [a,b] \to \mathbb{R}$ be such that $D^\alpha f(t)$ and $D^\alpha g(t)$ exist almost everywhere and let $a_1, a_2 \in \mathbb{R}$. Then $D^\alpha(a_1 f(t) + a_2 g(t))$ exists almost everywhere, and*

$$D^\alpha(a_1 f(t) + a_2 g(t)) = a_1 D^a f(t) + a_2 D^a g(t).$$

Further the Caputo fractional derivative for a constant function is zero [13].

Lemma 2. *Suppose that $f \in C[a,b]$ and $D^\alpha f \in C[a,b]$ with $0 < \alpha \leqslant 1$. Then there exists $\xi(x) \in [a,x]$, such that*

$$f(x) = f(a) + \frac{1}{\alpha} D^\alpha f(\xi)(x-a)^\alpha.$$

Based on the previous Lemma we have the following result [14].

Corollary 1. *Suppose that $f \in C([a,b])$ and $D^\alpha f \in C([a,b])$. If $D^\alpha f(t) \geqslant 0$, (resp: $D^\alpha f(t) \leqslant 0$) $\forall t \in (a,b)$, then f is non-decreasing (resp: non-increasing) in $[a,b]$.*

Definition 2. *The constant point x^* is a steady state of the fractional model*

$$D^\alpha x(t) = f(t, x(t)),$$

if and only if $f(t, x^*) = 0$ for all $t > 0$.

Lemma 3. *Let $\alpha \in (0,1)$ and consider a continuous function $x\colon [t_0,\infty) \to \mathbb{R}$ satisfying the following condition*

$$D^\alpha x(t) + \mu x(t) \leqslant v, \quad t \geqslant t_0, \quad \mu, v \in \mathbb{R}, \quad \mu \neq 0.$$

Then we have the inequality

$$x(t) \leqslant \left(x(t_0) - \frac{v}{\mu}\right) E_\alpha\left(-\mu(t-t_0)^\alpha\right) + \frac{v}{\mu},$$

for all $t \geqslant t_0$, where E_α is the Mittag–Leffler function of one parameter defined by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}.$$

We can now state the following existence result for fractional differential equations [15].

Theorem 1. *Let $\alpha \in (0,1]$, $\Omega \subset \mathbb{R}^n$ a domain and $f\colon [t_0,\infty) \times \Omega \to \mathbb{R}^n$ be a function satisfying the Lipschitz condition on x and consider the following fractional order equation*

$$D^{\alpha}x(t) = f(t, x(t)), \quad t > t_0,$$

with the initial condition $x(t_0) = x_0 \in \Omega$. Then the above system has a unique maximal solution [16].

Lemma 4. *Let $\tau > 0$, $\alpha \in (0,1]$, A, B two $(n \times n)$ square matrices and $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$. Consider the linear fractional delayed differential system with the Caputo derivative*

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + Bx(t-\tau), \quad t > 0, \\ x(t) = \varphi(t), t \in [-\tau, 0]. \end{cases} \tag{1}$$

We define the characteristic equation of system 1 by

$$\Delta(s) = \det\left(s^{\alpha} I_n - A - Be^{-st}\right) = 0.$$

If all the roots of the characteristic equation $\Delta(s) = 0$ have negative real parts, then the zero solution of system 1 is locally asymptotically stable [17].

Lemma 5. *(1) If all the eigenvalues λ of the matrix $M = A + B$ satisfy $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ and the characteristic equation $\Delta(s) = 0$ has a no purely imaginary roots for $\tau > 0$ then the zero solution of system* (1) *is locally asymptotically stable. (2) Suppose $\tau = 0$. If all the eigenvalues λ of M satisfy $|\arg(\lambda)| > \alpha\frac{\pi}{2}$, then the zero solution of 1 is locally asymptotically stable [14].*

Lemma 6. *Let $x^* \in \Omega \subset \mathbb{R}^n$ be an equilibrium point of the system*

$$D^{\alpha}x(t) = f(t, x(t)), \quad t \geqslant t_0,$$

and let $V(t,x)\colon [t_0,\infty) \times \Omega \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leqslant V(t,x) \leqslant W_2(x), \quad D^{\alpha}V(t,x) \leqslant -W_3(x),$$

for $t \geqslant t_0$ and $x \in \Omega$, where $W_i(x)$, $i = 1,2,3$ are continuous and positively defined functions on Ω. Then x^* is uniformly asymptotically stable [18].

3. Formulation of the mathematical model

The spatial spread of mealybug-infected cacti can be modeled initially using a four-compartment SIR-C system.

- **The compartment** S: The number of cacti belonging to the area that are not diseased but likely to become diseased.
- **The compartment** I: The number of cacti infected by the mealy bug.
- **The compartment** R: The number of cacti that once had the disease and are now immune to the mealy bug.
- **The compartment** C: The growth rate of the mealy bug.

$$\begin{cases} \dfrac{\partial^{\alpha} S}{\partial^{\alpha} t} = \Lambda - \gamma SC + \eta R - \mu_s S, \\ \dfrac{\partial^{\alpha} I}{\partial^{\alpha} t} = \gamma SC - \theta I - \lambda I, \\ \dfrac{\partial^{\alpha} R}{\partial^{\alpha} t} = -\eta R + \theta I, \\ \dfrac{\partial^{\alpha} C}{\partial^{\alpha} t} = \mu C\left(1 - \dfrac{C}{K}\right) - \beta C, \end{cases} \tag{2}$$

with the following non-negative initial conditions

$$S(0) \geqslant 0, \quad I(0) \geqslant 0, \quad R(0) \geqslant 0, \quad C(0) \geqslant 0,$$

where γ is rate of mushroom encounters with palms, β is fungal mortality rate, λ is death rate of palms due to infection, θ is palm healing rate, η is rate of healed palms that become susceptible due to loss of immunity, Λ is recruitment rate (birth rate), μ is natural mortality rate, μ_s is natural mortality rate of cacti.

4. Existence of solution

The model (2) is described by the first order nonlinear differential equations system. It can be rewritten in the following matrix form:

$$D^{\alpha}X(t) = F(X),$$

where

$$X(t) = \begin{pmatrix} S(t) \\ I(t) \\ R(t) \\ C(t) \end{pmatrix},$$

and F is the function of class $\mathcal{C}^{\infty}$ on $\mathbb{R}^4$ with values in $\mathbb{R}^4$ defined by

$$F(X) = \begin{pmatrix} F_1(X) = \Lambda - \gamma SC + \eta R - \mu_s S \\ F_2(X) = \gamma SC - \theta I - \lambda I \\ F_3(X) = -\eta R + \theta I \\ F_4(X) = \mu C\left(1 - \dfrac{C}{K}\right) - \beta C \end{pmatrix}.$$

We study the existence of the solution of the system 2 in the region $[0, T]$, where: $\max(|S|, |I|, |R|, |C|) \leqslant M$ and $T < +\infty$. We note $\bar{X} = (\bar{S}, \bar{I}, \bar{R}, \bar{C})$. For all $X, \bar{X} \in \mathbb{R}^4_+$, it follows that

$$\begin{aligned}
\|F(X) - F(\bar{X})\| &= \left|F_1(X) - F_1(\bar{X})\right| + \left|F_2(X) - F_2(\bar{X})\right| + \left|F_3(X) - F_3(\bar{X})\right| + \left|F_4(X) - F_4(\bar{X})\right| \\
&= \left|-\gamma SC + \eta R - \mu_s S + \gamma\bar{S}\bar{C} - \eta\bar{R} + \mu_s\bar{S}\right| \\
&\quad + \left|\gamma SC - \theta I - \lambda I - \gamma\bar{S}\bar{C} + \theta\bar{I} + \lambda\bar{I}\right| + \left| - \eta R + \theta I + \eta\bar{R} - \theta\bar{I}\right| \\
&\quad + \left|\mu C\left(1 - \frac{C}{K}\right) - \beta C - \mu\bar{C}\left(1 - \frac{\bar{C}}{K}\right) - \beta\bar{C}\right| \\
&= \left|\eta(R - \bar{R}) - \mu_s(S - \bar{S}) - \gamma(SC - \bar{S}\bar{C})\right| + |\gamma(SC - \bar{S}\bar{C}) - (\theta + \lambda)(I - \bar{I})| \\
&\quad + |\theta(I - \bar{I}) - \eta(R - \bar{R})| + |\mu(C - \bar{C})| - \frac{\mu}{K}(C^2 - \bar{C}^2) - \beta(C - \bar{C})| \\
&\leqslant 2\mu|R - \bar{R}| + \mu_s|S - \bar{S}| + 2\gamma|SC - \bar{S}\bar{C}| + (2\theta + \lambda)|I - \bar{I}| \\
&\quad + (\mu + \beta)|C - \bar{C}| + \frac{\mu}{K}|C^2 - \bar{C}^2| \\
&\leqslant 2\mu|R - \bar{R}| + (\mu_s + 2\gamma M)|S - \bar{S}| + (2\theta + \lambda)|I - \bar{I}| \\
&\quad + \left(2\gamma M + \mu + \beta + \frac{2\mu M}{K}\right)|C - \bar{C}| \\
&\leqslant L\|X - \bar{X}\|,
\end{aligned}$$

where

$$L = \max\left((\mu_s + 2\gamma M), (2\theta + \lambda), 2\eta, \left(2\gamma M + \mu + \beta + \frac{2\mu M}{K}\right)\right).$$

Thus, $F(X)$ satisfies the Lipschitz condition with respect to X, it follows from the theorem that for each $X_0 = (S_0, I_0, R_0, C_0) \in \mathbb{R}^4_+$, there is a unique solution $X(t) \in \mathbb{R}^4_+$ of the system (2) with the initial condition X_0, which is defined for any $t \geqslant 0$.

5. Stability of equilibrium points

The remaining equilibria of system (2) satisfy the following algebraic equations

$$\Lambda - \gamma SC + \eta R - \mu_s S = 0, \tag{3}$$

$$\gamma SC - \theta I - \lambda I = 0, \tag{4}$$

$$-\eta R + \theta I = 0, \tag{5}$$

$$\mu C\left(1-\frac{C}{K}\right)-\beta C=0, \tag{6}$$

From (6)

$$C\left(\mu\left(1-\frac{C}{K}\right)-\beta\right)=0 \Rightarrow C=0 \quad \text{or} \quad C=K-\frac{\beta K}{\mu}.$$

For $C=0$: (3) $\Rightarrow \Lambda+\eta R-\mu_s S=0 \Rightarrow S=\frac{\Lambda}{\mu_s}$, (4) $\Rightarrow I=0$, (5) $\Rightarrow R=0$, then $E_0=\left(\frac{\Lambda}{\mu_s},0,0,0\right)$.

For $C=C^*=K-\frac{\beta K}{\mu}$: From (3) we have $\gamma SC=\Lambda+\eta R-\mu_s S$. We replace in (4) $\Lambda+\eta R-\mu_s S-\theta I-\lambda I=0$, where we have $\eta R-\theta I=0$, $\Lambda-\mu_s S-\lambda I=0 \Rightarrow S=\frac{\Lambda-\lambda I}{\mu_s}$. From (4) on a $\gamma SC-\theta I-\lambda I=0$, then,

$$I^*=\frac{\Lambda K(\mu-\beta)}{\lambda(K\mu-\beta K)+\frac{\theta+\lambda}{\gamma}\mu\mu_s}.$$

From (5), $R=\frac{\theta}{\eta}I$, then,

$$R^*=\frac{\theta\Lambda K(\mu-\beta)}{\lambda\eta(K\mu-\beta K)+\eta\frac{\theta+\lambda}{\gamma}\mu_s\mu}, \quad \text{thus,} \quad S^*=\frac{\Lambda(\theta+\lambda)\mu}{\gamma\lambda(K\mu-\beta K)+(\theta+\lambda)\mu\mu_s}$$

therefore,

$$E^*=(S^*,I^*,R^*,C^*).$$

Local stability. Here we study local stability of both free E_0 and endemic steady states E^*.

Theorem 2. *If $\mu \leqslant \beta$, then E_0 is locally asymptotically stable.*

Proof. The Jacobian matrix for the system (2) evaluated at E_0 is

$$J_{E_0}=\begin{pmatrix}-\mu_s & 0 & \eta & 0\\ 0 & -\theta-\lambda & 0 & 0\\ 0 & \theta & -\eta & 0\\ 0 & 0 & 0 & \mu-\beta\end{pmatrix}.$$

One of the eigenvalues of $J(E_0)$ is $r_1=-\mu_s<0$. We put $d_1=\theta+\lambda$ and $d_2=\mu-\beta$. Then, we consider the following matrix

$$P_1=\begin{pmatrix}-\mu_s-r & 0 & \eta & 0\\ 0 & -d_1-r & 0 & 0\\ 0 & \theta & -\eta-r & 0\\ 0 & 0 & 0 & d_2-r\end{pmatrix}.$$

Hence, the eigenvalues of J_{E_0} are $\xi_1=-\mu_s$, $\xi_2=d_2$, $\xi_3=-d_1$ and $\xi_4=-\eta$. Clearly, ξ_2 satisfies condition (2) if $\mu \leqslant \beta$, since ξ_1, ξ_3 and ξ_4 are negative, proving the desired result. ∎

Theorem 3. *If $\mu \geqslant \beta$, then E^* is locally asymptotically stable.*

Proof. Let J_{E^*} be the Jacobian matrix of the system (2) evaluated at E_1,

$$J_{E^*}=\begin{pmatrix}-\gamma C^*-\mu_s & 0 & \eta & 0\\ \gamma C^* & -\theta-\lambda & 0 & 0\\ 0 & \theta & -\eta & 0\\ 0 & 0 & 0 & \mu-\dfrac{2\mu}{K}C^*-\beta\end{pmatrix}.$$

One of the eigenvalues of $J(E^*)$ is $r_1=-\gamma C^*-\mu_s<0$. We put $d_1=\gamma C^*+\mu_s$, $d_2=\theta+\lambda$ and $d_3=\mu-\beta-\frac{2\mu}{K}C^*$. Then, we consider the following matrix

$$P_2=\begin{pmatrix}-d_1-r & 0 & \eta & 0\\ \gamma C^* & -d_2-r & 0 & 0\\ 0 & \theta & -\eta-r & 0\\ 0 & 0 & 0 & d_3-r\end{pmatrix}.$$

The characteristic equation of P_2 can be written as

$$r^4 + h_3 r^3 + h_2 r^2 + h_1 r + h_0 = 0,$$

where

$$\begin{aligned} h_3 &= -d_3 + \eta + d_1 + d_2 > 0, \\ h_2 &= -d_3(\eta + d_1 + d_2) + \eta(d_1 + d_2) + d_1 d_2 > 0, \\ h_1 &= \eta\gamma\lambda C^* + \eta\mu_s d_2 - d_3\left(2d_1 d_2 + (\eta+1)d_1 + \eta d_2\right) > 0, \\ h_0 &= -d_3\left(\eta\lambda C^* + \eta\mu_s d_2\right) > 0. \end{aligned}$$

If $d_3 < 0 \Rightarrow \mu \geqslant \beta$, we have

$$\begin{gathered} h_3 > 0, \quad h_2 > 0, \quad h_1 > 0, \quad h_0 > 0, \\ h_2 h_3 > h_1, \quad h_1\left(h_2 h_3 - h_1\right) > h_0 h_3^2. \end{gathered}$$

Then by the FR-H all the eigenvalues of the P_2 matrix have negative real parts [19, 20].

Then from Theorem 3, E^* is locally asymptotically stable. ■

6. Numerical simulations

In this section, we validate our theoretical results by numerical simulations. Let Δt be the size of the time step such as $t_n = n\Delta t$ for $n \in \mathbb{N}$. The fractional derivative of Caputo can be approximated by

$$ {}^C D_t^\alpha X(t_n) \approx \frac{1}{\Delta t^\alpha} \sum_{j=0}^{n} \zeta_j^\alpha X(t_{n-j}) - \bar{X}_n, \tag{7}$$

where $\bar{X} = \frac{X(0)t_n^{-\alpha}}{\Gamma(1-\alpha)}$ and ζ_j^α is the fractional binomial coefficient with the recursive formula

$$\zeta_j^\alpha = \left(1 - \frac{1+\alpha}{j}\right)\zeta_{j-1}^\alpha, \quad \text{and} \quad \zeta_0^\alpha = 1.$$

For simplicity, we note the approximations of (S, I, R, C) solution of the system (2) at the discretized point t_n by (S^n, I^n, R^n, C^n). By applying (7), we find

$$\begin{aligned} \frac{1}{\Delta t^\alpha}\left(S^{n+1} + \sum_{j=0}^{n} \zeta_j^\alpha S^{n+1-j}\right) - \bar{S}^{n+1} &= \Lambda - \gamma S^n C^n - \eta R^n - \mu_s S^n, \\ \frac{1}{\Delta t^\alpha}\left(I^{n+1} + \sum_{j=0}^{n} \zeta_j^\alpha I^{n+1-j}\right) - \bar{I}^{n+1} &= \gamma S^n C^n - \theta I^n - \lambda I^n, \\ \frac{1}{\Delta t^\alpha}\left(R^{n+1} + \sum_{j=0}^{n} \zeta_j^\alpha R^{n+1-j}\right) - \bar{R}^{n+1} &= -\eta R^n + \theta I^n, \\ \frac{1}{\Delta t^\alpha}\left(C^{n+1} + \sum_{j=0}^{n} \zeta_j^\alpha C^{n+1-j}\right) - \bar{C}^{n+1} &= \mu C^n\left(1 - \frac{C^n}{k}\right) - \beta C^n. \end{aligned}$$

Thus,

$$\begin{aligned} S^{n-1} &= \Delta t^\alpha(\bar{S}^{n+1} + \Lambda - \gamma S^n C^n - \eta R^n - \mu_s S^n) - \sum_{j=0}^{n} \zeta_j^\alpha S^{n+1-j}, \\ I^{n-1} &= \Delta t^\alpha(\bar{I}^{n+1} + \gamma S^n C^n - \theta I^n - \lambda I^n) - \sum_{j=0}^{n} \zeta_j^\alpha I^{n+1-j}, \\ R^{n-1} &= \Delta t^\alpha(\bar{R}^{n+1} - \eta R^n + \theta I^n) - \sum_{j=0}^{n} \zeta_j^\alpha R^{n+1-j}, \\ C^{n-1} &= \Delta t^\alpha\left(\bar{C}^{n+1} + \mu C^n\left(1 - \frac{C^n}{k}\right) - \beta C^n\right) - \sum_{j=0}^{n} \zeta_j^\alpha C^{n+1-j}. \end{aligned}$$

For the numerical illustrations, we choose in all the section $\gamma = 0.01$, $\eta = 10^{-2}$, $\theta = 0.00002$, $\lambda = 0.00001$, $k = 0.2$, $\beta = 0.09$, $\mu = 0.00001$, $\mu_s = 0.09$, and $\Lambda = 10$. The initial conditions used are: $S(0) = 1000$, $I(0) = 8$, $R(0) = 2$, and $C(0) = 0.2$.

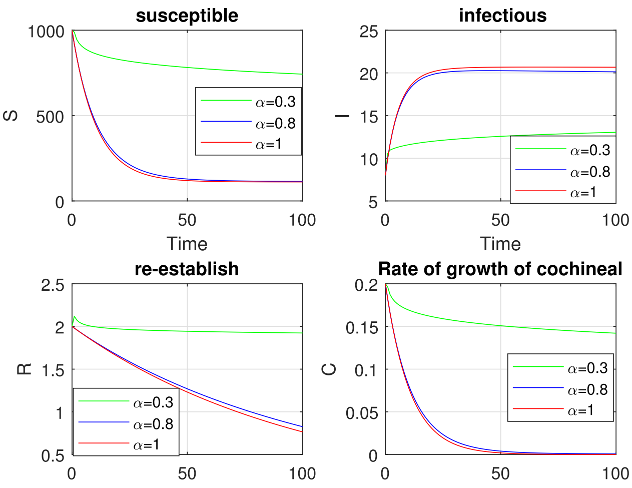

Fig. 1. Variations of the population number for different values of α.

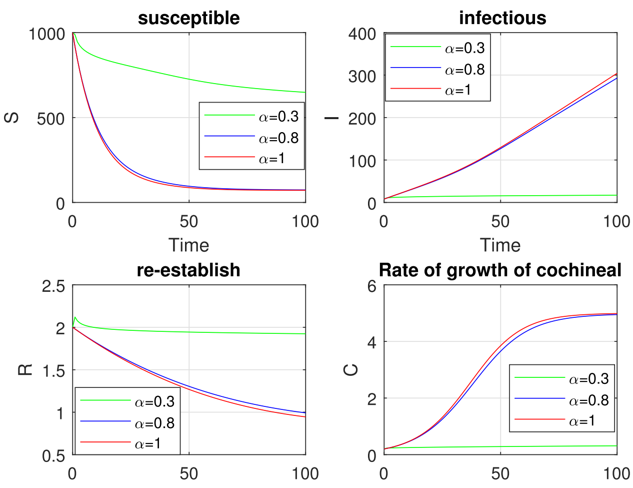

Fig. 2. Variations of the population for different values of α.

In this case, $\mathcal{R}_0 = 9 > 1$. According to Theorem 2, the infection-free equilibrium E_0 is locally asymptotically stable. Figure 1 confirms this result. In the case of $\mathcal{R}_0 < 1$, we chose $\mu = 0.00001$, $\beta = 0.09$ and leave the other values unchanged. By a simple calculation, we have $\mathcal{R}_0 = 1.11 \cdot 10^{-4} < 1$, and the system has a unique equilibrium point of free infection $E_0(1.1111 \cdot 10^2, 0, 0, 0)$, which is locally asymptotically stable, meaning that the mealy bug will disappear and the cactus will be completely cured. Figure 2 illustrates this result.

7. Conclusion

In this paper, we have studied a model that writes the growth dynamics of the mealy bug. We have shown that the global dynamics of the model is entirely determined by a threshold parameter which is the basic reproduction number $\mathcal{R}_{\prime}$. We proved that the equilibrium without infection E_* is locally asymptotically stable when $\mathcal{R}_{\prime} < 1$, which means that the growth rate of the mealybug decreases and thus leads to the eradication of the mealybug. Whereas, when $\mathcal{R}_{\prime} > 1$, the growth rate of cochineal increases. From the above analytical results, we deduce that the order α of the fractional derivative does not affect the stability of the equilibria, but from the numerical simulations, we observe that when the value of α decreases, the solutions of the model converge rapidly to the states, so the fractional order can affect the arrival time in the stable states.

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Аналіз стійкості дробової моделі передачі кошенілі

Ель Баз О.[1], Айт Ічоу М.[2], Лаарабі Х.[1], Рачік М.[1]

[1] *Лабораторія аналізу моделювання та симуляції, 20670, Касабланка, Марокко*
[2] *Лабораторія математики та застосунків, ENS, Касабланка, Марокко*

Лускокрилі — це паразитичні комахи, які уражують багато кімнатних і вуличних рослин, включаючи кактуси та сукуленти. Ці комахи є одними з частих причин захворювань кактусів: вони витривалі, розмножуються за рекордно короткий час і можуть бути згубними для цих рослин, хоча вони і вважаються стійкими. Борошнисті червці живляться соком рослин, висушуючи і знебарвлюючи їх. У цьому дослідженні пропонується та досліджується дробова модель передачі кошенілі. Спершу доводиться додатність і обмеженість розв'язків, щоб переконатися в коректності запропонованої моделі. Встановлюється локальна стійкість рівноваги без захворювання та рівноваги хронічної інфекції. Для підтвердження наших теоретичних результатів подано чисельне моделювання.

Ключові слова: *дробові диференціальні рівняння; кошеніль; дробова похідна Капуто; модель епідемії SIRC; локальна стійкість; чисельне моделювання.*